EJTIR

Issue 13(2), 2013
pp. 147-168
ISSN: 1567-7141
www.ejtir.tbm.tudelft.nl

Reference-Dependent Stochastic User Equilibrium with Endogenous Reference Points

Paolo Delle Site[1], Francesco Filippi and Claudia Castaldi
Department of Civil Architectural and Environmental Engineering, University of Rome "La Sapienza", and
Research Centre for Transport and Logistics, University of Rome "La Sapienza"

We consider the application of reference-dependent consumer choice theory to traffic assignment on transportation networks. Route choice is modelled based on random utility maximisation with systematic utility embodying loss aversion for the travel time and money expenditure attributes. Stochastic user equilibrium models found in the literature have considered exogenously given reference points. The paper proposes a model where reference points are determined consistently with the equilibrium flows and travel times. The reference-dependent stochastic user equilibrium (RDSUE) is defined as the condition where (i) no user can improve her utility by unilaterally changing path, (ii) each user has as reference point the current travel time and the money expenditure of one of the available paths, and (iii) if each user updates the reference point to her current path the observed path flows do not change. These conditions are formally equivalent to a multi-class stochastic equilibrium where each class is associated with a path and has as reference point the current state on the path, and the number of users in each class equals the current flow on the path. The RDSUE is formulated as a fixed point problem in the path flows. Existence of RDSUE is guaranteed under usual assumptions. A heuristic algorithm based on the method of successive averages is proposed to solve the problem. The model is illustrated by two numerical examples, one relates to a two-link network and another to the Nguyen-Dupuit network. A reference-dependent route choice model calibrated on stated preference data is used. The second example serves also to demonstrate the algorithm. The impact on the equilibrium of different assumptions on the degree of loss aversion with respect to the travel time attribute are investigated.

Keywords: Reference-dependent theory, Loss aversion, Stochastic user equilibrium, Endogenous reference point.

---

1. Introduction

Reference-dependent theory (Tversky and Kahneman, 1991) proposes a fundamental change of paradigm in choice theory because it assumes that carriers of utility are not states but gains and losses relative to a reference point. Loss aversion is central to the theory: losses are valued more

[1] Via Eudossiana 18, Rome 00184, Italy, T: +390644585136, F: +390644585774, E: paolo.dellesite@uniroma1.it

heavily than gains. The theory considers riskless choices with the utilities of alternatives being characterized by attributes whose outcomes are certain. The theory allows for both constant and diminishing sensitivity of the utilities with respect to changes in the attributes.

The experimental validation of reference-dependent theory in the case of the choice of the route on a road network has been the subject of research. Different authors (De Borger and Fosgerau, 2008; Hess et al., 2008; Delle Site and Filippi, 2011) have formulated reference-dependent route choice models following the framework proposed by Tversky and Kahneman (1991) and estimated random utility versions of these models based on multinomial logit assumptions for the stochastic terms. Evidence of asymmetrical response with respect to gains and losses in both travel time and money attributes of the route alternatives is found. A common finding in Hess et al. (2008) and in Delle Site and Filippi (2011) is that users are more loss averse in the money attribute than in the time attribute.

A natural subsequent step consists in the use of reference-dependent route choice models in network analysis. A variant of the classical stochastic user equilibrium (SUE) that assumes that the random utilities of route alternatives are reference-dependent has been proposed by Delle Site and Filippi (2011). They consider a reference-dependent route choice model with constant sensitivity of the utilities. They develop the equilibrium model on the basis of the assumption that the reference points are the status quo. They assume that there is a multiplicity of reference points for users of the same origin-destination pair because in the status quo users choose different routes, each having a distinct travel time and money expenditure. This gives rise to an equilibrium problem with multiple user classes, with each class having a distinct reference point determined by the travel time and money expenditure in the status quo.

