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NOTE ON VERMA BASES FOR REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

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Abstract. We discuss the construction of the Verma basis of the enveloping algebra and of finite dimensional representations of the Lie algebra A_n. We give an alternate proof of so-called Verma inequalities to the one given in [1] by P. Littelmann.

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1. Introduction

The theory of simple Lie groups and their representations (and corresponding representations of simple Lie algebras) has been at the center of interest of modern mathematics for a long time, because it has many relationships with other areas of mathematics and physics.

The simple Lie algebras over the field of complex numbers were classified in the famous works of Killing and Cartan in the 1930s. Since then we have known that there are four infinite series A_n, B_n, C_n, D_n, which are called the classical Lie algebras, and five Lie algebras E_6, E_7, E_8, F_4 and G_2, which we call exceptional Lie algebras. The structure of these Lie algebras is described in terms of special finite sets of elements in a Euclidean space, called roots, which generate a root system. Weyl's theorem assures that each finite dimensional reducible representation of such a Lie algebra is completely reducible. Therefore in the theory of finite dimensional representations of the semisimple Lie algebras, which are direct sums of simple ones, it is sufficient to restrict to irreducible finite dimensional representations. The complete classification of these irreducible finite dimensional representations is known. Their sets are parametrized by vectors of nonnegative integers called highest weights. Moreover, the characters and dimensions of such irreducible finite dimensional representations are explicitly known because of the Weyl formula [2–5].

Practical use of the simple Lie groups and Lie algebras, serving as a fundamental tool for studying the symmetries of systems examined in physics, often involve constructing the bases of the spaces on which their finite dimensional representations act.

The best known example of such constructions are two works of Gelfand and Tsetlin. In two famous papers (see [6] and [7]), they gave an explicit construction of bases for a general linear Lie algebra $\mathrm{gl}(n,\mathbb{C})$ (resp. special linear Lie algebra A_n) and for orthogonal Lie algebras B_n and D_n (for detailed comments, see also [8]). These papers contain no comments and no methods for deriving the explicit formulas. These papers also do not contain any references (the hint that one has to verify the commutation relations by direct calculation is not very useful for the proof). It is therefore no wonder that their formulas were re-derived and verified by other authors. Verification and/or an independent derivation of these formulas was given in the papers by Baird and Biedenharn (see [9, 10]), and also by other authors [11–14].

After Gelfand and Tsetlin's construction of representations, in the second half of the 20th century and later, a range of different approaches were developed and many techniques were adopted to construct the bases of the representations of classical series of Lie algebras. We can mention here the Gould paper [15], which made use of polynomial identities satisfied by the generators of the corresponding Lie group, an approach which was then generalized then to Kac-Moody algebras [16], and the approach of Asherova, Smirnov and Tolstoy involving projection operators [17, 18], which prove their usefulness also in the field of Lie superalgebras and quantum algebras [19]. The results of Tarasov and Nazarov [20] also belong to this group. Another approach, based on using Weyl realization [21] of the representations of corresponding groups in tensor spaces, was developed in many papers [22–26].

So-called special bases were constructed by de Concini and Kazhdan [27], and their q-analogs by Xi [28]. Proper bases were constructed by Gelfand and Zelevinsky [29], Retakh and Zelevinsky [30], and similar good bases were constructed by Mathieu [31]. Another well known group of bases are crystal bases. They were constructed by Lusztig [32, 33], Kashiwara [34], Du [35, 36], Kang [37], and others [38–41].

2. Verma bases

Besides these approaches, an important role is played by bases which have special properties as bases in the universal enveloping algebra of a given simple Lie algebra, and which can then be restricted by taking a suitable subset to the basis of a given representation of that Lie algebra. Such bases are called monomial bases, and were constructed in the standard monomial theories developed by Lakshmibai, Musili and Seshadri [42], by Littelmann [1, 43], its q-analogs by Chari and Xi [44] and others. One of the advantages of these bases is that the basis vectors are eigenvectors of Cartan subalgebra, and therefore such a basis is suitable for various modifications. On the other hand, there is no explicit form of matrix elements of operators expressed in these bases.

Verma bases as introduced in [45] are of this type. These bases were constructed for the Lie algebras A_n in [46] and for some concrete examples of other Lie algebras of low rank (see also [47]). In [48, 49] proof of the so-called Verma conjecture for the Lie algebra A_n was given by Raghavan and Sankaran. Note that the basis in enveloping algebra from which we can obtain the corresponding Verma basis by restriction was given in [1].

