Acta Polytechnica **53**(5):399–404, 2013
doi:10.14311/AP.2013.53.0399

available online at http://ojs.cvut.cz/ojs/index.php/ap

EXTREMAL VECTORS FOR VERMA TYPE REPRESENTATION OF B_2

Čestmír Burdík[a,*], Ondřej Navrátil[b]

[a] *Department of Mathematics, Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Trojanova 13, 120 00 Prague 2, Czech Republic*
[b] *Department of Mathematics, Czech Technical University in Prague, Faculty of Transportation Sciences, Na Florenci 25, 110 00 Prague, Czech Republic*
[*] corresponding author: burdices@kmlinux.fjfi.cvut.cz

Abstract. Starting from the Verma modules of the algebra B_2 we explicitly construct factor representations of the algebra B_2 which are connected with the unitary representation of the group $\mathrm{SO}(3,2)$. We find a full set of extremal vectors for representations of this kind. So we can explicitly resolve the problem of the irreducibility of these representations.

Keywords: Verma modules, height-weight representation, reducibility, extremal vectors.

Submitted: 16 July 2013. Accepted: 28 July 2013.

1. Introduction

Representations of Lie algebras are important in many physical models. It is therefore useful to study various methods for constructing them.

The general method of construction of the highest-weight representation for the semisimple Lie algebra was developed in [1, 2]. The irreducibility of such representations (now called Verma modules) was studied by Gelfand in [3]. The theory of these representations is included in Dixmier's book [4].

In the 1970's prof. Havlíček with his coworkers dealt with the construction of realizations of the classical Lie algebras, see [5]. Our aim in this paper is to show how one can use realizations of the Lie algebra to construct so called extremal vectors of the Verma modules. To work with a specific Lie algebra, we choose Lie algebra $\mathrm{so}(3,2)$, which plays an important role in physics, e.g. in AdS/CFT theory, see [6, 7].

In the construction of the Verma modules for B_2, the representations depend on parameters (λ_1, λ_2). For connection with irreducible unitary representations of $\mathrm{SO}(3,2)$ we take $\lambda_2 \in \mathbb{N}_0$, and in section 3 we explicitly construct the factor-Verma representation. Further, we construct a full set of extremal vectors. These vectors are called subsingular vectors in [8].

In this paper, we use an almost elementary partial differential equation approach to determine the extremal vectors in any factor-Verma module of B_2. It should be noted that our approach differs from a similar one used in [9]. First, we identify the factor-Verma modules with a space of polynomials, and the action of B_2 on the Verma module is identified with differential operators on the polynomials. Any extremal vector in the factor-Verma module becomes a polynomial solution of a system of variable-coefficient second-order linear partial differential equations.

2. The root system for Lie algebra B_2

In the Lie algebra $\mathfrak{g} = B_2$ we will take a basis composed by elements $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{E}_k$ and $\mathbf{F}_k$, where $k = 1, \ldots, 4$, which fulfill the commutation relations

$$
\begin{aligned}
&[\mathbf{H}_1, \mathbf{E}_1] = 2\mathbf{E}_1, && [\mathbf{H}_1, \mathbf{E}_2] = -\mathbf{E}_2,\\
&[\mathbf{H}_1, \mathbf{E}_3] = \mathbf{E}_3, && [\mathbf{H}_1, \mathbf{E}_4] = 0,\\
&[\mathbf{H}_2, \mathbf{E}_1] = -2\mathbf{E}_1, && [\mathbf{H}_2, \mathbf{E}_2] = 2\mathbf{E}_2,\\
&[\mathbf{H}_2, \mathbf{E}_3] = 0, && [\mathbf{H}_2, \mathbf{E}_4] = 2\mathbf{E}_4,\\
&[\mathbf{H}_1, \mathbf{F}_1] = -2\mathbf{F}_1, && [\mathbf{H}_1, \mathbf{F}_2] = \mathbf{F}_2,\\
&[\mathbf{H}_1, \mathbf{F}_3] = -\mathbf{F}_3, && [\mathbf{H}_1, \mathbf{F}_4] = 0,\\
&[\mathbf{H}_2, \mathbf{F}_1] = 2\mathbf{F}_1, && [\mathbf{H}_2, \mathbf{F}_2] = -2\mathbf{F}_2,\\
&[\mathbf{H}_2, \mathbf{F}_3] = 0, && [\mathbf{H}_2, \mathbf{F}_4] = -2\mathbf{F}_4,\\
&[\mathbf{E}_1, \mathbf{E}_2] = \mathbf{E}_3, && [\mathbf{E}_1, \mathbf{E}_3] = 0,\\
&[\mathbf{E}_1, \mathbf{E}_4] = 0, && [\mathbf{E}_2, \mathbf{E}_3] = 2\mathbf{E}_4,\\
&[\mathbf{E}_2, \mathbf{E}_4] = 0, && [\mathbf{E}_3, \mathbf{E}_4] = 0,\\
&[\mathbf{F}_1, \mathbf{F}_2] = -\mathbf{F}_3, && [\mathbf{F}_1, \mathbf{F}_3] = 0,\\
&[\mathbf{F}_1, \mathbf{F}_4] = 0, && [\mathbf{F}_2, \mathbf{F}_3] = -2\mathbf{F}_4,\\
&[\mathbf{F}_2, \mathbf{F}_4] = 0, && [\mathbf{F}_3, \mathbf{F}_4] = 0,\\
&[\mathbf{E}_1, \mathbf{F}_1] = \mathbf{H}_1, && [\mathbf{E}_1, \mathbf{F}_2] = 0,\\
&[\mathbf{E}_1, \mathbf{F}_3] = -\mathbf{F}_2, && [\mathbf{E}_1, \mathbf{F}_4] = 0,\\
&[\mathbf{E}_2, \mathbf{F}_1] = 0, && [\mathbf{E}_2, \mathbf{F}_2] = \mathbf{H}_2,\\
&[\mathbf{E}_2, \mathbf{F}_3] = 2\mathbf{F}_1, && [\mathbf{E}_2, \mathbf{F}_4] = -\mathbf{F}_3,\\
&[\mathbf{E}_3, \mathbf{F}_1] = -\mathbf{E}_2, && [\mathbf{E}_3, \mathbf{F}_2] = 2\mathbf{E}_1,\\
&[\mathbf{E}_3, \mathbf{F}_3] = 2\mathbf{H}_1 + \mathbf{H}_2, && [\mathbf{E}_3, \mathbf{F}_4] = \mathbf{F}_2,\\
&[\mathbf{E}_4, \mathbf{F}_1] = 0, && [\mathbf{E}_4, \mathbf{F}_2] = -\mathbf{E}_3,\\
&[\mathbf{E}_4, \mathbf{F}_3] = \mathbf{E}_2, && [\mathbf{E}_4, \mathbf{F}_4] = \mathbf{H}_1 + \mathbf{H}_2.
\end{aligned}
$$