The paper by Delle Site and Filippi (2011) deals also with the reference-dependent valuation of time changes over the network. It is shown that it is possible to attribute to each class of users a distinct valuation of time savings and time losses. Both time savings and losses can be values according to a compensating or an equivalent measure.

The status quo assumption is commonly adopted in reference-dependent models. There is limited research into which reference points should be employed in transportation applications, with existing literature relying pre-dominantly on using current trip conditions as reference (De Borger and Fosgerau, 2008; Hess et al., 2008; Delle Site and Filippi, 2011). There are, however, other possibilities. Chin and Knetsch (2006) and Knetsch (2007) consider the issue of which state the individuals regard as the norm for judging their satisfaction, arguing that their legitimate expectations, or what they feel is deserving or right, might be such norm. For road users they suggest the free-flowing traffic conditions as norm. Recently, Stathopoulos and Hess (2012) have considered referencing occurring against different plausible anchor points including current, ideal and acceptable travel conditions. They estimate discrete choice models and find that the best data fit is achieved with a model where the respondent-reported ideal value is assumed as reference point. Their investigation relates, however, to the intra-modal choices for bus and rail users, not to route choice. Also, they find evidence of asymmetric preferences only for the monetary cost attribute, not for travel time.

The status quo assumption implies that the reference points are the current travel conditions. In an equilibrium setting, the status quo assumption raises the issue whether the equilibrium is maintained when the reference points are updated to the new status quo. This property of a reference-dependent equilibrium is referred to as reflexivity in the economics literature on trade (Munro and Sugden, 2003; Munro, 2009). In a transportation network, Delle Site and Filippi (2011) have shown that the equilibrium is maintained when the reference points are updated to the new status quo if the additional assumption is made that the stochastic terms of the route choice model do not change with the updating. Thus the equilibrium model in Delle Site and Filippi (2011) treats the reference points as exogenous.

However, other assumptions on the dynamics of the stochastic terms can be considered to take into account changes in unobserved attributes of the routes or intra-personal preference variation. If the stochastic terms are redrawn, the relative convenience of the alternatives may change when the reference point is updated. A new equilibrium would need to be computed on the basis of the updated systematic utilities. The reference points would be further adjusted and a new equilibrium computed.

Therefore, the interest of the observer is in modelling the conditions where there is consistency between flows and travel times (this consistency identifies the conventional equilibrium over a congested network) and, at the same time, consistency between current choices and reference points. These extended consistency conditions represent the reference-dependent equilibrium with endogenous reference points which is the subject of the present paper. The approach to equilibrium is similar to the one found in Xu et al. (2011) who have considered a user equilibrium with endogenous reference points but in a prospect-theory based[2] and deterministic choice setting.

The contributions of the paper are the following:

- to define the equilibrium conditions over a network in a setting of random utility reference-dependent route choice under the status quo assumption with endogenous reference points;
- to provide the mathematical formulation of these reference-dependent equilibrium conditions;
- to provide a solution algorithm that computes flows, travel times and reference points at equilibrium;
- to illustrate the equilibrium model with numerical examples applying to networks of different size.

The paper is organised as follows. Section 2 provides the definition of the equilibrium conditions and presents the mathematical model and the solution algorithm. Section 3 presents some numerical results relating to two illustrative networks. Section 4 provides a few concluding remarks.

2. Network equilibrium

2.1 Network representation and assumptions

Let $G=(N,A)$ be a strongly connected road transportation network, with N and A being the sets of nodes and links, respectively. Let a be the link index. Origins (O) and destinations (D) constitute a subset of N. Let R be the set of OD pairs and r the OD pair index. Let K^r be the set of simple paths of OD pair r, and k the path index.

For each path $k \in K^r$, $F^{k,r}$ denotes the corresponding path flow. We denote by z_a the flow on link $a \in A$. The link flows are obtained from the path flows by:

$$z_a = \sum_{r \in R} \sum_{k \in K^r} \delta_a^{k,r} \cdot F^{k,r} \qquad a \in A \tag{1}$$

where $\delta_a^{k,r}$ is the element of the link-path incidence matrix whose value is 1 if path k includes link a, is 0 otherwise.