The basis vectors of the Verma basis are constructed from the highest weight vector (vacuum state) in a way consisting of the action of some specified sequence of the elements corresponding to simple roots. Each set of basis vectors is constructed using sequences given by a certain set of inequalities (called Verma inequalities). Let us briefly describe the main result for the Lie algebra A_n.

Let $A_n = \mathrm{sl}(n+1,\mathbb{C}) = n^+ \oplus h \oplus n^-$ be the decomposition of the Lie algebra $\mathrm{sl}(n+1,\mathbb{C})$ into strictly upper triangular, diagonal and strictly lower triangular matrices. Denote $U(A_n)$, $U(A_n^+)$ and $U(A_n^-)$ corresponding enveloping algebras of A_n, n^+ and n^-. Let Φ be the root system of A_n and fixed h such that $\Phi = \Phi^+ \cup \Phi^-$, where $n^+ = \bigoplus_{\beta\in\Phi^+} g_\beta$. For the positive root $\beta \in \Phi^+$ denote by $f_\beta \in g_\beta$ and $e_\beta \in g_{-\beta}$ fixed elements of the Chevalley basis of A_n. For a fixed ordering of simple roots $\{\beta_1,\dots,\beta_n\}$ denote f_{β_j} by f_j and the corresponding e_{β_j} by e_j. Put $h_j = [e_j, f_j]$. Then the set of the following monomials (so-called Verma monomials),

$$f_1^{a_1^1}\big(f_2^{a_2^2} f_1^{a_1^2}\big)\big(f_3^{a_3^3} f_2^{a_2^3} f_1^{a_1^3}\big)\cdots\big(f_n^{a_n^n} f_{n-1}^{a_{n-1}^n}\cdots f_2^{a_2^n} f_1^{a_1^n}\big),$$

where

$$a_k^c \le a_{k+1}^c \tag{1}$$

is a linear basis of $U(A_n^-)$. A similar basis consisting of vectors generated by appropriate sequences of e_j's spans $U(A_n^+)$. Together with the enveloping algebra of h one can obtain a basis of the whole $U(A_n)$.

If we now restrict to the elements generated by sequences fulfilling Verma inequalities

$$0 \le a_k^c \le \min\{a_{k-1}^c + \lambda_{n-c+k}, a_{k+1}^{c+1}\},$$

where $a_k^{n+1} = +\infty$ and $a_0^k = 0$ for all k, acting on the highest weight vector $|0\rangle$ (vacuum state) with the highest weight $(\lambda_1,\dots,\lambda_n)$, where λ_j are nonnegative integers, we obtain a basis of the representation space of the corresponding finite dimensional representation.

3. Verma monomials inequalities

As a contribution to the above discussion, we give an alternate proof of (1) to the proof given in [1].

Lemma 3.1. *For any $n, m \ge 1$ and $k \ge 0$ we have*

$$\begin{aligned} f_i^n f_{i-1}^k f_i^m \in \operatorname{span}\{& f_{i-1}^k f_i^{n+m},\\ & f_{i-1}^{k-1} f_i^{n+m} f_{i-1},\dots, f_i^{n+m} f_{i-1}^k\}, \end{aligned} \tag{2}$$

$$\begin{aligned} f_{i-1}^{n+m} f_i^n \in \operatorname{span}\{& f_{i-1}^n f_i^n f_{i-1}^m,\\ & f_{i-1}^{n-1} f_i^n f_{i-1}^{m+1},\dots, f_i^n f_{i-1}^{m+n}\}, \end{aligned} \tag{3}$$

and (2), (3) in which f_i and f_{i-1} are interchanged.

Proof. To show (2) we first prove the following identity: for any $m \ge 1$ and $i = 2, 3, \dots, n$ we have

$$f_i f_{i-1} f_i^m = \frac{1}{m+1}\left(f_i^{m+1} f_{i-1} + m f_{i-1} f_i^{m+1}\right). \tag{4}$$

For $m = 1$, (4) follows from the fact that $\big[f_i,[f_i,f_{i-1}]\big] = 0$. Now let us assume validity for m and calculate the equation

$$\begin{aligned} f_i f_{i-1} f_i^{m+1} &= \frac{1}{2}(f_i^2 f_{i-1} + f_{i-1} f_i^2) f_i^m\\ &= \frac{1}{2(m+1)} f_i (f_i^{m+1} f_{i-1} + m f_{i-1} f_i^{m+1})\\ &\quad + \frac{1}{2} f_{i-1} f_i^{m+2} = \frac{1}{2(m+1)} f_i^{m+2} f_{i-1}\\ &\quad + \frac{m}{2(m+1)} f_i f_{i-1} f_i^{m+1} + \frac{1}{2} f_{i-1} f_i^{m+2}, \end{aligned}$$

from which, isolating the term $f_i f_{i-1} f_i^{m+1}$ we get the desired result.

