More on $\mathcal{PT}$-Symmetry in (Generalized) Effect Algebras and Partial Groups

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Abstract

We continue in the direction of our paper on $\mathcal{PT}$-Symmetry in (Generalized) Effect Algebras and Partial Groups. Namely we extend our considerations to the setting of weakly ordered partial groups. In this setting, any operator weakly ordered partial group is a pasting of its partially ordered commutative subgroups of linear operators with a fixed dense domain over bounded operators. Moreover, applications of our approach for generalized effect algebras are mentioned.

Keywords: (generalized) effect algebra, partially ordered commutative group, weakly ordered partial group, Hilbert space, (unbounded) linear operators, $\mathcal{PT}$-symmetry, Pseudo-Hermitian Quantum Mechanics.

1 Introduction

It is a well known fact that unbounded linear operators play the role of the *observable* in the mathematical formulation of quantum mechanics. Examples of such observables corresponding to the momentum and position observables, respectively, are the following self-adjoint unbounded linear operators on the Hilbert space $L^2(\mathbb{R})$:

(i) The *differential operator* A defined by

$$(Af)(x) = i\frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

where i is the imaginary unit and f is a differentiable function with compact support. Then $D(A) \neq L^2(\mathbb{R})$, since otherwise the derivative need not exist.

(ii) $(Bf)(x) = xf(x)$, *multiplication by* x and again $D(B) \neq L^2(\mathbb{R})$, since otherwise $xf(x)$ need not be square integrable.

Note that in both cases the possible domains are dense sub-spaces of $L^2(\mathbb{R})$, i.e., $\overline{D(A)} = \overline{D(B)} = L^2(\mathbb{R})$.

The same is true in general, since for any unbounded linear operator there is no standard way to extend it to the whole space $\mathcal{H}$. By the Hellinger-Toeplitz theorem, every symmetric operator A with $D(A) = \mathcal{H}$ is bounded.

An important attempt at an alternative formulation of quantum mechanics started in the seminal paper [1] by Bender and Boettcher in 1998. Bender and others adopted all the axioms of quantum mechanics except the axiom that restricted the Hamiltonian to be Hermitian. They replaced this condition with the requirement that the Hamiltonian must have an exact $\mathcal{PT}$-symmetry. Later, A. Mostafazadeh [6] showed that $\mathcal{PT}$-symmetric quantum mechanics is an example of a more general class of theories, called *Pseudo-Hermitian Quantum Mechanics*.

In [4] Foulis and Bennett introduced the notion of effect algebras that generalized the algebraic structure of the set $\mathcal{E}(\mathcal{H})$ of Hilbert space effects. In such a case the set $\mathcal{E}(\mathcal{H})$ of effects is the set of all self-adjoint operators A on a Hilbert space $\mathcal{H}$ between the null operator 0 and the identity operator 1 and endowed with the partial operation $+$ defined iff $A + B$ is in $\mathcal{E}(\mathcal{H})$, where $+$ is the usual operator sum. Recently, M. Polakovič and Z. Riečanová [8] established new examples of generalized effect algebras of positive operators on a Hilbert space.

In [7] we showed how the standard effect algebra $\mathcal{E}(\mathcal{H})$ and the latter generalized effect algebras of positive operators are related to the type of attempt mentioned above. As a by-product, we placed some of the results from [8] under the common roof of partially ordered commutative groups.

The aim of the present note is to continue in this direction.

The paper is organized in the following way. In Section 1 we recall the basic notions concerning the theory of (generalized) effect algebras and partially ordered commutative groups. In Section 2 we show that the linear operators on $\mathcal{H}$ and symmetric linear operators on $\mathcal{H}$ are equipped with the structure of a weakly ordered commutative partial group. In Section 3 we manifest the fact that each of these operator structures is a pasting of partially ordered commutative groups of the respective operators with a fixed dense domain. In the last section we show that our results concerning the application of renormalization due to the $\mathcal{PT}$-symmetry of an operator from [7] remain true for weakly ordered commutative partial groups.

2 Basic definitions and some known facts

The basic reference for the present text is the classic book by A. Dvurečenskij and S. Pulmannová [3], where the interested reader can find unexplained terms and notation concerning the subject.

We now review some terminology concerning (generalized) effect algebras and weakly ordered partial commutative groups.

Definition 1 ([4]) A partial algebra $(E; +, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements and $+$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x + y = y + x$ if $x + y$ is defined,

(Eii) $(x + y) + z = x + (y + z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x + y = 1$ (we put $x' = y$),

(Eiv) if $1 + x$ is defined then $x = 0$.

Definition 2 ([5]) A partial algebra $(E; +, 0)$ is called a *generalized effect algebra* if $0 \in E$ is a distinguished element and $+$ is a partially defined binary operation on E which satisfies the following conditions for any $x, y, z \in E$:

(GEi) $x + y = y + x$, if one side is defined,

(GEii) $(x+y)+z = x+(y+z)$, if one side is defined,

(GEiii) $x + 0 = x$,

(GEiv) $x + y = x + z$ implies $y = z$ (cancellation law),

(GEv) $x + y = 0$ implies $x = y = 0$.

In every generalized effect algebra E the partial binary operation $\ominus$ and relation $\leq$ can be defined by

(ED) $x \leq y$ and $y \ominus x = z$ iff $x + z$ is defined and $x + z = y$.

Then $\leq$ is a partial order on E under which 0 is the least element of E.

Note that every effect algebra satisfies the axioms of a generalized effect algebra, and if a generalized effect algebra has the greatest element then it is an effect algebra.

Definition 3 A partial algebra $(G; +, 0)$ is called a *commutative partial group* if $0 \in E$ is a distinguished element and $+$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(Gi) $x + y = y + x$ if $x + y$ is defined,

(Gii) $(x + y) + z = x + (y + z)$ if both sides are defined,

(Giii) $x + 0$ is defined and $x + 0 = x$,

(Giv) for every $x \in E$ there exists a unique $y \in E$ such that $x + y = 0$ (we put $-x = y$),

(Gv) $x + y = x + z$ implies $y = z$.

We will put $\perp_G = \{(x, y) \in G \times G \mid x + y \text{ is defined}\}$.