[2] Prospect theory (Kahneman and Tversky, 1979) differs from reference-dependent theory because it assumes, in addition to loss aversion, that outcomes of the attributes are uncertain thus calling for the need to model risk.

The demand flow of the OD pair *r* is denoted by q^r. We have the demand constraints:

$$q^r = \sum_{k \in K^r} F^{k,r} \qquad r \in R \tag{2}$$

The feasible path flows are all the non-negative $F^{k,r}$ satisfying the demand constraints (2). Therefore, the set of feasible path flows is non empty, compact and convex.

Let $T^{k,r}$ denote the travel time on path *k* of OD pair *r*. Let t_a denote the travel time on link *a*. The link travel times are continuous functions of the link flows: $t_a = t_a(z_a,\ a \in A)$. The path travel times are obtained from the link travel times by the standard link-additive model:

$$T^{k,r} = \sum_{a \in A} \delta_a^{k,r} \cdot t_a(z_a,\ a \in A) \quad k \in K^r,\ r \in R \tag{3}$$

2.2 Reference-dependent route choice

The users of an OD pair perceive a utility on each path. This path utility is a random variable given by the sum of a systematic, i.e. deterministic, component and a stochastic term. The stochastic terms summarise factors that are unobserved by the modeller. The stochastic terms are interpreted as individual specific thus accounting for both inter-individual and intra-individual variability of tastes. The individual-specific stochastic terms may change across repeated choices.

A reference-dependent model is adopted for the path systematic utility according to the following hypotheses. The path systematic utility

(i) depends on two attributes: expenditure in travel time *T* and expenditure in money *M*;

(ii) depends on gains *G* and losses *L* in the two attributes defined relative to a reference point, and increases with gains and decreases with losses;

(iii) is linear in gains and losses (constant sensitivity) and steeper for losses than for gains (loss aversion).

The users of each OD pair *r* are grouped into classes, with each class denoted by *j* and identified by a reference point in terms of path travel time and money spent. Let J^r be the set of classes of OD pair *r*.

The path utilities have the additive form:

$$\begin{aligned}
&U_j^{k,r} = V_j^{k,r} + \varepsilon_j^{k,r} \\
&V_j^{k,r} = \beta_{GT} \cdot GT_j^{k,r} + \beta_{LT} \cdot LT_j^{k,r} + \beta_{GM} \cdot GM_j^{k,r} + \beta_{LM} \cdot LM_j^{k,r} \\
&GT_j^{k,r} = \max\left(T_j^r - T^{k,r}, 0\right) \\
&LT_j^{k,r} = \max\left(T^{k,r} - T_j^r, 0\right) \qquad\qquad k \in K^r,\ j \in J^r,\ r \in R \\
&GM_j^{k,r} = \max\left(M_j^r - M^{k,r}, 0\right) \\
&LM_j^{k,r} = \max\left(M^{k,r} - M_j^r, 0\right)
\end{aligned} \tag{4}$$

where:

$U_j^{k,r}$ is the path perceived utility,

$V_j^{k,r}$ is the path systematic utility,

$\varepsilon_j^{k,r}$ is the stochastic term,

β_{GT}, β_{GM} are the gain coefficients,

β_{LT}, β_{LM} are the loss coefficients,

$GT_j^{k,r}, GM_j^{k,r}$ are the gain, respectively, in travel time and in money,

$LT_j^{k,r}, LM_j^{k,r}$ are the loss, respectively, in travel time and in money,

$M^{k,r}$ is the money spent on the path;

T_j^r, M_j^r are the reference point for, respectively, the travel time and the money spent.