We now generalize formula (4) to the form

$$\begin{aligned} f_i f_{i-1}^k f_i^m = \frac{1}{m+1}\big(& k f_{i-1}^{k-1} f_i^{m+1} f_{i-1}\\ & + (m-k+1) f_{i-1}^k f_i^{m+1}\big). \end{aligned} \tag{5}$$

This formula is proved similarly by induction on k. Multiplying both sides of (5) by f_{i-1}, we obtain

$$\begin{aligned} f_{i-1} f_i f_{i-1}^k f_i^m &= \frac{1}{m+1}\big(k f_{i-1}^k f_i^{m+1} f_{i-1}\\ &\quad + (m-k+1) f_{i-1}^{k+1} f_i^{m+1}\big),\\ f_{i-1} f_i f_{i-1}^k f_i^m &= \frac{1}{2}\left(f_{i-1}^2 f_i f_i f_{i-1}^2\right) f_{i-1}^{k-1} f_i^m\\ &= \frac{1}{2} f_{i-1}^2 \frac{1}{m+1}\big((k-1) f_{i-1}^{k-2} f_i^{m+1} f_{i-1}\\ &\quad + (m-k+2) f_{i-1}^{k-1} f_i^{m+1}\big) + \frac{1}{2} f_i f_{i-1}^{k+1} f_i^m. \end{aligned}$$

Extracting terms $f_i f_{i-1}^{k+1} f_i^m$ we obtain (5) for $k+1$.

The last step is to prove the following identity: for any $n, k, m \geq 0$ we have

$$f_i^n f_{i-1}^k f_i^m = \frac{1}{\binom{m+n}{n}} \sum_{l=0}^{n} \binom{k}{l} \binom{m-k+n}{n-l} \times f_{i-1}^{k-l} f_i^{m+n} f_{i-1}^l. \quad (6)$$

This can be shown by induction on n.

To show (3) we apply Dixmier's antiisomorphism (see [3], 2.2.18., p. 73) to (6) to obtain the relation

$$f_i^m f_{i-1}^k f_i = \frac{1}{m+1} \big(k f_{i-1} f_i^{m+1} f_{i-1}^{k-1} + (m-k+1) f_i^{m+1} f_{i-1}^k\big),$$

which allows inductively to prove

$$f_{i-1}^{n+1} f_i^n = \sum_{l=1}^{n+1} (-1)^{l+1} \binom{n+1}{l} f_{i-1}^{n+1-l} f_i^n f_{i-1}^l. \quad (7)$$

Multiplying (7) by f_{i-1} and repeatedly applying (7) to the right-hand side we finally obtain (3).

Replacing f_i and f_{i-1} and using a similar approach, we subsequently obtain the following formulas:

$$f_{i-1} f_i f_{i-1}^m = \frac{1}{m+1} \left(f_{i-1}^{m+1} f_i + m f_i f_{i-1}^{m+1}\right),$$
$$f_{i-1} f_i^k f_{i-1}^m = \frac{1}{m+1} \Big(k f_i^{k-1} f_{i-1}^{m+1} f_i + (m-k+1) f_i^k f_{i-1}^{m+1}\Big),$$
$$f_{i-1}^n f_i^k f_{i-1}^m = \frac{1}{\binom{m+n}{n}} \sum_{l=0}^{n} \binom{k}{l} \binom{m-k+n}{n-l} \times f_i^{k-l} f_{i-1}^{m+n} f_i^l,$$
$$f_i^{n+1} f_{i-1}^n = \sum_{l=1}^{n+1} (-1)^{l+1} \binom{n+1}{l} f_i^{n+1-l} f_{i-1}^n f_i^l. \quad \blacksquare$$