We can take as $\mathfrak{h}$ the Cartan subalgebra with the bases $\mathbf{H}_1$ and $\mathbf{H}_2$.

We will denote $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{h}^*$, for which we have

$$\lambda(\mathbf{H}_1) = \lambda_1, \qquad \lambda(\mathbf{H}_2) = \lambda_2.$$

The root systems $\mathfrak{g} = B_2$ with respect to these bases $\mathbf{H}_1$ and $\mathbf{H}_2$ are $R = \{\pm\boldsymbol{\alpha}_k\,;\ k = 1,2,3,4\}$, where

$$\boldsymbol{\alpha}_1 = (2,-2), \qquad \boldsymbol{\alpha}_2 = (-1,2),$$
$$\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = (1,0), \quad \boldsymbol{\alpha}_4 = \boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 = (0,2).$$

If we choose positive roots $R_+ = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}$, the basis in root system R is $B = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$.

If we define $\mathbf{H}_3 = 2\mathbf{H}_1 + \mathbf{H}_2$ and $\mathbf{H}_4 = \mathbf{H}_1 + \mathbf{H}_2$, the following relations

$$[\mathbf{H}_k, \mathbf{E}_k] = 2\mathbf{E}_k, \quad [\mathbf{H}_k, \mathbf{F}_k] = -2\mathbf{F}_k, \quad [\mathbf{E}_k, \mathbf{F}_k] = \mathbf{H}_k$$

are valid for any $k = 1, \dots, 4$.

3. The extremal vectors for Verma type representation

We denote by $\mathfrak{n}_+$, and $\mathfrak{n}_-$ the Lie subalgebras generated by elements $\mathbf{E}_k$, and $\mathbf{F}_k$, respectively, where $k = 1, \dots, 4$, and $\mathfrak{b}_+ = \mathfrak{h} + \mathbf{n}_+$. Let us further consider $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{h}^*$ the one-dimensional representation $\tau_{\boldsymbol{\lambda}}$ for the Lie algebra $\mathfrak{b}_+$ such that for any $\mathbf{H} \in \mathfrak{h}$ and $\mathbf{E} \in \mathfrak{n}_+$

$$\tau_{\boldsymbol{\lambda}}(\mathbf{H} + \mathbf{E})|0\rangle = \boldsymbol{\lambda}(\mathbf{H})|0\rangle.$$

The element $|0\rangle$ will be called the lowest-weight vector. Let further be

$$W(\boldsymbol{\lambda}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}|0\rangle,$$

where $\mathfrak{b}_+$-module $\mathbb{C}|0\rangle$ is defined by $\tau_{\boldsymbol{\lambda}}$.

