Quantized Solitons in the Extended Skyrme-Faddeev Model

L. A. Ferreira, S. Kato, N. Sawado, K. Toda

Abstract

The construction of axially symmetric soliton solutions with non-zero Hopf topological charges according to a theory known as the extended Skyrme-Faddeev model, was performed in [1]. In this paper we show how masses of glueballs are predicted within this model.

Keywords: integrable systems, solitons, monopoles, instantons, semiclassical quantizations.

We construct static soliton solutions carrying non-trivial Hopf topological charges for a field theory that has found interesting applications in many areas of physics. It is a $(3+1)$-dimensional Lorentz invariant field theory for a triplet of scalar fields $\vec{n}$, living on the two-sphere S^2, $\vec{n}^2 = 1$, and defined by the Lagrangian density

$$\mathcal{L} = M^2\,\partial_\mu\vec{n}\cdot\partial^\mu\vec{n} - \frac{1}{e^2}\,(\partial_\mu\vec{n}\wedge\partial_\nu\vec{n})^2 + \frac{\beta}{2}\,(\partial_\mu\vec{n}\cdot\partial^\mu\vec{n})^2\,, \tag{1}$$

where the coupling constants e^2 and β are dimensionless, and M has a mass dimension. The first two terms correspond to the so-called Skyrme-Faddeev (SF) model [2, 3, 4] proposed by the following idea of Skyrme [5]. It was conjectured by Faddeev and Niemi [6] that the SF model describes the low energy (strong coupling) regime of the pure SU(2) Yang-Mills theory. This was based on the so-called Cho-Faddeev-Niemi-Shabanov decomposition [6, 7, 8] of the SU(2) Yang-Mills field $\vec{A}_\mu$, where its six physical degrees of freedom are encoded into a triplet of scalars $\vec{n}$ ($\vec{n}^2 = 1$), a massless U(1) gauge field, and two real scalar fields. Gies has computed the one-loop Wilsonian effective action for the SU(2) Yang-Mills theory and has found agreements with the conjecture, which is provided that the SF model is modified by additional quartic terms in derivatives of the $\vec{n}$ field [9].

One can now stereographically project S^2 on a plane and work with a complex scalar field u related to the triplet $\vec{n}$ by

$$\vec{n} = \frac{1}{1+|\,u\,|^2}\,(u+u^*, -i(u-u^*), |\,u\,|^2 - 1). \tag{2}$$

The static Hamiltonian associated to (1) is

$$\mathcal{H}_{\rm static} = 4\,M^2\,\frac{\partial_i u\,\partial_i u^*}{(1+|\,u\,|^2)^2} - \frac{8}{e^2}\left[\frac{(\partial_i u)^2(\partial_j u^*)^2}{(1+|\,u\,|^2)^4} + (\beta\,e^2 - 1)\,\frac{(\partial_i u\,\partial_i u^*)^2}{(1+|\,u\,|^2)^4}\right]. \tag{3}$$

Therefore, it is positive definite for $M^2 > 0$, $e^2 < 0$, $\beta < 0$, $\beta e^2 \geq 1$. The Euler-Lagrange equations from (1) read

$$(1+|\,u\,|^2)\,\partial^\mu\mathcal{K}_\mu - 2\,u^*\,\mathcal{K}_\mu\,\partial^\mu u = 0, \tag{4}$$

together with its complex conjugate, where

$$\mathcal{K}_\mu := M^2\,\partial_\mu u + \frac{4}{e^2}\,\frac{\left[(\beta\,e^2 - 1)\,(\partial_\nu u\,\partial^\nu u^*)\,\partial_\mu u + (\partial_\nu u\partial^\nu u)\partial_\mu u^*\right]}{(1+|\,u\,|^2)^2}. \tag{5}$$