Due to the Poincaré-Birkhoff-Witt theorem, ordered monomials

$$e_{12}^{s_{12}} e_{13}^{s_{13}} \cdots e_{1,n+1}^{s_{1,n+1}} e_{23}^{s_{23}} e_{24}^{s_{24}} \cdots e_{2,n+1}^{s_{2,n+1}} \cdots e_{n-1,n+1}^{s_{n-1,n+1}} e_{n,n+1}^{s_{n,n+1}}, \quad s_{ij} \geq 0, \quad (8)$$

where $e_{i,i+1} = f_i$ and

$$e_{ik} = [e_{i,k-1}, e_{k-1,k}], \quad i+1 < k, \quad (9)$$

form a basis of $U(A_n^-)$. Let us consider any such monomial and denote it by v. The relations (9) express any generator e_{ik}, $i < k$ as a commutator of simple ones $f_1, \ldots, f_n$. Therefore v is a linear combination of (unordered) monomials from these simple generators and such monomials can be written in the form

$$v' = v_1 f_n^{r_1} v_2 f_n^{r_2} \cdots v_m f_n^{r_m} v_{m+1}, \quad (10)$$

where $v_i \in U(A_{n-1}^-) \subset U(A_n^-)$ and the monomials $v_2, v_3, \ldots, v_m \notin U(A_{n-2}^-)$ (i. e. they contain generator f_{n-1}, otherwise we use the relation $f_n^r v_j f_n^s = f_n^{r+s} v_j$ and the product can be truncated).

Theorem 3.2. *Let us denote*

$$V_{R,m} = \mathrm{span}\big\{v_1 f_n^{r_1} v_2 f_n^{r_2} \cdots v_m f_n^{r_m} v_{m+1} \big| v_i \in U(A_{n-1}^-),\ r_1 + r_2 + \cdots + r_m = R\big\}. \quad (11)$$

Then we have $v' \in V_{R,1}$.

Proof. By induction on n. If $n = 2$, we have

$$v' = f_1^{s_1} f_2^{r_1} f_1^{s_2} f_2^{r_2} \cdots f_1^{s_m} f_2^{r_m} f_1^{s_{m+1}}. \quad (12)$$

If $m \geq 2$, we use formula (2) from lemma 3.1 applied to the product $f_2^{r_1} f_1^{s_2} f_2^{r_2}$ and we obtain $v' \in V_{R,m-1}$. In the general case we write

$$v_i = w_i f_n^{s_i} w_i', \quad w, w' \in U(A_{n-1}^-),$$

and, therefore,

$$[w, f_{n+1}] = [w', f_{n+1}] = 0.$$

For monomial v' we can write

$$v' = v_1 f_{n+1}^{r_1} w_2 f_n^{s_2} w_2' f_{n+1}^{r_2} \cdots = v_1 w_2 f_{n+1}^{r_1} f_n^{s_2} f_{n+1}^{r_2} w_2' \cdots$$

and we use the same argument as in the case $n = 2$. $\blacksquare$

It follows from the above theorem that the set $U(A_n^-)$ is spanned by monomials of a special type. When $n = 2$ these monomials are of the form

$$\{f_1^{s_1} f_2^{r_1} f_1^{s_2} | s_1, s_2, r_1 \geq 0\}.$$

Due to formula (3) from lemma 3.1 the monomials $f_1^{s_1} f_2^{r_1} f_1^{s_2}$, where $s_1 > r_1$ are linearly dependent on those having $s_1 \leq r_1$, therefore we can restrict to the set

$$\{f_1^{s_1} f_2^{r_1} f_1^{s_2} | s_1, s_2, r_1 \geq 0,\ s_1 \leq r_1\}. \quad (13)$$

We can generalize this assertion for $n > 2$ as follows.

Theorem 3.3.

$$U(A_n^-) = \mathrm{span}\big\{f_1^{k_{1n}} f_2^{k_{2n}} \cdots f_n^{k_{nn}} f_1^{k_{1,n-1}} f_2^{k_{2,n-1}} \cdots f_{n-1}^{k_{n-1,n-1}} \cdots f_1^{k_{12}} f_2^{k_{22}} f_1^{k_{11}} \big| k_{ij} \leq k_{i+1,j},\ i = 1, \ldots, n-1,\ j = 1, \ldots, n\big\}. \quad (14)$$

(We call Verma monomials those monomials appearing on the right hand side of this equality.)