A commutative partial group $(G; +, 0)$ is called *weakly ordered* (shortly a *wop-group*) with respect to a reflexive and antisymmetric relation $\leq$ on G if $\leq$ is compatible w.r.t. partial addition, i.e., for all $x, y, z \in G$, $x \leq y$ and both $x + z$ and $y + z$ are defined implies $x+z \leq y+z$. We will denote by $Pos(G)$ the set $\{x \in G \mid x \geq 0\}$.

Recall that wop-groups equipped with a total operation $+$ such that $\leq$ is an order are exactly partially ordered commutative groups.

Throughout the paper we assume that $\mathcal{H}$ is an infinite-dimensional complex Hilbert space, i.e., a linear space with inner product $\langle \cdot, \cdot \rangle$ which is complete in the induced metric. Recall that here for any $x, y \in \mathcal{H}$ we have $\langle x, y \rangle \in \mathbb{C}$ (the set of complex numbers) such that $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathcal{H}$. Moreover, $\overline{\langle x, y \rangle} = \langle y, x \rangle$ and finally $\langle x, x \rangle \geq 0$ at which $\langle x, x \rangle = 0$ iff $x = 0$. The term *dimension* of $\mathcal{H}$ in the following always means the *Hilbertian dimension* defined as the cardinality of any orthonormal basis of $\mathcal{H}$ (see [2]).

Moreover, we will assume that all considered linear operators A (i.e., linear maps $A : D(A) \to \mathcal{H}$) have a domain $D(A)$ a linear subspace dense in $\mathcal{H}$ with respect to the metric topology induced by the inner product, so $\overline{D(A)} = \mathcal{H}$ (we say that A is *densely defined*). We denote by $\mathcal{D}$ the set of all dense linear subspaces of $\mathcal{H}$. Moreover, by *positive linear operators* A, (denoted by $A \geq 0$) it means that $\langle Ax, x \rangle \geq 0$ for all $x \in D(A)$, therefore operators A are also *symmetric*, i.e., $\langle y, Ax \rangle = \langle Ay, x \rangle$ for all $x, y \in D(A)$ (for more details see [2]).

To every linear operator $A : D(A) \to \mathcal{H}$ with $\overline{D(A)} = \mathcal{H}$ there exists the *adjoint operator* A^* *of* A such that $D(A^*) = \{y \in \mathcal{H} \mid \text{there exists } y^* \in \mathcal{H} \text{ such that } (y^*, x) = (y, Ax) \text{ for every } x \in D(A)\}$ and $A^* y = y^*$ for every $y \in D(A^*)$. If $A^* = A$ then A is called *self-adjoint*.

Recall that $A : D(A) \to \mathcal{H}$ is called a *bounded operator* if there exists a real constant $C \geq 0$ such that $\|Ax\| \leq C\|x\|$ for all $x \in D(A)$ and hence A is an *unbounded operator* if to every $C \in \mathbb{R}$, $C \geq 0$ there exists $x_C \in D(A)$ with $\|Ax_C\| > C\|x_C\|$. The set of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For every bounded operator $A : D(A) \to \mathcal{H}$ densely defined on $D(A) = D \subset \mathcal{H}$ exists a unique extension B such as $D(B) = \mathcal{H}$ and $Ax = Bx$ for every $x \in D(A)$. We will denote this extension $B = A^{\mathbf{b}}$ (for more details see [2]). Bounded and symmetric operators are called *Hermitian operators*.

We also write, for linear operators $A : D(A) \to \mathcal{H}$ and $B : D(B) \to \mathcal{H}$, $A \subset B$ iff $D(A) \subseteq D(B)$ and $Ax = Bx$ for every $x \in D(A)$.

3 Operator wop-groups as a pasting of operator sub-groups equipped with the usual sum of operators

Definition 4 Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let us define the following set of linear operators densely defined in $\mathcal{H}$:

$$\mathcal{G}r(\mathcal{H}) = \{A : D(A) \to \mathcal{H} \mid \overline{D(A)} = \mathcal{H}$$

and

$$D(A) = \mathcal{H} \text{ if } A \text{ is bounded}\}.$$

Theorem 1 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\oplus_{\mathcal{D}}$ be a partial operation on $\mathcal{G}r(\mathcal{H})$ defined for $A, B \in \mathcal{G}r(\mathcal{H})$ by*

$$A \oplus_{\mathcal{D}} B = \begin{cases} A + B & \text{(the usual sum)} \\ & \text{if } A+B \text{ is unbounded and} \\ & (D(A) = D(B) \text{ or} \\ & \text{one out of } A, B \text{ is bounded}), \\ (A+B)^{\mathbf{b}} & \text{if } A+B \text{ is bounded and} \\ & D(A) = D(B), \\ \text{undefined} & \text{otherwise} \end{cases}$$

and $\leq$ be a relation on $\mathcal{G}r(\mathcal{H})$ defined for $A, B \in \mathcal{G}r(\mathcal{H})$ by

$A \leq B$ *iff there is a positive linear operator $C \in \mathcal{G}r(\mathcal{H})$ such that $B = A \oplus_{\mathcal{D}} C$.*

Then $\mathcal{G}r(\mathcal{H}) = (\mathcal{G}r(\mathcal{H}); \oplus_{\mathcal{D}}, 0)$ is a wop-group with respect to $\leq$.

Proof. Let $A, B, C \in \mathcal{G}r(\mathcal{H})$. Then (Gi) is valid since $A \oplus_{\mathcal{D}} B$ is defined iff $B \oplus_{\mathcal{D}} A$ is defined and because the usual sum is commutative we get that $A \oplus_{\mathcal{D}} B = B \oplus_{\mathcal{D}} A$. Moreover, (Giii) is valid since $A \oplus_{\mathcal{D}} 0$ is always defined and $A \oplus_{\mathcal{D}} 0 = A$. Clearly, (Giv) follows from the fact that for every $A \in \mathcal{G}r(\mathcal{H})$ there exists a unique $B \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathcal{D}} B = 0$ (namely we put $-A = B$ and evidently $A + B = 0_{/D(A)}$ yields $0^{\mathbf{b}}_{/D(A)} = 0$). It remains to check (Gii) and (Gv). This will be proved by cases. Assume that $(A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ is defined and $A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$ is defined.