We choose to use the toroidal coordinates defined as

$$\begin{aligned} x^1 &= \frac{r_0}{p}\,\sqrt{z}\,\cos\varphi, \\ x^2 &= \frac{r_0}{p}\,\sqrt{z}\,\sin\varphi, \\ x^3 &= \frac{r_0}{p}\,\sqrt{1-z}\,\sin\xi, \end{aligned} \tag{6}$$

where $p = 1 - \cos\xi\,\sqrt{1-z}$, x^i ($i = 1, 2, 3$) are the Cartesian coordinates in $\mathbb{R}^3$, and (z, ξ, φ) are the toroidal coordinates. We have $0 \leq z \leq 1$, $-\pi \leq \xi \leq \pi$, $0 \leq \varphi \leq 2\pi$, and r_0 is a free parameter with dimension of length. We use the ansatz for the solution

$$u = \sqrt{\frac{1-g(z,\xi)}{g(z,\xi)}}\,e^{i\,\Theta(z,\xi) + i\,n\,\varphi}, \tag{7}$$

with n being an integer. We now impose the boundary conditions

$$\begin{aligned} g(z=0,\xi) &= 0, \\ g(z=1,\xi) &= 1, \quad \text{for} \quad -\pi \leq \xi \leq \pi \end{aligned} \tag{8}$$

and

$$\begin{aligned} \Theta(z,\xi=-\pi) &= -m\,\pi, \\ \Theta(z,\xi=\pi) &= m\,\pi, \quad \text{for} \quad 0 \leq z \leq 1 \end{aligned} \tag{9}$$

with m being an integer.

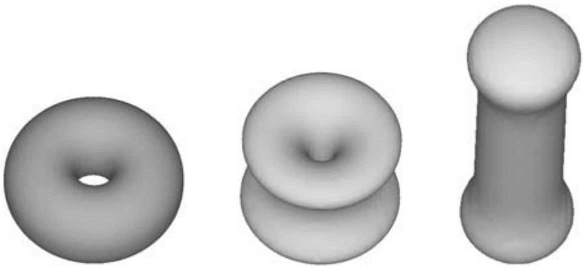

Fig. 1: The Hopf charge density isosurfaces for solutions with various charges $(m, n) = (1, 2)$ (left), $(2, 2)$ (middle) and $(4, 1)$ (right) for $\beta e^2 = 1.1$

The finite energy solutions of the theory (1) define maps from the three dimensional space $\mathbb{R}^3 \sim S^3$ to the target space S^2. These are classified into homotopy classes labeled by an integer Q_H called the Hopf index, which has values of $Q_H = m\,n$.

Substituting (7) into (4) and (5), we get two coupled non-linear partial differential equations in two variables. We have constructed numerical solutions with several Hopf charges up to four by a successive over-relaxation method. In Fig. 1, we present some of the results of the Hopf charge density for the charge $(m, n) = (1, 2)$, $(2, 2)$ and $(4, 1)$.

Probably the main interest in studying the class of the model is to get an insight into the mass of the glueballs. Since our finding solutions are classical, they must be properly quantized. Making the following replacement for the fields [10]

$$\begin{aligned} &\vec{n}(\vec{r}) \cdot \vec{\tau} \rightarrow \vec{n}'(\vec{r}, t) \cdot \vec{\tau} := \\ &A(t) \left[\vec{n}(R(B(t))\vec{r} - \vec{X}) \cdot \vec{\tau} \right] A^{\dagger}(t), \end{aligned} \tag{10}$$

one obtains the kinetic contribution to the energy

$$\mathcal{T} = \frac{1}{2} a_i U_{ij} a_j - a_i W_{ij} b_j + \frac{1}{2} b_i V_{ij} b_j \tag{11}$$

with $a_i := -i\mathrm{tr}(\tau_i A^{\dagger} \dot{A})$ and $b_i := i\mathrm{tr}(\tau_i \dot{B} B^{\dagger})$. Here A and B are matrices in SU(2), and B works through SO(3) form $R_{ij}(B) = \frac{1}{2}\mathrm{tr}(\tau_i B \tau_j B^{-1})$. $\vec{b}$ is an angular velocity, and $\vec{a}$ means an angular velocity in an isospace. Quantizing these coordinates, one finally finds the quantized mass of the Hopf soliton

$$\begin{aligned} E_{\mathrm{quanta}} := &\frac{M}{|e|} E_{\mathrm{static}} - \\ &M e^2 |e| \left[\frac{i(i+1)}{2U_{11}} + \frac{j(j+1)}{2V_{11}} + \right. \\ &\left. \frac{k_3^2}{2} \left(\frac{1}{U_{33}} - \frac{1}{U_{11}} - \frac{n^2}{V_{11}} \right) \right], \end{aligned} \tag{12}$$

where E_{static} is the energy corresponding to the hamiltonian (3) and the inertia tensors U_{ab}, V_{ab} are the function of the classical fields $\vec{n}$. Numerically U_{11} has a quite large value, so we omit the terms with U_{11}. As a result, the states are essentially labeled by two quantum numbers (j, k_3).