Hypothesis (ii) implies that the systematic utility is decreasing in each attribute, i.e. the gain coefficients are positive and the loss coefficients are negative. Hypothesis (iii) implies loss aversion, i.e. $|\beta_{LT}| > |\beta_{GT}|$ and $|\beta_{LM}| > |\beta_{GM}|$: in absolute values, the loss coefficient is larger than the gain coefficient for each attribute.

The systematic utility in eqns (4) has two terms for each attribute: a gain term and a loss term. If there is a gain in the attribute the gain term is positive and the loss term is zero. Conversely, if there is a loss the loss term is positive and the gain term is zero.

The single-attribute part of the systematic utility is piecewise linear in the attribute with a kink in the reference point. Thus the function is everywhere continuous in the attribute but non differentiable in the reference point. If the absolute values of the gain and loss coefficients were equal, the function would be symmetric about the reference point. For different coefficients the function is asymmetric with slope steeper in losses than in gains if the coefficients satisfy loss aversion. This is shown in Figure 1 where the attribute, i.e. the expenditure in travel time or in money, is denoted by *X*.

A constant additive term may be included in the systematic utility in eqns (4), e.g. to represent other time-independent path attributes; however, without loss of generality it is left out because it does not affect the developments below.

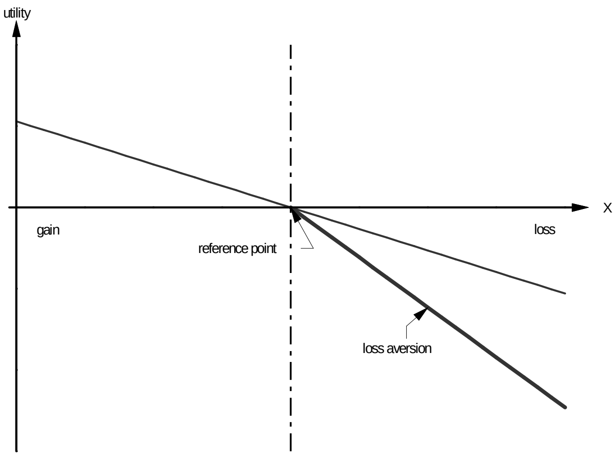

Figure 1. Single-attribute systematic utility

Users of class *j* of OD pair *r* who choose path *k* are those who perceive this path to maximise their utility. The choice probabilities are defined as:

$$P_j^{k,r} = \Pr\left(U_j^{k,r} \geq U_j^{m,r} \quad \forall m \neq k \in K^r\right) \qquad k \in K^r,\ j \in J^r,\ r \in R \tag{5}$$

We assume that the stochastic terms $\varepsilon_j^{k,r}$ have a non-degenerate joint probability density function that is continuous, strictly positive, and independent of the path systematic utility. We assume that the choice probabilities are single-valued and continuous in the path systematic utilities:

$$P_j^{k,r} = P_j^{k,r}\left(V_j^{k,r},\ \ k \in K^r\right) \qquad k \in K^r,\ j \in J^r,\ r \in R \tag{6}$$

The hypotheses are sufficiently general to admit a range of behavioural assumptions through the form of the joint distribution for the stochastic terms, thus encompassing various additive models, including, but not restricting to, multinomial logit. In the case of multinomial logit the probability function takes an asymmetric "S" shape with a kink in the reference point due to the loss aversion assumption (this is illustrated graphically in Suzuki et al., 2001).

Let $f_j^{k,r}$ denote the flow on path *k* of class *j* of OD pair *r*. The choice model is expressed in terms of these class-specific path flows as:

$$f_j^{k,r} = q_j^r \cdot P_j^{k,r} \qquad k \in K^r,\ j \in J^r,\ r \in R \tag{7}$$

where q_j^r denotes the number of users of class *j* of OD pair *r*, with

$$\sum_{j \in J^r} q_j^r = q^r \qquad r \in R \tag{8}$$

2.3 Reference-dependent stochastic user equilibrium with endogenous reference points

In a stochastic setting each class of users is assigned to all available paths. The reference-dependent stochastic user equilibrium (RDSUE) with endogenous reference points is defined as the condition where:

- no user can improve her reference-dependent utility by unilaterally changing path,
- each user has as reference point the current travel time and the money expenditure of one of the available paths,,
- if each user updates the reference point to her current path the observed path flows do not change.