Proof. We proceed by induction on n. For $n = 2$ the assertion is true, now assume validity for n and we prove for $n+1$. It follows from the preceeding lemma that it is sufficient to consider $v \in U(A_{n+1}^-)$ such that it can be written as $v_1 f_{n+1}^R v_2$, where $v_1, v_2 \in U(A_n^-)$ and v_1 is a Verma monomial of the form

$$v_1 = f_1^{k_{1n}} f_2^{k_{2n}} \cdots f_n^{k_{nn}} f_1^{k_{1,n-1}} f_2^{k_{2,n-1}} \cdots \times f_{n-1}^{k_{n-1,n-1}} \cdots f_1^{k_{12}} f_2^{k_{22}} f_1^{k_{11}}.$$

Therefore

$$v_1 f_{n+1}^R v_2 = f_1^{k_{1n}} f_2^{k_{2n}} \cdots f_n^{k_{nn}} f_{n+1}^R \times \underbrace{f_1^{k_{1,n-1}} f_2^{k_{2,n-1}} \cdots f_{n-1}^{k_{n-1,n-1}} \cdots f_1^{k_{12}} f_2^{k_{22}} f_1^{k_{11}} v_2}_{v'}$$

and, due to the induction hypothesis, v' is a linear combination of Verma monomials. Now if $R \geq k_{nn}$, then the product

$$f_1^{k_{1n}} f_2^{k_{2n}} \cdots f_n^{k_{nn}} f_{n+1}^R$$

is already a Verma monomial and the proof is finished. When $R < k_{nn}$, we rewrite the product $f_n^{k_{nn}} f_{n+1}^R$ using (2) from lemma 3.1 to the form

$$f_n^{k_{nn}} f_{n+1}^R = a_0 f_n^R f_{n+1}^R f_n^{k_{nn}-R} + a_1 f_n^{R-1} f_{n+1}^R f_n^{k_{nn}-R+1} + \cdots + a_R f_{n+1}^R f_n^{k_{nn}},$$

a_i being suitable complex constants. From this we conclude that

$$f_1^{k_{1n}} f_2^{k_{2n}} \cdots f_n^{k_{nn}} f_{n+1}^R \in \operatorname{span}\{u_1 f_{n+1}^R w_1, u_2 f_{n+1}^R w_2, \ldots, u_n f_{n+1}^R w_n\}, \quad (15)$$

where u_i, w_i are Verma monomials from $U(A_n^-)$ and the highest degree of the simple root f_n in v_i is less or equal to R. Therefore the product (15) is a linear combination of Verma monomials from $U(A_{n+1}^-)$, as desired. ∎

Linear independence of Verma monomials can be shown as follows. We make use of commuting operators $\operatorname{ad} h_i : U(A_n^-) \to U(A_n^-)$ defined by

$$\operatorname{ad} h_i v = [h_i, v], \quad i = 1, \ldots, n. \quad (16)$$

The algebra $U(A_n^-)$ decomposes into the direct sum

$$U(A_n^-) = \bigoplus_{z_1,\ldots,z_n} V(z_1, \ldots, z_n)$$

of common eigenspaces of operators $\operatorname{ad} h_i$

$$V(z_1, \ldots, z_n) = \{v \in U(A_n^-) | \operatorname{ad} h_i v = z_i v,\ i = 1, \ldots, n\}.$$

Two vectors belonging to different subspaces are linearly independent.

Lemma 3.4. (1.) *PBW monomials (8) of $U(A_n^-)$ are eigenvectors of all $\operatorname{ad} h_i$.*

(2.) *$v \in V(z_1, \ldots, z_n)$ iff*

$$\begin{aligned} s_{12} + s_{13} + \cdots + s_{1,n+1} &= m_1, \\ s_{23} + \cdots + s_{2,n+1} &= m_2 + s_{12}, \\ &\vdots \\ s_{n,n+1} &= m_n + s_{1n} + s_{2n} \\ &\quad + \cdots + s_{n-1,n}, \end{aligned} \quad (17)$$

where

$$m = -\frac{1}{n+1} C_1 z, \quad (18)$$

and

$$m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_{n-1} \\ m_n \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -n & -(n-1) & -(n-2) & \cdots & -2 & -1 \\ 1 & -(n-1) & -(n-2) & \cdots & -2 & -1 \\ 1 & 2 & -(n-2) & \cdots & -2 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & -2 & -1 \\ 1 & 2 & 3 & \cdots & (n-1) & -1 \end{pmatrix},$$

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix}.$$

Proof. (1.) The assertion is direct consequence of the fact that generators e_{jk}, $j < k$ are eigenvectors of $\operatorname{ad} h_i$.