First, let $(A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ be of the form $(A+B)+C$, hence $(A+B)+C$ is unbounded. Then $D(A+B) = D(A) \cap D(B) \in \{D(A), D(B)\}$ and $D((A+B)+C) = D(A+B) \cap D(C) \in \{D(A), D(B), D(C)\}$. Assume for the moment that $D((A+B)+C) = D(A) \neq \mathcal{H}$ (the other cases follow by a symmetric argument). We have the following possibilities:

(α_1): $D(A) = D(B)$, hence $A + B$ is unbounded and $D((A+B)+C) = D(A+B) = D(A) = D(B)$. Then either C is unbounded and $D(A+B) = D(C)$ or C is bounded and $D(A+B) \subset D(C)$. In both cases we have that $D(B+C) = D(B) = D(A)$. But this yields that $(A+B)+C = A+(B+C) = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$.

(β_1): $D(A) \neq D(B)$, hence $A+B$ is unbounded, $D(A+B) = D(A)$ and B is bounded. As in (α_1), either C is unbounded and $D(A+B) = D(C)$ or C is bounded and $D(A+B) \subset D(C)$, $D(B) = D(C) = \mathcal{H}$. In both cases we have that $D(B+C) = D(C) \supseteq D(A)$. Hence again $(A+B)+C = A+(B+C) = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$.

Similarly, let $(A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ be of the form $((A+B)+C)^{\mathbf{b}}$, hence $(A+B)+C$ is bounded. Because $(A+B)$ is unbounded, C is also unbounded. Then $D(A+B) = D(A) \cap D(B) \in \{D(A), D(B)\}$ and $D(A+B) = D(C)$. Assume that $D(A) \neq \mathcal{H}$ (the other case where $D(B) \neq \mathcal{H}$ is symmetric). We will distinguish the following cases:

(α_2): $D(A) = D(B)$, then $D(A+B) = D(A) = D(B)$ and because $D(C) = D(A+B)$ we have $D(A) = D(B) = D(C)$. So $((A+B)+C)^{\mathbf{b}} = (A+(B+C))^{\mathbf{b}} = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$.

(β_2): $D(A) \neq D(B)$, then B is bounded and $D(B) = \mathcal{H}$, hence $D(A) = D(A+B) = D(C)$. Then $D(B+C) = D(C) = D(A)$, which yields that $((A+B)+C)^{\mathbf{b}} = (A+(B+C))^{\mathbf{b}} = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$.

Let $(A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ be of the form $((A+B)^{\mathbf{b}}+C)^{\mathbf{b}}$, hence $(A+B)^{\mathbf{b}} + C$ is bounded $A+B$ is bounded and then C is also bounded. We will verify each of the following cases:

(α_3): $D(A) = D(B) \neq \mathcal{H}$, then $D(B+C) = D(B) = D(A)$. And then $((A+B)^{\mathbf{b}} + C)^{\mathbf{b}} = (A+(B+C))^{\mathbf{b}} = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$.

(β_3): $D(A) = D(B) = \mathcal{H}$, therefore $D(B+C) = \mathcal{H} = D(A)$ and $((A+B)^{\mathbf{b}}+C)^{\mathbf{b}} = (A+(B+C)^{\mathbf{b}})^{\mathbf{b}} = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$.

And in the last case, let $(A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ be of the form $((A+B)^{\mathbf{b}} + C)$. That is, $(A+B)$ is bounded and C is unbounded. Then we prove:

(α_4): $D(A) = D(B) = \mathcal{H}$ i.e. A and B are bounded. Then $D((A+B)^{\mathbf{b}} + C) = D(C)$ and $D(C) = D(B+C) = D(A+(B+C))$. Hence $((A+B)^{\mathbf{b}} + C) = (A+(B+C)) = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$.

(β_4): $D(A) = D(B) \neq \mathcal{H}$, i.e. A and B are unbounded. Then if $D(B) \neq D(C)$, $((A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C) = ((A+B)^{\mathbf{b}} + C)$ is defined and $D((A+B)^{\mathbf{b}} + C) = D(C)$, but $(B \oplus_{\mathcal{D}} C)$ is not defined, so $(A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C))$ is not defined. In the case that $D(B) = D(C)$ we have $D(B) = D(C) = D(A)$, so $((A+B)^{\mathbf{b}} + C) = (A+(B+C)) = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$.

Table 1:

$D(C_1)$	$D(C_2)$	$C_1 \oplus_{\mathcal{D}} C_2$	$D(C_1 \oplus_{\mathcal{D}} C_2)$
$= \mathcal{H}$	$= \mathcal{H}$	$(C_1 + C_2)^{\mathbf{b}}$	$= \mathcal{H}$
$= D(B)$	$= \mathcal{H}$	$C_1 + C_2$	$= D(C_1) = D(B)$
$= \mathcal{H}$	$= D(A)$	$C_1 + C_2$	$= D(C_2) = D(A)$
$= D(B) = D(A)$	$= D(C_1)$	$C_1 + C_2$ if $C_1 + C_2$ is unbounded $(C_1 + C_2)^{\mathbf{b}}$ if $C_1 + C_2$ is bounded	$\neq \mathcal{H}$ $= D(A) = D(B)$

Now, assume that $A \oplus_{\mathcal{D}} B = A \oplus_{\mathcal{D}} C$. First, assume that A is bounded. Then $D(B) = D(C) \subseteq D(A)$. Hence $A + B = A + C$. This yields by (Gii) that $C = -A + (A + C) = -A + (A + B) = B$. Now, assume that A is unbounded.

If B is unbounded then $D(B) = D(A)$ and we will distinguish the following cases:

(γ_1): $A + B$ is bounded and hence also $A + C$ is bounded. It follows that C is unbounded and hence $D(C) = D(A)$. Therefore also $A + B = A + C$.

(δ_1): $A + B$ is unbounded and hence also $A + C$ is unbounded. We get that $D(C) = D(A)$, i.e. $A + B = A + C$.

In both cases we get as above from (Gii) that $C = B$.

If B is bounded we have that $A+B$ is unbounded. This implies that $A+C$ is unbounded as well. Therefore $D(A + B) = D(A) \subseteq D(B)$ and $D(A) \subseteq D(C)$. We then have $A + B = A + C$. It follows again by (Gii) that $C_{/D(A)}$ is bounded and $B_{/D(A)} = C_{/D(A)}$. Therefore $B = C$.