This condition is obtained when:

- each user chooses the path with the maximum utility,
- each user class is associated with a path and the reference point of the class is the travel time of the current state and the money expenditure on that path,
- the number of users in each class equals the current flow on the corresponding path.

In fact, the flow of a path of an OD pair is given by the union of the following two sets of users: the users who have as reference that path and choose it, and the users who choose the path while having as reference other paths. The number of users in the second set equals the number of users who have as reference the path while choosing other paths. Therefore, if each user updates the reference point to the current path, or, in other words, if each user shifts class, the path flows do not change.

It is noteworthy that in the RDSUE here the property that the path flows do not change if each user updates the reference point to her current path holds under any assumption on the dynamics of the stochastic terms. In Delle Site and Filippi (2011) it has been proved that the property holds in a reference-dependent stochastic equilibrium with exogenous reference points if the stochastic terms are unchanged with the updating. The RDSUE here is defined as the equilibrium where the property holds for any change of the stochastic terms.

Mathematically, the equilibrium conditions are defined by the following:

$$J^r \equiv K^r \qquad r \in R \tag{9}$$

$$q_j^r = F^{j,r} \qquad j \in K^r,\ r \in R \tag{10}$$

where:

$$F^{j,r} = \sum_{k \in K^r} f_k^{j,r} \qquad j \in K^r,\ r \in R \tag{11}$$

Eqns (9) imply that the reference points in terms of travel time and money expenditure of each class that appear in the systematic utilities of eqns (4) are given by the following:

$$T_j^r = T^{j,r} \qquad j \in K^r,\ r \in R \tag{12}$$

$$M_j^r = M^{j,r} \qquad j \in K^r,\ r \in R \tag{13}$$

A RDSUE is a solution to the fixed point problem in the path flows $F^{j,r}$:

$$F^{j,r} = \sum_{k \in K^r} f_k^{j,r} \qquad j = 1,\ldots,\left|K^r\right| - 1 \qquad r \in R \tag{14}$$

$$F^{\left|K^r\right|,r} = q^r - \sum_{j=1}^{\left|K^r\right|-1} F^{j,r} \qquad r \in R \tag{15}$$

with

$$\begin{aligned} & f_j^{k,r} = F^{j,r} \cdot P_j^{k,r}\left(F^{j,r},\ j \in K^r,\ r \in R\right) \qquad k \in K^r, j \in K^r,\ r \in R \\ & F^{j,r} \geq 0 \qquad j \in K^r,\ r \in R \end{aligned} \tag{16}$$

where $\left|K^r\right|$ denotes the cardinality of the set K^r.

The dependence of the probabilities on the path flows which appears in eqns (16) is obtained by chaining the expressions (4) of the systematic utilities in the path travel times, the expressions (3) of the path travel times in the link travel times, the link travel times in the link flows, and the expressions (1) of the link flows in the path flows.

A solution, not necessarily unique, to the fixed point problem (14), (15) and (16) determines uniquely the link flows z_a, the link travel times t_a, the path travel times $T^{k,r}$, as well as the class-specific path flows $f_j^{k,r}$.

The structure of the problem is as follows: eqns (14) and (15) are a system of non-linear equations, the inequalities in (16) restrict the set of feasible path flows to the non negative orthant. Eqns (14) are from the equilibrium conditions, eqns (15) from the demand constraints. The equilibrium conditions (14) are written, for each OD pair, only for $\left|K^r\right| - 1$ paths because only $\left|K^r\right| - 1$ conditions are independent, as the condition for one path is a linear combination of the other $\left|K^r\right| - 1$. By adding, for each OD pair, the demand constraint, a well defined system of equations, i.e. a system with as many equations as unknowns, is obtained.