It is clear that $W(\boldsymbol{\lambda}) \sim U(\mathfrak{n}_-)|0\rangle$ and it is the $U(\mathfrak{g})$-module for the left regular representation, which will be called the Verma module. [1]

It is a well-known fact that every $U(\mathfrak{g})$-submodule of the module $W(\boldsymbol{\lambda})$ is isomorphic to module $W(\boldsymbol{\mu})$, where

$$\boldsymbol{\mu} = \boldsymbol{\lambda} - n_1\boldsymbol{\alpha}_1 - n_2\boldsymbol{\alpha}_2,$$

for $n_1, n_2 \in \mathbb{N}_0 = \{0,1,2,\dots\}$. For the lowest-weight vector of the representation $W(\boldsymbol{\mu}) \subset W(\boldsymbol{\lambda})$, $|0\rangle_{\boldsymbol{\mu}}$, is fulfilled

$$\mathbf{H}|0\rangle_{\boldsymbol{\mu}} = \boldsymbol{\mu}(\mathbf{H})|0\rangle_{\boldsymbol{\mu}}, \quad \mathbf{H} \in \mathfrak{h}, \quad \mathbf{E}|0\rangle_{\boldsymbol{\mu}} = 0, \quad \mathbf{E} \in \mathfrak{n}_+.$$

Such vectors $|0\rangle_{\boldsymbol{\mu}}$ will be called extremal vectors $W(\boldsymbol{\lambda})$.

From the well-known result for the Verma modules we know that the Verma module $W(\boldsymbol{\lambda})$ is irreducible iff

$$\lambda_1 \notin \mathbb{N}_0, \qquad \lambda_2 \notin \mathbb{N}_0,$$
$$\lambda_1 + \lambda_2 + 1 \notin \mathbb{N}_0, \qquad 2\lambda_1 + \lambda_2 + 2 \notin \mathbb{N}_0.$$

If $\lambda_1 \in \mathbb{N}_0$, resp. $\lambda_2 \in \mathbb{N}_0$, then the extremal vectors are

$$\mathbf{F}_1^{\lambda_1+1}|0\rangle = |0\rangle_{\boldsymbol{\mu}_1}, \quad \text{resp.} \quad \mathbf{F}_2^{\lambda_2+1}|0\rangle = |0\rangle_{\boldsymbol{\mu}_2},$$

where

$$\boldsymbol{\mu}_1 = \boldsymbol{\lambda} - (\lambda_1+1)\boldsymbol{\alpha}_1 = (-\lambda_1 - 2, 2\lambda_1 + \lambda_2 + 2),$$
$$\boldsymbol{\mu}_2 = \boldsymbol{\lambda} - (\lambda_2+1)\boldsymbol{\alpha}_2 = (\lambda_1 + \lambda_2 + 1, -\lambda_2 - 2). \quad (1)$$

If $W(\boldsymbol{\mu})$ is a submodule $W(\boldsymbol{\lambda})$, we will define the $U(\mathfrak{g})$-factor-module

$$W(\boldsymbol{\lambda}|\boldsymbol{\mu}) = W(\boldsymbol{\lambda})/W(\boldsymbol{\mu}).$$

Now we can study the reducibility of a representation like that.

Again, the extremal vector is called any nonzero vector $\mathbf{v} \in W(\boldsymbol{\lambda}|\boldsymbol{\mu})$ for which there exists $\boldsymbol{\nu} \in \mathfrak{h}^*$ such that

$$\mathbf{H}_k\mathbf{v} = \nu_k\mathbf{v}, \quad \mathbf{E}_k\mathbf{v} = 0, \quad k = 1,2. \quad (2)$$

It is clear that $\mathbf{E}_k\mathbf{v} = 0$ for $k = 1,2,3,4$.

In this paper, we find all such extremal vectors in the space $W(\boldsymbol{\lambda}|\boldsymbol{\mu}_2)$, where $\lambda_2 \in \mathbb{N}_0$ and $\boldsymbol{\mu}_2$ is given by (1).

4. Differential equations for extremal vectors

Let $\lambda_2 \in \mathbb{N}_0$ and $\boldsymbol{\mu}_2$ be given by equation (1). It is easy to see that the basis in the space $W(\boldsymbol{\lambda}|\boldsymbol{\mu}_2)$ is given by the vectors

$$|\mathbf{n}\rangle = |n_1, n_3, n_4, n_2\rangle = (\lambda_2 - n_2)!\,\mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle,$$

where $n_1, n_3, n_4 \in \mathbb{N}_0$ and $n_2 = 0,1,\dots,\lambda_2$.[2]

Now by direct calculation we obtain

$$\mathbf{H}_1|\mathbf{n}\rangle = (\lambda_1 - 2n_1 + n_2 - n_3)|\mathbf{n}\rangle,$$
$$\mathbf{H}_2|\mathbf{n}\rangle = (\lambda_2 + 2n_1 - 2n_2 - 2n_4)|\mathbf{n}\rangle,$$
$$\mathbf{E}_1|\mathbf{n}\rangle = n_1(\lambda_1 - n_1 + n_2 - n_3 + 1)|n_1 - 1, n_3, n_4, n_2\rangle - (\lambda_2 - n_2)n_3|n_1, n_3 - 1, n_4, n_2 + 1\rangle + n_3(n_3 - 1)|n_1, n_3 - 2, n_4 + 1, n_2\rangle,$$
$$\mathbf{E}_2|\mathbf{n}\rangle = n_2|n_1, n_3, n_4, n_2 - 1\rangle + 2n_3|n_1 + 1, n_3 - 1, n_4, n_2\rangle - n_4|n_1, n_3 + 1, n_4 - 1, n_2\rangle. \quad (3)$$