Let us check that $\leq$ is reflexive and antisymmetric. Let $A, B \in \mathcal{G}r(\mathcal{H})$. Evidently, $A \leq A$ since $A = A \oplus_{\mathcal{D}} 0$ and 0 is a positive bounded linear operator on $\mathcal{H}$. Now, assume that $B = A \oplus_{\mathcal{D}} C_1$ and $A = B \oplus_{\mathcal{D}} C_2$ for some positive linear operators C_1, C_2 on $\mathcal{H}$. By cases we have that ($A+C_1 = B$ with B unbounded or $(A + C_1)^{\mathbf{b}} = B$ with B bounded) and ($B+C_2 = A$ with A unbounded or $(B+C_2)^{\mathbf{b}} = A$ with A bounded). Assume first that $A+C_1 = B$ with B unbounded and $B + C_2 = A$ with A unbounded. We have the following possibilities for C_1 and C_2 (see Tab. 1).

Now assume that $A+C_1 = B$ with B unbounded and $(B + C_2)^{\mathbf{b}} = A$ with A bounded. Then C_1 has to be unbounded with $D(C_1) = D(B)$ and C_2 can only also be unbounded with $D(C_2) = D(B)$. When C_1+C_2 is bounded then $C_1 \oplus_{\mathcal{D}} C_2 = (C_1+C_2)^{\mathbf{b}}$ and $D(C_1 \oplus_{\mathcal{D}} C_2) = \mathcal{H}$. For $C_1 + C_2$ unbounded we have $D(C_1 \oplus_{\mathcal{D}} C_2) = D(B)$.

The situation for $(A+C_1)^{\mathbf{b}} = B$ with B bounded and $(B + C_2) = A$ with A unbounded is symmetric to the previous case with $D(C_1 \oplus_{\mathcal{D}} C_2) = \mathcal{H}$ when $C_1 + C_2$ is bounded and $D(C_1 \oplus_{\mathcal{D}} C_2) = D(A)$ when $C_1 + C_2$ is unbounded.

The last case is $(A + C_1)^{\mathbf{b}} = B$ with B bounded and $(B + C_2)^{\mathbf{b}} = A$ with A bounded too. Hence C_1 and C_2 are bounded as well and $(C_1 \oplus_{\mathcal{D}} C_2) = (C_1 + C_2)^{\mathbf{b}}$ with $D(C_1 \oplus_{\mathcal{D}} C_2) = \mathcal{H}$.

This yields that $A \oplus_{\mathcal{D}} (C_1 \oplus_{\mathcal{D}} C_2)$ is defined and $A \oplus_{\mathcal{D}} (C_1 \oplus_{\mathcal{D}} C_2) = (A \oplus_{\mathcal{D}} C_1) \oplus_{\mathcal{D}} C_2 = B \oplus_{\mathcal{D}} C_2 = A$. Hence by (Giii) and (Giv) we obtain that $C_1 \oplus_{\mathcal{D}} C_2 = 0$, hence $C_1 + C_2 = 0_{/D(C_1)}$. By [7, Theorem 2] we have that $C_1 = C_2 = 0$.

So, it remains to check that $\leq$ is compatible with addition, i.e., for all $A, B, C \in \mathcal{G}r(\mathcal{H})$ such that $A \leq B$, $C \oplus_{\mathcal{D}} A$ and $C \oplus_{\mathcal{D}} B$ are defined we have that $C \oplus_{\mathcal{D}} A \leq C \oplus_{\mathcal{D}} B$. Again by cases we have that ($C \oplus_{\mathcal{D}} A = C + A$ with $C + A$ unbounded or $C \oplus_{\mathcal{D}} A = (C + A)^{\mathbf{b}}$ with $C + A$ bounded) and ($C \oplus_{\mathcal{D}} B = C + B$ with $C + B$ unbounded or $C \oplus_{\mathcal{D}} B = (C + B)^{\mathbf{b}}$ with $C + B$ bounded). Since $A \leq B$ there is $E \in \mathcal{G}r(\mathcal{H})$, E positive such that $A \oplus_{\mathcal{D}} E = B$. But $C \oplus_{\mathcal{D}} B = C \oplus_{\mathcal{D}} (A \oplus_{\mathcal{D}} E)$.

For the case when E is bounded, clearly, $(C \oplus_{\mathcal{D}} A) \oplus_{\mathcal{D}} E$ is always defined. Now assume that E is unbounded.

In the case when A is unbounded and B is bounded we get that $D(E) = D(A)$ and $D((E + A)^{\mathbf{b}}) = \mathcal{H}$. We have the following possibilities:
(a): C is bounded. Then $D(C + A) = D(A) = D(E)$.
(b): C is unbounded. Then $D(C) = D(A) = D(E)$.
In the case when both A, B are unbounded we obtain that $D(E + A) = D(B) = D(A) = D(E)$. We distinguish:
(c): C is bounded. Then $D(C + A) = D(A) = D(E)$.
(d): C is unbounded. Then $D(C) = D(A) = D(E)$.
In the last case assume that A is bounded and B is unbounded, hence $D(E + A) = D(B) = D(E)$.
(e): C is bounded. Then $D(C + A) = \mathcal{H}$.
(f): C is unbounded. Then $D(C + A) = D(C) = D(B) = D(E)$.
Hence in all cases $(C \oplus_{\mathcal{D}} A) \oplus_{\mathcal{D}} E$ is defined.

Recall that we have the following result of [9].

Theorem 2 [9, Theorem 1] *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let us define the following set of positive linear operators densely defined in $\mathcal{H}$:*

$$\begin{aligned}\mathcal{V}(\mathcal{H}) = \{&A : D(A) \to \mathcal{H} \mid A \geq 0,\\ &\overline{D(A)} = \mathcal{H} \text{ and}\\ &D(A) = \mathcal{H} \text{ if } A \text{ is bounded}\}.\end{aligned}$$

Let $\oplus_{\mathcal{D}}$ be defined for $A, B \in \mathcal{V}(\mathcal{H})$ by $A \oplus_{\mathcal{D}} B = A + B$ (the usual sum) iff

1. *either at least one out of A, B is bounded*
2. *or both A, B are unbounded and $D(A) = D(B)$.*

Then $\mathcal{V}_{\mathcal{D}}(\mathcal{H}) = (\mathcal{V}(\mathcal{H}); \oplus_{\mathcal{D}}, 0)$ is a generalized effect algebra such that $\oplus_{\mathcal{D}}$ extends the operation $\oplus$.