It is convenient to re-write in compact form the fixed point problem using a vector notation (vectors in bold):

$$\mathbf{F} = \boldsymbol{\Psi}(\mathbf{F}) \qquad \mathbf{F} \geq 0 \tag{17}$$

where **F** is the $\left[\left(\sum_{r\in R}\left|K^r\right|\right)\times 1\right]$ vector of path flows, and $\boldsymbol{\Psi}$ the $\left[\left(\sum_{r\in R}\left|K^r\right|\right)\times 1\right]$ vector mapping representing the functions at the right-hand sides of eqns (14) and (15).

2.3.1 Existence and uniqueness properties
In the light of the Brouwer's fixed point theorem, a solution to RDSUE exists since the feasible set is non empty, compact and convex (having taken into account the demand and the non-negativity constraints) and all the functions composed to form the fixed point formulation are continuous. We have not established uniqueness of the RDSUE solution, which, however, cannot be excluded.

2.3.2 Reduction to conventional SUE
The RDSUE collapses to a conventional SUE when the absolute values of the loss and gain coefficients are equal for both the travel time and the money expenditure attributes, i.e. $|\beta_{\mathrm{LT}}| = |\beta_{\mathrm{GT}}|$ and $|\beta_{\mathrm{LM}}| = |\beta_{\mathrm{GM}}|$. In fact, due to the model additivity, when these conditions occur choice probabilities are not affected by reference points:

$$P_j^{k,r} = P^{k,r} \qquad k\in K^r,\ j\in J^r,\ r\in R \tag{18}$$

The RDSUE fixed point problem (14), (15) and (16) reduces then to the conventional SUE fixed point problem:

$$F^{j,r} = q^r \cdot P^{j,r} \qquad j\in K^r,\ r\in R \tag{19}$$

which can be re-written in compact form:

$$\mathbf{F} = \tilde{\mathbf{q}} \circ \mathbf{P} \tag{20}$$

where $\tilde{\mathbf{q}}$ is the $\left[\left(\sum_{r\in R}\left|K^r\right|\right)\times 1\right]$ path-based expanded version of the demand vector $\mathbf{q} = \left[q^1,\ldots,q^R\right]^T$, **P** is the $\left[\left(\sum_{r\in R}\left|K^r\right|\right)\times 1\right]$ vector mapping of probabilities, and "$\circ$" denotes the Hadamard, i.e. componentwise, product ($\mathbf{x}\circ\mathbf{y}$ is the vector whose *i*-component is $x_i \cdot y_i$). The non-negativity constraints become redundant because the probabilities are non-negative. It is possible to prove that in this case, under an additional assumption on the monotonicity of the link time-flow functions, the solution is unique (see, among the others, Cascetta, 2009).

2.3.3 Dealing with the route overlapping problem
It is possible in a RDSUE to take into account the problem of overlapping routes. To this aim, it is sufficient to consider suitable route choice models, other than the simple multinomial logit model which suffers from the limitation of the independence of irrelevant alternatives property. As a result, in a traffic assignment with multinomial logit route choice the overlapping parts of routes that share common links are overloaded.

To tackle the route overlapping problem, different formulations of the route choice model have been proposed. These can be grouped into two classes according to the level at which the similarity of routes is taken into account. A review of the different models in the context of equilibrium problems is found in Chen et al. (2012).

The first class considers the similarity at the level of the systematic part of the route utility and keeps for the stochastic terms the independently and identically Gumbel distributed assumption of the multinomial logit. This class includes the c-logit proposed by Cascetta et al. (1996) and the path-size logit proposed by Ben-Akiva and Bierlaire (1999). These models have the advantage that the route choice probabilities have the simple closed-form expression of the multinomial logit.