It is possible to rewrite the action by the second order differential operators (see [10, 11]) on the polynomial functions z_1, z_2, z_3 a z_4, which are in variable z_2 up to the level λ_2. If we put

$$|n_1, n_3, n_4, n_2\rangle = (\lambda_2 - n_2)!\,\mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle \leftrightarrow z_1^{n_1}z_2^{n_2}z_3^{n_3}z_4^{n_4},$$

we obtain from equations (3) for the action on polynomials $f = f(z_1, z_2, z_3, z_4)$

$$\mathbf{H}_1 f = \lambda_1 f - 2z_1 f_1 + z_2 f_2 - z_3 f_3,$$
$$\mathbf{H}_2 f = \lambda_2 f + 2z_1 f_1 - 2z_2 f_2 - 2z_4 f_4,$$
$$\mathbf{E}_1 f = \lambda_1 f_1 - z_1 f_{11} + z_2 f_{12} - z_3 f_{13} - \lambda_2 z_2 f_3 + z_2^2 f_{23} + z_4 f_{33},$$
$$\mathbf{E}_2 f = f_2 + 2z_1 f_3 - z_3 f_4, \quad (4)$$

[1] In Dixmier's book the Verma module $M(\boldsymbol{\lambda})$ is defined with respect to $\tau_{\boldsymbol{\lambda}-\boldsymbol{\delta}}$, where $\boldsymbol{\delta} = \frac{1}{2}\sum_{k=1}^{4}\boldsymbol{\alpha}_k = (1,1)$. So we have $W(\boldsymbol{\lambda}) = M(\boldsymbol{\lambda} + \boldsymbol{\delta})$.

[2] If $\lambda_2 \notin \mathbb{Z}$ we can use a similar construction with basis $|\mathbf{n}\rangle = \Gamma(\lambda_2 - n_2 + 1)\mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle$, where $n_1, n_2, n_3, n_4 \in \mathbb{N}_0$.

where $f_k = \dfrac{\partial f}{\partial z_k}$.

The conditions for extremal vectors (2) are now

$$\begin{aligned}\lambda_1 f - 2z_1 f_1 + z_2 f_2 - z_3 f_3 &= \nu_1 f,\\ \lambda_2 f + 2z_1 f_1 - 2z_2 f_2 - 2z_4 f_4 &= \nu_2 f,\\ \lambda_1 f_1 - z_1 f_{11} + z_2 f_{12} & \\ - z_3 f_{13} - \lambda_2 z_2 f_3 + z_2^2 f_{23} + z_4 f_{33} &= 0,\\ f_2 + 2z_1 f_3 - z_3 f_4 &= 0, \qquad (5)\end{aligned}$$

where ν_1 and ν_2 are complex numbers.

The condition on the degree of the polynomial $f(z_1, z_2, z_3, z_4)$ in variable z_2 can be rewritten in the following way

$$\frac{\partial^{\lambda_2+1} f}{\partial z_2^{\lambda_2+1}} = 0.$$

5. The extremal vectors

The extremal vectors are in one-to-one correspondence to polynomial solutions of the systems of equations (5), which are in variable z_2 of maximal degree λ_2. You can find all such solutions in the appendix.

For any λ_1 and λ_2 there exists a constant solution $f(z_1, z_2, z_3, z_4) = 1$. But such a solution gives $\mathbf{v} = |0\rangle$, which is not interesting.

A further solution exists only in the cases $\lambda_1 \in \mathbb{N}_0$, $\lambda_1 + \lambda_2 + 1 \in \mathbb{N}_0$ or $2\lambda_1 + \lambda_2 + 2 \in \mathbb{N}_0$.

For $\lambda_1 \in \mathbb{N}_0$ there is a function $f(z_1, z_2, z_3, z_4) = z_1^{\lambda_1+1}$, and we obtain the extremal vector

$$\mathbf{v} = \mathbf{F}_1^{\lambda_1+1}|0\rangle.$$

For $\lambda_1 + \lambda_2 + 1 \in \mathbb{N}_0$ and $2\lambda_1 + \lambda_2 + 4 \le 0$ we find the solution

$$\begin{aligned} f(z_1, z_2, z_3, z_4) &= \big(z_4 + z_2 z_3 - z_1 z_2^2\big)^{\lambda_1+\lambda_2+2}\\ &= \sum_{(n_1,n_3)\in\mathcal{D}_\lambda} \frac{(-1)^{n_1}(\lambda_1+\lambda_2+2)!}{n_1!\,n_3!\,(\lambda_1+\lambda_2-n_1-n_3+2)!}\\ &\qquad \times z_1^{n_1} z_2^{2n_1+n_3} z_3^{n_3} z_4^{\lambda_1+\lambda_2-n_1-n_3+2},\end{aligned}$$

where $\mathcal{D}_\lambda = \big\{(n_1, n_3) \in \mathbb{N}_0^2\,;\, n_1 + n_3 \le \lambda_1 + \lambda_2 + 2\big\}$. The extremal vector corresponding to this solution is

$$\begin{aligned}\mathbf{v} = \sum_{(n_1,n_3)\in\mathcal{D}_\lambda} &\frac{(-1)^{n_1}(\lambda_2 - 2n_1 - n_3)!}{n_1!\,n_3!\,(\lambda_1+\lambda_2-n_1-n_3+2)!}\\ &\times \mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{\lambda_1+\lambda_2-n_1-n_3+2}\mathbf{F}_2^{2n_1+n_3}|0\rangle.\end{aligned}$$