Krusch applied the basic FR constraints to the Skyrme-Faddeev model, and finally found that the quantum states with even topological charge Q_H could be bosonic[11]. They may be possible candidates for glueballs.

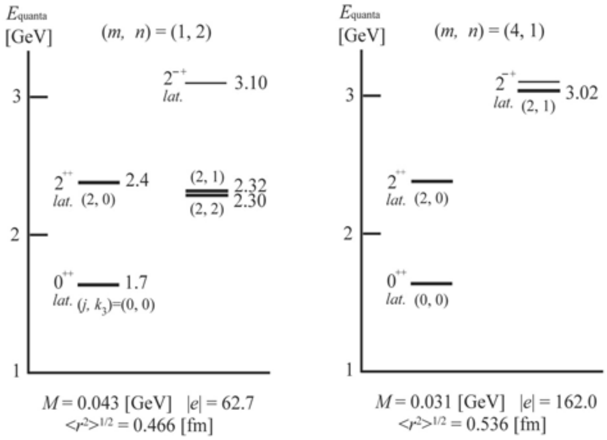

Fig. 2: The quantized spectra for the topological charge $(m, n) = (1, 2)$ and $(4, 1)$ (bold line), compared with result of the lattice simulation [12]

We plot the lowest three states of $(m, n) = (1, 2)$ and $(4, 1)$. First, we have fitted the first two spectra to the result of the lattice simulation [12] and have computed the third lowest spectrum corresponding to $J^{PC} = 2^{-+}$. For $(1, 2)$, the third state is lower energy than the second state because the second term on the right hand side of (12) has a negative contribution to the quantum energy. On the other hand, if we employ the solution of $(4, 1)$, the third state appears near to the prediction of the lattice. This is quite promising. In [12], the authors also predicted the root mean square radius $\sqrt{\langle r^2 \rangle} \sim 0.481$ [fm]. In our calculation, the radius is $\sqrt{\langle r^2 \rangle} = 0.466$ (for $(1, 2)$) or 0.536 (for $(4, 1)$) [fm]; the results are consistent.

We summarize our analysis. We have found new Hopf soliton solutions for the extended Skyrme-Faddeev model. The model is a low energy effective model of QCD, and the solutions are possible candidates for the glueballs. We have performed the collective coordinate quantization for the obtained classical solutions and have computed the quantum energies. Some of our results are in good agreement with the study of the lattice gauge simulation.

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About the author

Nobuyuki Sawado is currently working as an Associate Professor at the Institute of Science and Technology, Department of Physics at Tokyo University of Science. His research interest is in topological solitons and related areas, such as effective models of QCD, brane world scenarios and some topics in condensed matter physics.

Luiz Agostinho Ferreira
Institute of Physics of São Carlos
IFSC/USP, University of São Paulo – USP
Caixa Postal 369, CEP 13560-970, São Carlos-SP, Brazil

Satoru Kato
Department of Physics
Institute of Science and Technology
Tokyo University of Science, Noda, Chiba 278-8510, Japan

Nobuyuki Sawado
E-mail: sawado@ph.noda.tus.ac.jp
Department of Physics
Institute of Science and Technology
Tokyo University of Science
Noda, Chiba 278-8510, Japan

Kouichi Toda
Department of Mathematical Physics
Toyama Prefectural University
Kurokawa 5180, Imizu, Toyama, 939-0398, Japan
and
Research and Education Center for Natural Sciences
Hiyoshi Campus, Keio University
4-1-1 Hiyoshi, Kouhoku-ku, Yokohama, 223-8521, Japan