The second class considers the similarity at the level of the stochastic part of the route utility and, therefore, departs from multinomial logit assumptions because correlation among alternatives is accounted for. This class includes the probit model, which has been specialized to tackle the overlapping routes problems by Yai et al. (1997), and generalised extreme value (GEV) models. The latter include the cross-nested logit (Vovsha and Bekhor, 1998) and the paired-combinatorial logit (Prashker and Bekhor, 1998). The route choice probabilities need to be computed numerically in the case of probit, have closed-form expressions in the cases of cross-nested logit and paired-combinatorial logit.

All the models reduce the probabilities of choice of the overlapping routes. For application in the case of RDSUE, it needs to be considered that the systematic utility is reference-dependent. The c-logit and path-size logit models can be adapted by adding to the reference-dependent systematic utility of eqns (4) the reference-independent term that accounts for overlapping. The probit, cross-nested logit and paired-combinatorial logit can be adapted by simply taking as systematic utilities the reference-dependent expressions of eqns (4).

2.3.4 Comparison with the model with exogenous reference points

The RDSUE with endogenous reference points here is formulated as a fixed point problem in the path flows with $\sum_{r\in R}\left|K^r\right|$ eqns and $\sum_{r\in R}\left|K^r\right|$ unknowns. In contrast, the reference-dependent equilibrium model with exogenous reference points in Delle Site and Filippi (2011) is formulated as a fixed point problem in the class-specific path flows and has, consequently, a higher number of eqns and unknowns.

Another difference with the model in Delle Site and Filippi (2011) relates to the policy path independence property. The RDSUE with endogenous reference points here is independent of the initial network conditions and, therefore, of the initial reference points. This implies that the RDSUE does not change if the interventions on the network are implemented stepwise or simultaneously (policy path independence). In contrast, the equilibrium with exogenous reference points in Delle Site and Filippi (2011) depends on the initial network conditions and reference points and, therefore, a different equilibrium is obtained depending on how the interventions are phased (policy path dependence).

2.3.5 Solution algorithm

To solve the problem we use a heuristic approach based on the method of successive averages (MSA). The formulation of RDSUE as a fixed point in the path flows suggests a path flow-based MSA. A path-based algorithm is in any case the only viable option since the path systematic utilities of eqns (4) are not additive in the constituent links. The main steps are, in compact notation, as follows.

Step 1. *Initialisation*.
Set link flows equal to zero.
Compute link and path travel times.
Set the reference point of each OD pair equal to the travel time of an arbitrary path (e.g. the path with minimum time in free-flow conditions).
Set iteration counter: t=1.
Compute the probabilities $\mathbf{P}_t$ where the reference point is set as above.
Set initial path flows: $\mathbf{F}_t = \tilde{\mathbf{q}} \circ \mathbf{P}_t$

Step 2. *Convergence check*.
If $\|\mathbf{F}_t - \mathbf{\Psi}(\mathbf{F}_t)\|_\infty < \delta$
then stop and provide outputs,
otherwise go to step 3.

Step 3. *Updating of path flows*.
Set path flows: $\mathbf{F}_{t+1} = \mathbf{F}_t + \frac{1}{t} \cdot [\mathbf{\Psi}(\mathbf{F}_t) - \mathbf{F}_t]$
Increment iteration counter: t=t+1.
Go to step 2.

The symbol $\|\mathbf{x}\|_\infty$ denotes the infinity, or maximum, norm of the vector **x** with components x_i, i.e. $\|\mathbf{x}\|_\infty = \max_i |x_i|$. The convergence tolerance δ can be taken equal to unity, which means a difference in flows less than 1 vehicle per unit of time, because a higher accuracy is practically irrelevant.

The algorithm generates a sequence of feasible path flows, i.e. satisfying both the demand and non-negativity constraints. At each iteration the solution $\mathbf{F}_{t+1}$ is the average of the first t solutions $\mathbf{\Psi}(\mathbf{F}_t)$, hence the name successive averages. As in the MSA for conventional SUE, the convergence can be slow because of the decreasing flow correction which depends on the factor $1/t$. Enumeration of paths is required, therefore in large networks the path set may need to be reduced selectively (reviews of selection criteria are in Bekhor and Toledo, 2005, and in Cascetta, 2009).