If $2\lambda_1 + \lambda_2 + 2 \in \mathbb{N}_0$, we introduce

$$N = 2\lambda_1 + \lambda_2 + 3, \quad \ell_2 = \big[\tfrac12\lambda_2\big], \quad M = \big[\tfrac12 N\big].$$

Then we can rewrite the solution from the appendix in the following way:

For λ_1 being a half integer, i.e. $\lambda_1 = \ell_1 - \frac12$, where $\ell_1 \in \mathbb{Z}$, we have

$$\begin{aligned} f = \sum_{n_4=0}^{M} \sum_{n_2=0}^{\min(\lambda_2, N-2n_4)} &(-1)^{n_2}\frac{c_{n_2,n_4}}{n_2!\,n_4!}\\ &\times z_1^{n_2+n_4} z_2^{n_2} z_3^{N-n_2-2n_4} z_4^{n_4},\end{aligned}$$

where

$$c_{n_2,n_4} = \begin{cases} \sum_{n=[\frac12(n_2+1)]}^{\min(\ell_2,M-n_4)} 2^{2n+n_2+2n_4} \times \frac{\ell_2!\,M!}{(2n-n_2)!\,(\ell_2-n)!\,(M-n-n_4)!}, & \lambda_2 \text{ even},\\ \sum_{n=[\frac12 n_2]}^{\min(\ell_2,M-n_4)} 2^{2n+n_2+2n_4} \times \frac{\ell_2!\,M!}{(2n-n_2+1)!\,(\ell_2-n)!\,(M-n-n_4)!}, & \lambda_2 \text{ odd}.\end{cases}$$

For these solutions we obtain the extremal vectors

$$\begin{aligned}\mathbf{v} = \sum_{n_4=0}^{M} \sum_{n_2=0}^{\min(\lambda_2,N-2n_4)} &(-1)^{n_2}\frac{(\lambda_2-n_2)!}{n_2!\,n_4!}c_{n_2,n_4}\\ &\times \mathbf{F}_1^{n_2+n_4}\mathbf{F}_3^{N-n_2-2n_4}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle.\end{aligned}$$

If λ_1 is an integer we have $\lambda_1 \le -2$. The solution of the differential equations in this case is

$$\begin{aligned} f = \sum_{n_4=0}^{M} \sum_{n_2=0}^{N-2n_4} &(-1)^{n_2}\frac{d_{n_2,n_4}}{n_2!\,n_4!}\\ &\times z_1^{n_2+n_4} z_2^{n_2} z_3^{N-n_2-2n_4} z_4^{n_4},\end{aligned}$$

where

$$d_{n_2,n_4} = \begin{cases} \sum_{n=[\frac12(n_2+1)]}^{M-n_4} 2^{n+n_2+2n_4}\frac{(2\ell_2-1)!!}{(2\ell_2-2n-1)!!} \times \frac{M!}{(2n-n_2)!\,(M-n-n_4)!}, & \lambda_2 \text{ even},\\ \sum_{n=[\frac12 n_2]}^{M-n_4} 2^{n+n_2+2n_4}\frac{(2\ell_2-1)!!}{(2\ell_2-2n-1)!!} \times \frac{M!}{(2n-n_2+1)!\,(M-n-n_4)!}, & \lambda_2 \text{ odd},\end{cases}$$

and the extremal vectors are

$$\begin{aligned}\mathbf{v} = \sum_{n_4=0}^{M} \sum_{n_2=0}^{N-2n_4} &(-1)^{n_2}\frac{(\lambda_2-n_2)!}{n_2!\,n_4!}d_{n_2,n_4}\\ &\times \mathbf{F}_1^{n_2+n_4}\mathbf{F}_3^{N-n_2-2n_4}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle.\end{aligned}$$

6. Appendix: Polynomial solutions of differential equations

To obtain extremal vectors we need to find the polynomial solutions

$$f(z_1, z_2, z_3, z_4) = \sum_{n_1,n_2,n_3,n_4\ge 0} c_{n_1,n_2,n_3,n_4} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4}$$

of the system of equations (5), which are of less degree than $(\lambda_2 + 1)$ in the variable z_2.