Then $Pos(\mathcal{G}r(\mathcal{H})) = \mathcal{V}(\mathcal{H})$, hence the generalized effect algebra $\mathcal{V}(\mathcal{H})$ is the positive cone of $\mathcal{G}r(\mathcal{H})$ and, for all $A, B, C \in \mathcal{V}(\mathcal{H})$, $(A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ exists iff $A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C)$ exists.

Definition 5 Let $(G, +, 0)$ be a commutative partial group and let S be a subset of G such as:

(Si) $0 \in S$,

(Sii) $-x \in S$ for all $x \in S$,

(Siii) for every $x, y \in S$ such $x + y$ is defined also $x + y \in S$.

Then we call S a *commutative partial subgroup* of G.

Let G be a wop-group with respect to a partial order $\leq_G$ and let $\leq_S$ be a partial order on a commutative partial subgroup $S \subseteq G$. If for all $x, y \in S$ holds: $x \leq_S y$ if and only if $x \leq_G y$, we call S a *wop-subgroup* of G.

For a commutative partial group $G = (G, +, 0)$ and a commutative partial subgroup S, we denote $+_S = +_{/S^2}$. We will omit an index and we will write $S = (S, +, 0)$ instead of $(S, +_S, 0)$ where no confusion can result.

Lemma 1 *Let $G = (G, +, 0)$ be a commutative partial group and let S be a commutative partial subgroup of G. Then $(S, +, 0)$ is a commutative partial group.*

Let G be a wop-group and let S be a wop-subgroup of G. Then S is a wop-group.

Proof. Conditions (Gi), (Gii) and (Gv) follow immediately from (Siii). Condition (Giii) follows from (Si) and (Siii) and condition (Giv) follows from (Sii).

Assume now that G is a wop-group such that S is a wop-subgroup of G. If $x, y, z \in S, x \leq_S y$ and $x + z, y + z$ are defined, then $x + z, y + z \in S$ and $x + z \leq_G y + z$ hence $x + z \leq_S y + z$.

Lemma 2 *Let $G = (G, +, 0)$ be a commutative partial group and S_1, S_2 commutative partial subgroups of G. Then $S = S_1 \cap S_2$ is also a commutative partial subgroup of G.*

Proof. Condition (Si) is clear. (Sii): If $x \in S$ then $x \in S_1$ and $x \in S_2$ hence $-x \in S_1$ and $-x \in S_2$. Therefore $-x \in S$.

(Siii): Assume that $x, y \in S$ such that $x + y$ is defined. Then $x, y \in S_1$ and $x, y \in S_2$. Hence $x + y \in S_1$ and $x + y \in S_2$. This yields $x + y \in S$.

Definition 6 Let $G_1 = (G_1, +_1, 0_1)$ and $G_2 = (G_2, +_2, 0_2)$ be commutative partial groups. A *morphism* is a map $\varphi : G_1 \to G_2$ such that, for any $x, y \in G_1$, whenever $x +_1 y$ exists then $\varphi(x) +_2 \varphi(y)$ exists, in which case $\varphi(x +_1 y) = \varphi(x) +_2 \varphi(y)$. If φ is a bijection such that φ and φ^{-1} are morphisms we say that φ is an *isomorphism* of commutative partial groups and G_1 and G_2 are *isomorphic*.

Moreover, let $\leq_1$ on G_1 and $\leq_2$ on G_2 be partial orders such that G_1 and G_2 are wop-groups. Let $\varphi : G_1 \to G_2$ be a morphism between commutative partial groups. If for every $x, y \in G_1 : x \leq_1 y$ implies $\varphi(x) \leq_2 \varphi(y)$, then φ is a *morphism* between wop-groups. If φ is a bijection, φ and φ^{-1} are morphisms we say that φ is an *isomorphism* of wop-groups and G_1 and G_2 are *isomorphic* as wop-groups.

Definition 7 Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let us define the following sets of linear operators densely defined in $\mathcal{H}$:

$$\begin{aligned}\mathcal{SG}r(\mathcal{H}) &= \{A \in \mathcal{G}r(\mathcal{H}) \mid A \subset A^*\}\\ \mathcal{HG}r(\mathcal{H}) &= \{A \in \mathcal{G}r(\mathcal{H}) \mid A \subset A^*, D(A) = \mathcal{H}\}.\end{aligned}$$

i.e. $\mathcal{SG}r(\mathcal{H})$ is the set of all symmetric operators and $\mathcal{HG}r(\mathcal{H})$ is the set of all Hermitian operators.

From the definition we can see that $\mathcal{HG}r(\mathcal{H}) \subseteq \mathcal{SG}r(\mathcal{H})$. It is a well known fact that every positive operator is symmetric and every positive bounded operator is both self-adjoint and Hermitian (see [2]).

Theorem 3 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\leq_S$ be a relation on $\mathcal{SG}r(\mathcal{H})$ defined for $A, B \in \mathcal{SG}r(\mathcal{H})$ by $A \leq_S B$ if and only if there exists a positive operator $C \in \mathcal{SG}r(\mathcal{H})$ such as $A \oplus_{\mathcal{D}} C = B$. Then $(\mathcal{SG}r(\mathcal{H}); \oplus_{\mathcal{D}}, 0)$ equipped with $\leq_S$ forms a wop-subgroup of $\mathcal{G}r(\mathcal{H})$.*

Proof. Conditions (Si) and (Sii) are clearly satisfied. We have to verify that $\mathcal{SG}r(\mathcal{H})$ is closed under addition. Let $A, B \in \mathcal{SG}r(\mathcal{H})$ be bounded, then they are Hermitian and it is well known that the sum of two Hermitian operators is also Hermitian.

Recall that $A \subset A^*$ iff for all $x, y \in D(A)$: $\langle x, Ay\rangle = \langle Ax, y\rangle$. If A is bounded and B is unbounded then $D(A + B) = D(B)$ and, for all $x, y \in D(B)$, it holds $\langle x, (A + B)y\rangle = \langle x, Ay\rangle + \langle x, By\rangle =$

$\langle Ax, y\rangle + \langle Bx, y\rangle = \langle (A+B)x, y\rangle$ hence $(A+B) \subset (A+B)^*$.