(2.) System (17) was obtained using generators e_{ii},

$$e_{ii} = \frac{1}{n+1}\Big(c_1 - \sum_{j=1}^{n+1-i} j h_{n+1-j} + \sum_{j=n+2-i}^{n} (n+1-j) h_{n+1-j}\Big),$$

$$\operatorname{ad} e_{ii} v = \frac{1}{n+1}\Big(-\sum_{j=1}^{n+1-j} j z_{n+1-j} + \sum_{j=n+2-i}^{n} (n+1-j) z_{n+1-j}\Big) v \equiv m_i v.$$

$c_1 = e_{11} + \cdots + e_{nn}$ stands for the Casimir operator. ∎

Note that the matrix which appears in (18) is the inverse of the Cartan matrix of the algebra A_n. From equations (17) we see that m_i are all nonnegative integers. The dimension of $V(z_1,\dots,z_n)$ is finite since for fixed right hand sides of equations (18) there is only a finite number of decompositions of m_1 into the sum of s_{12}, s_{13}, …, $s_{1,n+1}$, etc. For each of the possibilities

$$(s_{12},\dots,s_{1,n+1},s_{21},\dots,s_{2,n+1},\dots,\dots,s_{n,n+1}) \quad (19)$$

we obtain a basis vector $v \in V(z_1,\dots,z_n)$. By exhausting all these possibilities we obtain the basis of $V(z_1,\dots,z_n)$.

Lemma 3.5. *Verma monomial*

$$\begin{aligned} v = \big(f_1^{l_{1n}} \cdots f_{n-1}^{l_{n-1,n}} f_n^{k_n}\big)\big(f_1^{l_{1,n-1}} \cdots f_{n-2}^{l_{n-2,n-1}} f_{n-1}^{k_{n-1}}\big) \\ \cdots \big(f_1^{l_{12}} f_2^{k_2}\big)\big(f_1^{k_1}\big) \in V(z_1,\dots,z_n) \quad (20) \end{aligned}$$

iff

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & -2 \end{pmatrix} \times \begin{pmatrix} l_{1n} + l_{1,n-1} + \cdots + l_{12} + k_1 \\ l_{2n} + l_{2,n-1} + \cdots + l_{2,3} + k_2 \\ \vdots \\ l_{n-1,n} + k_{n-1} \\ k_n \end{pmatrix}. \quad (21)$$

Proof. By direct calculation using relations

$$\begin{aligned} [f_i, h_j] &= c_{ij} f_i, \\ c_{ij} &= 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}, \\ c_{ij} &= 0 \text{ for } |i-j| > 1 \end{aligned}$$

and, consequently

$$[f_i^\alpha, h_j] = \alpha c_{ij} f_i^\alpha. \qquad \blacksquare$$

Note that inverting the Cartan matrix and matrix from eq. (21) we can rewrite system (21) to the form

$$\begin{aligned} l_{1n} + l_{1,n-1} + \cdots + l_{12} + k_1 &= m_1, \\ l_{2n} + l_{2,n-1} + \cdots + k_2 &= m_1 + m_2, \\ &\vdots \\ l_{n-1,n} + k_{n-1} &= m_1 + m_2 + \cdots + m_{n-1}, \\ k_n &= m_1 + m_2 + \cdots + m_n. \end{aligned} \quad (22)$$

Theorem 3.6. *If the numbers $m_1,\dots,m_n$ are fixed, then there is a bijective mapping between the set of all solutions (19) of (17) and the set of all solutions*

$$\Big(l_{1n}, l_{1,n-1},\dots,l_{12},k_1,l_{2n},l_{2,n-1},\dots,l_{23},k_2, \dots,\dots,l_{n-1,n},k_{n-1}\Big)$$

of the system (22).

Proof. The explicit form of this bijection is

$$\begin{aligned} s_{12} &= k_1 \\ s_{1t} &= l_{1,t-1}, \quad t = 3,\dots,n+1, \\ s_{r,r+1} &= k_r - l_{r-1,r}, \\ s_{rt} &= l_{r,t-1} - l_{r-1,t-1}, \quad t = r+2,\dots,n+1, \\ s_{n,n+1} &= m_n + k_{n-1}. \end{aligned}$$

Bijectivity is the consequence of the fact that Verma monomials form the spanning set of $U(A_n^-)$. ■

Corollary 3.7. *All Verma monomials are linearly independent.*

4. Conclusions

Problems of unified construction of Verma bases for other series of simple Lie algebras (namely orthogonal and symplectic) and of an effective determination of matrix elements in these cases are still open. In [50, 51], Kang and Lee developed the notion of Gröbner-Shirshov pairs. In this way, the reduction problem in representation theory was solved and monomial bases of representations of various associative algebras could be constructed. The algebra A_n was among the first examples. Note that the bases obtained there are different from Verma bases. It is an interesting question whether Verma bases can be derived this way.

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