To simplify the solution of the first equations, we put

$$\begin{aligned} f(z_1&, z_2, z_3, z_4)\\ &= z_1^{-\rho_2}(4z_1z_4 + z_3^2)^{\rho_2+\rho_1/2} g(t, x_1, x_2, x_3),\end{aligned}$$

where $\rho_1 = \lambda_1 - \nu_1$, $\rho_2 = \frac12(\lambda_2 - \nu_2)$, $x_2 = z_1$, $x_3 = z_2$ and

$$t = \frac{(2z_1z_2 - z_3)^2}{4z_1z_4 + z_3^2}, \quad x_1 = \frac{2z_1z_2 - z_3}{z_3},$$

or $z_1 = x_2$, $z_2 = x_3$ and

$$z_3 = \frac{2x_2x_3}{1+x_1}, \quad z_4 = \frac{x_2x_3^2(x_1^2-t)}{t(1+x_1)^2}.$$

The first order equations are equivalent to the conditions

$$g_{x_1} = g_{x_2} = g_{x_3} = 0,$$

and so $g(t, x_1, x_2, x_3) = g(t)$.

The equations of the second order give the system of three equations

$$\begin{aligned}&(2\lambda_1\rho_1 + 2\lambda_1\rho_2 + \lambda_2\rho_1 + 2\lambda_2\rho_2\\ &\quad - \rho_1^2 - 2\rho_1\rho_2 - 2\rho_2^2 + 3\rho_1 + 4\rho_2)g = 0,\\ &(2\lambda_1 + \lambda_2 - \rho_1 - 2\rho_2 + 3)(1-t)g'\\ &\quad + \rho_2(\lambda_1 - \rho_1 - \rho_2 + 1)g = 0,\\ &4t(1-t)g'' + 2\big(1 + (2\lambda_1 + 2\lambda_2 + 1)t\big)g'\\ &\quad + (2\lambda_1\rho_1 - \rho_1^2 + 3\rho_1 + 2\rho_2)g = 0. \qquad (6)\end{aligned}$$

As we want to obtain polynomial solutions $f(z_1, z_2, z_3, z_4)$, which are in variable z_2 of less or equal degree $\lambda_2 \in \mathbb{N}_0$, there must be solution $g(t)$ of the system (6), which is the polynomial in $\sqrt{t}$ of less or equal degree λ_2.

If we exclude derivatives of g from the second and the third equations,we find that nonzero solutions can exist only in the following six cases:

(1.) $\rho_1 = 0$, $\rho_2 = 0$;

(2.) $\rho_1 = 2\lambda_1 + 2$, $\rho_2 = -\lambda_1 - 1$;

(3.) $\rho_1 = 0$, $\rho_2 = \lambda_1 + \lambda_2 + 2$;

(4.) $\rho_1 = 2\lambda_1 + 2$, $\rho_2 = \lambda_2 + 1$;

(5.) $\rho_1 = 2\lambda_1 + \lambda_2 + 3$, $\rho_2 = 0$;

(6.) $\rho_1 = -\lambda_2 - 1$, $\rho_2 = \lambda_1 + \lambda_2 + 2$.

Case 1 ($\rho_1 = \rho_2 = 0$). A function that corresponds to the extremal vector is $f(z_1, z_2, z_3, z_4) = g(t)$, where $g(t)$ is the solution of the system

$$\begin{aligned}(2\lambda_1 + \lambda_2 + 3)(1-t)g' = 0,\\ 2t(1-t)g'' + \big(1 + (2\lambda_1 + 2\lambda_2 + 1)t\big)g' = 0. \qquad (7)\end{aligned}$$

For each λ_1 and λ_2 this system has the solution $g(t) = 1$ which corresponds to the extremal vector

$$f(z_1, z_2, z_3, z_4) = 1.$$

But for $2\lambda_1 + \lambda_2 + 3 = 0$ we obtained for $g(t)$ the equation

$$2t(1-t)g'' + \big(1 + (\lambda_2 - 2)t\big)g' = 0,$$

which also has a non-constant solution

$$g(t) = G(\sqrt{t}), \quad \text{where} \quad G(x) = \int \big(1-x^2\big)^{(\lambda_2-1)/2} \mathrm{d}x.$$

However this solution does not give a polynomial function $f(z_1, z_2, z_3, z_4)$ for any λ_2.

Case 2 ($\rho_1 = 2\lambda_1 + 2$, $\rho_2 = -\lambda_1 - 1$). The function that corresponds to the extremal vector is in this case

$$f(z_1, z_2, z_3, z_4) = z_1^{\lambda_1+1} g(t),$$

where $g(t)$ is the solution of system (7). As in event 1 we find that the non-constant polynomial solutions

$$f(z_1, z_2, z_3, z_4) = z_1^{\lambda_1+1}$$

get only $\lambda_1 \in \mathbb{N}_0$.