A similar argument holds for A unbounded, B unbounded and $A+B$ unbounded, where $D(A+B) = D(A) = D(B)$.

If A and B are unbounded and $A+B$ is bounded, then, for all $x, y \in D(A) = D(B)$, we have $\langle x, (A+B)y\rangle = \langle x, Ay\rangle + \langle x, By\rangle = \langle Ax, y\rangle + \langle Bx, y\rangle = \langle (A+B)x, y\rangle$. Therefore $(A+B) \subset (A+B)^*$. From $(A+B) \subset (A+B)^{\mathbf{b}}$ we get $((A+B)^{\mathbf{b}})^* \subset (A+B)^*$ and then $\mathcal{H} = D((A+B)^{\mathbf{b}*}) = D((A+B)^*)$. Hence $(A+B)^* = ((A+B)^{\mathbf{b}})^*$. Because $(A+B)^*$ is symmetric one obtains that $(A+B)^* = ((A+B)^{\mathbf{b}})^* = (A+B)^{\mathbf{b}}$. Hence $(A+B)^{\mathbf{b}} = (A \oplus_{\mathcal{D}} B) \in \mathcal{SG}r(\mathcal{H})$.

Now let $A \oplus_{\mathcal{D}} C = B$ where $A, B \in \mathcal{SG}r(\mathcal{H})$ and $C \in \mathcal{G}r(\mathcal{H})$, C positive. Since C is a positive operator we get that $C \in \mathcal{SG}r(\mathcal{H})$. Therefore $\leq_{\mathcal{S}} = \leq_{/\mathcal{SG}r(\mathcal{H})^2}$.

4 Operator weakly ordered partial groups as a pasting of operator sub-groups equipped with the usual sum of operators

Let us recall the following theorem from [7] that was our basic motivation for investigating the set of linear operators on a Hilbert space.

Theorem 4 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $D \in \mathcal{D}$. Let*

$$Lin_D(\mathcal{H}) = \{A : D \to \mathcal{H} \mid A \text{ is a linear operator defined on } D\}.$$

Then $(Lin_D(\mathcal{H}); +, \leq, 0)$ is a partially ordered commutative group where 0 is the null operator, $+$ is the usual sum of operators defined on D and $\leq$ is defined for all $A, B \in Lin_D(\mathcal{H})$ by $A \leq B$ iff $B - A$ is positive.

Definition 8 Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $D \in \mathcal{D}$. Let

$$\begin{aligned} \mathcal{G}r_D(\mathcal{H}) &= \{A \in \mathcal{G}r(\mathcal{H}) \mid D(A) = D \text{ or } A \\ &\quad \text{is bounded}\}. \\ \mathcal{SG}r_D(\mathcal{H}) &= \{A \in \mathcal{SG}r(\mathcal{H}) \mid D(A) = D \text{ or } A \\ &\quad \text{is bounded}\}. \end{aligned}$$

Now, we are going to show that the set $\mathcal{G}r_D(\mathcal{H})$ equipped with the prescription $\oplus_D = \oplus_{\mathcal{D}/(\mathcal{G}r_D(\mathcal{H}))^2}$ and the relation $\leq_D = \leq_{/(\mathcal{G}r_D(\mathcal{H}))^2}$ is a partially ordered commutative group isomorphic to $Lin_D(\mathcal{H})$.

Theorem 5 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $D \in \mathcal{D}$. Then $\mathcal{G}r_D(\mathcal{H}) = (\mathcal{G}r_D(\mathcal{H}); \oplus_D, 0)$ with respect to $\leq_D$ is a wop-subgroup of $\mathcal{G}r(\mathcal{H})$ such that the induced operation $\oplus_D$ is total. Moreover, $\mathcal{G}r_D(\mathcal{H})$ is isomorphic to $Lin_D(\mathcal{H})$ and hence a partially ordered commutative group.*

Proof. Conditions (Si) and (Sii) are clearly satisfied. Let us check condition (Siii).

For $A, B \in \mathcal{G}r_D(\mathcal{H})$, first assume that $D(A) = D(B) \in \{D, \mathcal{H}\}$. Then $D(A) = D(B) = D(A+B) \in \{D, \mathcal{H}\}$, hence $A \oplus_D B$ exists in $\mathcal{G}r_D(\mathcal{H})$. On the other hand, let $D(A) \neq D(B)$. Then either $D(A) \subset D(B) = \mathcal{H}$, in which case $D(A) = D(A+B) = D$ or $D(B) \subset D(A) = \mathcal{H}$ with $D(B) = D(A+B) = D$ hence $A \oplus_D B$ also exists in $\mathcal{G}r_D(\mathcal{H})$. Hence for all $A, B \in \mathcal{G}r_D(\mathcal{H})$ we have that $A \oplus_D B \in \mathcal{G}r_D(\mathcal{H})$ i.e. $\oplus_D$ is a total operation on $\mathcal{G}r_D(\mathcal{H})$.

Now let $A \oplus_D B = C$ where $A, C \in \mathcal{G}r_D(\mathcal{H})$ and $B \in \mathcal{G}r(\mathcal{H})$, B positive. Since $A \oplus_D B$ is defined and B is positive we have that $B \in \mathcal{G}r_D(\mathcal{H}) \cap \mathcal{V}(\mathcal{H})$.

We can define a map $\varphi : Lin_D \to \mathcal{G}r_D(\mathcal{H})$ where:

$$\varphi(A) = \begin{cases} A & \text{if } A \text{ is unbounded,} \\ A^{\mathbf{b}} & \text{if } A \text{ is bounded.} \end{cases}$$

For $A \in Lin_D$ unbounded it holds $D(A) = D(\varphi(A)) = D$ hence $\varphi(A) \in \mathcal{G}r_D(\mathcal{H})$. For A bounded we have $D(\varphi(A)) = D(A^{\mathbf{b}}) = \mathcal{H}$ hence $A \in \mathcal{G}r_D(\mathcal{H})$.

We can define $\psi : \mathcal{G}r_D(\mathcal{H}) \to Lin_D$ as $\psi(B) = B$ for B unbounded and $\psi(B) = B_{/D}$ for B bounded. Then clearly $\psi \circ \varphi = id_{Lin_D}$ and $\varphi \circ \psi = id_{\mathcal{G}r_D(\mathcal{H})}$ hence φ is a bijection.