Case 3 ($\rho_1 = 0$, $\rho_2 = \lambda_1 + \lambda_2 + 2$.). The function for the extremal vectors is

$$f(z_1, z_2, z_3, z_4) = \left(\frac{4z_1z_4 + z_3^2}{z_1}\right)^{\lambda_1+\lambda_2+2} g(t),$$

where $g(t)$ is the solution of the system

$$\begin{aligned}(\lambda_2 + 1)\big((1-t)g' + (\lambda_1 + \lambda_2 + 2)g\big) = 0,\\ 2t(1-t)g'' + \big(1 + (2\lambda_1 + 2\lambda_2 + 1)t\big)g'\\ +(\lambda_1 + \lambda_2 + 2)g = 0. \qquad (8)\end{aligned}$$

As we assume that $\lambda_2 \in \mathbb{N}_0$,for each λ_1, λ_2 this system has the solution

$$g(t) = (1-t)^{\lambda_1+\lambda_2+2}.$$

This solution corresponds to the function

$$f(z_1, z_2, z_3, z_4) = \big(z_4 + z_2z_3 - z_1z_2^2\big)^{\lambda_1+\lambda_2+2}, \qquad (9)$$

which is a non-constant polynomial for $\lambda_1 + \lambda_2 + 1 \in \mathbb{N}_0$.

This function is a polynomial in the variable z_2 of degree $2\lambda_1 + 2\lambda_2 + 2$. It gives sought solutions for $2\lambda_1 + \lambda_2 + 4 \leq 0$.

Thus, function (9) provides a permissible solution for the $\lambda_2 \in \mathbb{N}_0$ only if $\lambda_1 \in \mathbb{Z}$, $-\lambda_2 - 1 \leq \lambda_1 \leq -\frac{1}{2}\lambda_2 - 2$, from which follows $\lambda_2 \geq 2$.

Case 4 ($\rho_1 = 2\lambda_1 + 2$, $\rho_2 = \lambda_2 + 1$.). In this case, the function that can match the extremal vector is

$$f(z_1, z_2, z_3, z_4) = z_1^{-\lambda_2-1}(4z_1z_4 + z_3^2)^{\lambda_1+\lambda_2+2} g(t),$$

where $g(t)$ is the solution of system (8). So

$$f(z_1, z_2, z_3, z_4) = z_1^{\lambda_1+1}\big(z_4 + z_2z_3 - z_1z_2^2\big)^{\lambda_1+\lambda_2+2}.$$

To give a polynomial solution, which we have found, to this function there must be $\lambda_1 \in \mathbb{N}_0$ and $\lambda_1 + \lambda_2 + 1 \in \mathbb{N}_0$. But in this case, the degree of polynomial f in the variable z_2 is greater than λ_2 and, therefore, is not a permissible solution.

Case 5 ($\rho_1 = 2\lambda_1 + \lambda_2 + 3$, $\rho_2 = 0$.). The function corresponding to the possible extremal vectors is

$$f(z_1, z_2, z_3, z_4) = (4z_1 z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2}} g(t),$$

where function $g(t)$ meets the equation

$$4t(1-t)g'' + 2\big(1 + (2\lambda_1 + 2\lambda_2 + 1)t\big)g' - \lambda_2(2\lambda_1 + \lambda_2 + 3)g = 0. \quad (10)$$

This equation has two linearly independent solutions

$$g_1(t) = F\big(-\tfrac{1}{2}\lambda_2, -\lambda_1 - \tfrac{1}{2}\lambda_2 - \tfrac{3}{2}; \tfrac{1}{2}; t\big),$$
$$g_2(t) = \sqrt{t}\, F\big(\tfrac{1}{2} - \tfrac{1}{2}\lambda_2, -\lambda_1 - \tfrac{1}{2}\lambda_2 - 1; \tfrac{3}{2}; t\big),$$

where $F(\alpha, \beta; \gamma; t)$ is the hypergeometric function

$$F(\alpha, \beta; \gamma; t) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!\,(\gamma)_n} t^n,$$

where

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \alpha(\alpha + 1)\dots(\alpha + n - 1).$$

These solutions correspond to the functions

$$f_1 = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - \frac{3}{2})_n}{n!\,(\frac{1}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + \frac{3}{2}},$$
$$f_2 = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - \frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - 1)_n}{n!\,(\frac{3}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + 1}.$$

For at least one of these functions to be a nonconstant polynomial, must be $2\lambda_1+\lambda_2+3 \in \mathbb{N}$, i.e. $2\lambda_1+\lambda_2+2 \in \mathbb{N}_0$.

If $2\lambda_1 + \lambda_2 + 3$ is even, we get the solution

$$f_1 = \sum_{n=0}^{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2}} \frac{(-\frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - \frac{3}{2})_n}{n!\,(\frac{1}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + \frac{3}{2}},$$

and for $2\lambda_1 + \lambda_2 + 3$ odd, we have the solution

$$f_2 = \sum_{n=0}^{\lambda_1 + \frac{1}{2}\lambda_2 + 1} \frac{(\frac{1}{2} - \frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - 1)_n}{n!\,(\frac{3}{2})_n} \times (2z_1z_2 - z_3)^{2n+1}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + 1}.$$

If $2\lambda_1 + \lambda_2 + 3$ is even and λ_2 is even, then λ_1 is a half integer, i.e. $\lambda_1 = \ell_1 - \frac{1}{2}$, where $\ell_1 \in \mathbb{Z}$, counts in f_1 only to $n \le \frac{1}{2}\lambda_2$, i.e.

$$f = \sum_{n=0}^{\min(\frac{1}{2}\lambda_2, \lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2})} \frac{(-\frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - \frac{3}{2})_n}{n!\,(\frac{1}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + \frac{3}{2}},$$

and, therefore, f is in the variable z_2 of a polynomial of degree not exceeding λ_2.