It is evident that $\varphi(0) = 0$ and $+$ on Lin_D is total. For $A, B \in Lin_D$, let us assume that:

(a): A, B be bounded. Then $\varphi(A+B) = (A+B)^{\mathbf{b}} = A^{\mathbf{b}} + B^{\mathbf{b}} = \varphi(A) \oplus_D \varphi(B)$.

(b): A be bounded, B be unbounded. Then $D(A+B) = D(B) = D$ and $\varphi(A+B) = A+B = A^{\mathbf{b}} + B = \varphi(A) \oplus_D \varphi(B)$.

(c): A be unbounded, B be unbounded, $A+B$ be unbounded. Then $\varphi(A+B) = A+B = \varphi(A) \oplus_D \varphi(B)$

(d): A be unbounded, B be unbounded, $A+B$ be bounded. Then $\varphi(A+B) = (A+B)^{\mathbf{b}} = (\varphi(A) + \varphi(B))^{\mathbf{b}} = \varphi(A) \oplus_D \varphi(B)$.

Now, we should verify order preservation, but it is clear that φ and ψ preserve order.

Theorem 6 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $D \in \mathcal{D}$. Then $\mathcal{SG}r_D(\mathcal{H})$ with the induced total operation $\oplus_D$ and the induced partial order $\leq_{\mathcal{SG}r_D(\mathcal{H})}$ is a wop-subgroup of $\mathcal{G}r_D(\mathcal{H})$ and hence a partially ordered commutative subgroup of $\mathcal{G}r_D(\mathcal{H})$.*

Proof. $\mathcal{SG}r_D(\mathcal{H})$ is a commutative subgroup of $\mathcal{G}r_D(\mathcal{H})$ because of Lemma 2 and $\mathcal{SG}r_D(\mathcal{H}) =$

$\mathcal{G}r_D(\mathcal{H}) \cap \mathcal{SG}r(\mathcal{H})$. We have to check order preservation. For any $A, B \in \mathcal{SG}r_D(\mathcal{H})$ such that $A \leq_{\mathcal{G}r_D(\mathcal{H})} B$ we have that there exists positive $C \in \mathcal{G}r_D(\mathcal{H})$ such that $A + C = B$. Since every positive operator is symmetric we have that $C \in \mathcal{SG}r_D(\mathcal{H})$. This yields that $\leq_{\mathcal{SG}r_D(\mathcal{H})} = (\leq_{\mathcal{G}r_D(\mathcal{H})})_{/\mathcal{SG}r_D(\mathcal{H})^2}$.

Theorem 7 (The pasting theorem for $\mathcal{G}r(\mathcal{H})$) *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Then the wop-group $\mathcal{G}r(\mathcal{H})$ pastes their partially ordered commutative subgroups $\mathcal{G}r_D(\mathcal{H})$, $D \subseteq \mathcal{H}$ a dense linear subspace of $\mathcal{H}$, together over $\mathcal{B}(\mathcal{H})$, i.e. $\mathcal{G}r_{D_1}(\mathcal{H}) \cap \mathcal{G}r_{D_2}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ for every pair D_1, D_2 of dense linear subspaces of $\mathcal{H}$, $D_1 \neq D_2$, and*

$$\mathcal{G}r(\mathcal{H}) = \bigcup\{\mathcal{G}r_D(\mathcal{H}) \mid D \in \mathcal{D}\}.$$

Proof. Straightforward from definition, for $D \in \mathcal{D}$, every bounded $A \in \mathcal{G}r(\mathcal{H})$ lies in $\mathcal{G}r_D(\mathcal{H})$. For any unbounded $B \in \mathcal{G}r(\mathcal{H})$, $B \in \mathcal{G}r_D(\mathcal{H})$ if and only if $D(B) = D$ hence there is unique $\mathcal{G}r_D(\mathcal{H})$ in which B lies. Hence $\mathcal{G}r_{D_1}(\mathcal{H}) \cap \mathcal{G}r_{D_2}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ for all $D_1 \neq D_2$, $D_1, D_2 \in \mathcal{D}$. And because $\mathcal{G}r_D(\mathcal{H})$ in which B lies exists for every $B \in \mathcal{G}r(\mathcal{H})$, we have $\mathcal{G}r(\mathcal{H}) = \bigcup\{\mathcal{G}r_D(\mathcal{H}) \mid D \in \mathcal{D}\}$.

Theorem 8 (The pasting theorem for $\mathcal{SG}r(\mathcal{H})$) *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Then the wop-group $\mathcal{SG}r(\mathcal{H})$ pastes their partially ordered commutative subgroups $\mathcal{SG}r_D(\mathcal{H})$, $D \in \mathcal{D}$, together over $\mathcal{HG}r(\mathcal{H})$, i.e., for every pair D_1, D_2 of dense linear subspaces of $\mathcal{H}$, $D_1 \neq D_2$, $\mathcal{SG}r_{D_1}(\mathcal{H}) \cap \mathcal{SG}r_{D_2}(\mathcal{H}) = \mathcal{HG}r(\mathcal{H})$ and*

$$\mathcal{SG}r(\mathcal{H}) = \bigcup\{\mathcal{SG}r_D(\mathcal{H}) \mid D \in \mathcal{D}\}.$$

Proof. Let $D \in \mathcal{D}$. Since $\mathcal{SG}r_D(\mathcal{H}) = \mathcal{G}r_D(\mathcal{H}) \cap \mathcal{SG}r(\mathcal{H})$, with previous theorem $\bigcup\limits_{D\in\mathcal{D}} \mathcal{SG}r_D(\mathcal{H}) = \bigcup\limits_{D\in\mathcal{D}} (\mathcal{G}r_D(\mathcal{H}) \cap \mathcal{SG}r(\mathcal{H})) = \left(\bigcup\limits_{D\in\mathcal{D}} \mathcal{G}r_D(\mathcal{H})\right) \cap \mathcal{SG}r(\mathcal{H}) = \mathcal{G}r(\mathcal{H}) \cap \mathcal{SG}r(\mathcal{H}) = \mathcal{SG}r(\mathcal{H})$. Similarly we have $\mathcal{SG}r_{D_1}(\mathcal{H}) \cap \mathcal{SG}r_{D_2}(\mathcal{H}) = (\mathcal{SG}r(\mathcal{H}) \cap \mathcal{G}r_{D_1}(\mathcal{H})) \cap (\mathcal{SG}r(\mathcal{H}) \cap \mathcal{G}r_{D_2}(\mathcal{H})) = \mathcal{SG}r(\mathcal{H}) \cap (\mathcal{G}r_{D_1}(\mathcal{H}) \cap \mathcal{G}r_{D_2}(\mathcal{H})) = \mathcal{SG}r(\mathcal{H}) \cap \mathcal{B}(\mathcal{H}) = \mathcal{HG}r(\mathcal{H})$ for all $D_1 \neq D_2$, $D_1, D_2 \in \mathcal{D}$.