If $2\lambda_1 + \lambda_2 + 3$ is even and λ_2 is odd, i.e. λ_1 is an integer, the function

$$f = \sum_{n=0}^{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2}} \frac{(-\frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - \frac{3}{2})_n}{n!\,(\frac{1}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + \frac{3}{2}},$$

in the variable z_2 is a polynomial of degree $2\lambda_1+\lambda_2+3$. Thus admissible solutions get only $\lambda_1 \le -2$.

If $2\lambda_1 + \lambda_2 + 3$ is odd, then solution f_2 comes into play. If $\frac{1}{2}(\lambda_2 - 1) \in \mathbb{N}_0$, i.e. for odd λ_2 and half integer λ_1 sum in f_2 only $n \le \frac{1}{2}(\lambda_2 - 1)$, then the solutions are

$$f = \sum_{n=0}^{\min(\frac{1}{2}\lambda_2 - \frac{1}{2}, \lambda_1 + \frac{1}{2}\lambda_2 + 1)} \frac{(\frac{1}{2} - \frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - 1)_n}{n!\,(\frac{3}{2})_n} \times (2z_1z_2 - z_3)^{2n+1}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + 1}$$

in the z_2 polynomial of degree not exceeding λ_2.

But for $2\lambda_1 + \lambda_2 + 3$ odd and λ_2 even, i.e. $\lambda_1 \in \mathbb{Z}$, the solution is

$$f = \sum_{n=0}^{\lambda_1 + \frac{1}{2}\lambda_2 + 1} \frac{(\frac{1}{2} - \frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - 1)_n}{n!\,(\frac{3}{2})_n} \times (2z_1z_2 - z_3)^{2n+1}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + 1}.$$

In the variable z_2 it is a polynomial of degree $2\lambda_1 + \lambda_2 + 3$. Therefore we get a permissible solution for $2\lambda_1 + \lambda_2 + 3 \le \lambda_2$, i.e. $\lambda_1 \le -2$.

Case 6 ($\rho_1 = -\lambda_2 - 1$, $\rho_2 = \lambda_1 + \lambda_2 + 2$.). In this case,

$$f(z_1, z_2, z_3, z_4) = z_1^{-\lambda_1 - \lambda_2 - 2}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2}} g(t),$$

where function $g(t)$ is the solution of equation (10).

For this function f to be polynomial, must be $2\lambda_1 + \lambda_2 + 3 \in \mathbb{N}_0$ and $-\lambda_1 - \lambda_2 - 2 \in \mathbb{N}_0$. But these conditions are not fulfilled for any $\lambda_2 \in \mathbb{N}_0$.

Acknowledgements

The work of Č.B. and O.N. was supported in part by the GACR-P201/10/1509 grant, and by research plan MSM6840770039.

References

[1] D. Verma. *Structure of certain induced representations of complex semisimple Lie algebras.* Yale University, 1966.

[2] D. Verma. Structure of certain induced representations of complex semisimple lie algebras. *Bull Amer Math Soc* **74**:160–166, 1968.

[3] I. Bernstein, I.M. Gel'fand and S.I. Gel'fand. Structure of representations generated by highest weight vectors. *Funct Anal Appl* **5**:1–8, 1971.

[4] J. Dixmier. *Enveloping Algebras*. New York: North Holland, 1977.

[5] P. Exner, M. Havlíček and W. Lassner. Canonical Realizations of Clasical Lie Algebras. *Czech. J. Phys.* **26B**:1213–1228, 1976.

[6] E. Witten. Anti–de Sitter Space and Holography. *Adv. Theor. Math. Phys.* **2**:253–291, 1998.

[7] O. Aharony, S.S Gubser, J. Maldacena, H. Ooguri and Y. Oz. Large N Field Theories, String Theory and Gravity. *Phys. Rept.* **323**:183–386, 2000.

[8] V. Dobrev. Subsingular vectors and conditionally invariant (q-deformed) equations. *J Phys A: Math Gen* **28**:7135–7155, 1995.

[9] X. Xu. Differential equations for singular vectors of $\mathrm{sl}(n)$. `arXiv:math/0305180v1`.

[10] Č. Burdík. Realization of the real semisimple Lie algebras: method of construction. *J Phys A: Math Gen* **15**:3101–3111, 1985.

[11] Č. Burdík, P. Grozman, D. Leites and A. Sergeev. Realizations of Lie algebras and superalgebras via creation and annihilation operators I. *Theoretical and Math Physics* **124**(2), 2000.