5 $\mathcal{PT}$-symmetry and related effect algebras

Let us repeat some of the notions concerning the basics of $\mathcal{PT}$-symmetry from [7]. Let $\mathcal{H}$ be a Hilbert space equipped with an inner product $\langle \psi, \phi \rangle$. Let $\Omega : \mathcal{H} \to \mathcal{H}$ be an invertible linear operator. Then we obtain a new inner product $\langle\!\langle -, - \rangle\!\rangle$ on $\mathcal{H}$ which will have the form:

$$\langle\!\langle \psi, \varphi \rangle\!\rangle = \langle \Omega\psi, \Omega\varphi \rangle, \qquad \forall \psi, \varphi \in \mathcal{H}.$$

Clearly, our new inner product space is complete with respect to $\langle\!\langle -, - \rangle\!\rangle$. Let us denote $\mathcal{H}_\Omega$ the corresponding Hilbert space. Hence $\Omega : \mathcal{H}_\Omega \to \mathcal{H}$ and $\Omega^{-1} : \mathcal{H} \to \mathcal{H}_\Omega$ provide a realization of the unitary-equivalence of the Hilbert spaces $\mathcal{H}_\Omega$ and $\mathcal{H}$.

Let us define a map $(\cdot)^D_\Omega : \mathcal{G}r_D(\mathcal{H}) \to \mathcal{G}r_{\Omega^{-1}(D)}(\mathcal{H}_\Omega)$ by $A_\Omega = \Omega^{-1} \circ A \circ \Omega$ for a linear map $A \in \mathcal{G}r_D(\mathcal{H})$, $D \in \mathcal{D}$. We then have

Proposition 1 [7, Proposition 3] *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Assume moreover that $\Omega : \mathcal{H} \to \mathcal{H}$ is an invertible linear operator and $D \in \mathcal{D}$. Then*

1. *$(A)^D_\Omega$ is a positive operator on $\mathcal{H}_\Omega$ iff A is a positive operator on $\mathcal{H}$.*
2. *$(\cdot)^D_\Omega$ is an isomorphism of partially ordered commutative groups.*
3. *$(A)^D_\Omega$ is a Hermitian operator on $\mathcal{H}_\Omega$ iff A is a Hermitian operator on $\mathcal{H}$.*
4. *$(I)^D_\Omega = I$.*

The preceding proposition immediately yields that

Theorem 9 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\Omega : \mathcal{H} \to \mathcal{H}$ an invertible linear operator. Then the map $(\cdot)_\Omega : \mathcal{G}r(\mathcal{H}) \to \mathcal{G}r(\mathcal{H}_\Omega)$ defined by $(A)_\Omega := (A)^{D(A)}_\Omega$ for all $A \in \mathcal{G}r(\mathcal{H})$ is an isomorphism of wop-groups.*

Proof. It follows from Proposition 1 and Theorem 7.

Corollary 1 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\Omega : \mathcal{H} \to \mathcal{H}$ an invertible linear operator. Then $\mathcal{V}(\mathcal{H})$ and $\mathcal{V}(\mathcal{H}_\Omega)$ are isomorphic generalized effect algebras.*

Proof. It follows immediately from the fact that $(\cdot)_\Omega$ preserves and reflects positive operators and from Theorem 9.

We say that an operator $H : D \to \mathcal{H}$ defined on a dense linear subspace D of a Hilbert space $\mathcal{H}$ is *η_+-pseudo-Hermitian* and η_+ is a *metric operator* if $\eta_+ : \mathcal{H} \to \mathcal{H}$ is a positive, Hermitian, invertible, linear operator such that $H^* = \eta_+ H \eta_+^{-1}$ (see also [6]).

Theorem 10 *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space, let $D \subseteq \mathcal{H}$ be a linear subspace dense in $\mathcal{H}$ and let $H : D \to \mathcal{H}$ be a η_+-pseudo-Hermitian operator for some metric operator $\eta_+ : \mathcal{H} \to \mathcal{H}$ such that $\eta_+ = \rho_+^2$. Then*

1. *$\mathcal{G}r(\mathcal{H}_{\rho_+})$ and $\mathcal{G}r(\mathcal{H})$ are mutually isomorphic wop-groups such that $H \in \mathcal{G}r(\mathcal{H}_{\rho_+})$ and H is a self-adjoint operator with respect to the positive-definite inner product $\langle\!\langle -, - \rangle\!\rangle = \langle \rho_+ -, \rho_+ - \rangle$ on $\mathcal{H}_{\rho_+}$.*

2. $\mathcal{V}(\mathcal{H})$ and $\mathcal{V}(\mathcal{H}_{\rho_+})$ are mutually isomorphic generalized effect algebras. If moreover H is a positive operator with respect to $\langle\!\langle -, - \rangle\!\rangle$ (i.e., its real spectrum will be contained in the interval $[0, \infty)$) then $H \in \mathcal{V}(\mathcal{H}_{\rho_+})$.

Proof. It follows from the above considerations and [7, Theorem 3].

6 Conclusion

In this paper we have shown that a η_+-pseudo-Hermitian operator for some metric operator η_+ of a quantum system described by a Hilbert space $\mathcal{H}$ yields an isomorphism between the weakly ordered commutative partial group of linear maps on $\mathcal{H}$ and the weakly ordered commutative partial group of linear maps on $\mathcal{H}_{\rho_+}$. The same applies to the generalized effect algebras of positive operators introduced in [9]. Hence, from the standpoint of (generalized) effect algebra theory the two representations of our quantum system coincide.

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