When the absolute values of the loss and gain coefficients are equal, the algorithm reduces to the conventional path flow-based MSA for SUE. It is possible to prove that the MSA algorithm converges to SUE if probabilities are logit (Powell and Sheffi, 1982). Conditions of convergence of MSA to SUE for other random utility models are investigated in Cantarella and Velonà (2010).

3. Illustrative examples

3.1 Route choice model

We use a multinomial logit route choice model estimated on the basis of data from a stated preference survey which took place in Rome in 2007. This model is found in Delle Site and Filippi (2011). Table 1 shows the results of the estimation. The hypothesis of loss aversion is supported by data because, in absolute value, the loss coefficient is higher than the gain coefficient for both travel time and money.

Table 1. Estimation results for the route choice model

	Coefficient	t statistic	t statistic for difference in absolute values
time gain (minutes)	0.10545	9.521	- 1.5318
time loss	-0.12270	-9.827	
money gain (EUR)	1.25287	9.481	-3.302
money loss	-1.67346	-14.075	
number of observations:	1068		
final log likelihood	-413.4574		
rho-squared	0.4414		
parameters estimated	4		
rho-squared adjusted	0.4393		

3.2 Two-link network

In the first example, we consider a two-link network (Figure 2) representing a town centre route and a bypass route.

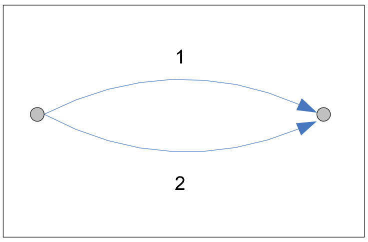

Figure 2. Two-link network

We assume a total demand of 1200 veh/h. For supply, Bureau of Public Roads (BPR) time-flow functions derived empirically for similar routes are used. The functions (in hours) are $t = 0.057 \cdot [1 + (z/800)^{5.2}]$ for the town centre route, and $t = 0.045 \cdot [1 + 0.68 \cdot (z/1230)^{4.6}]$ for the bypass route.

The RDSUE fixed point problem reduces to the non-linear equation in the town centre route flow F^1:

$$F^1 = F^1 \cdot P_1^1\left(F^1, 1200 - F^1\right) + \left(1200 - F^1\right) \cdot P_2^1\left(F^1, 1200 - F^1\right) \tag{21}$$

which in the case of a multinomial logit route choice model specializes to:

$$F^1 = F^1 \cdot \frac{\exp V_1^1(F^1)}{\exp V_1^1(F^1) + \exp V_1^2(1200 - F^1)} + \left(1200 - F^1\right) \cdot \frac{\exp V_2^1(F^1)}{\exp V_2^1(F^1) + \exp V_2^2(1200 - F^1)} \tag{22}$$

Table 2 provides the results in the case where a toll of 1 EUR is charged on the bypass. Table 3 provides the class-specific path flows. Of the 858 veh/h which are found on the town centre route, 641 veh/h have as reference point the current state on the route, while the remaining 217 veh/h are those having as reference point the current state on the bypass route. At the same time, there are 217 veh/h which have as reference point the current state on the town centre route and choose the bypass route. Therefore, if the 217 veh/h having as reference point one route and choosing the other update their reference point to their current route, the total flow on each route does not change.

Table 2. RDSUE results with toll on the bypass

town centre route		bypass route		time expenditure
flow (veh/h)	**time (minutes)**	**flow (veh/h)**	**time (minutes)**	**(h)**
858	8.3	342	2.7	134.7

Table 3. RDSUE results with toll on the bypass – class-specific and total path flows (veh/h)

		chosen path		total
		town centre route	**bypass route**	
Reference path	town centre route	641	217	858
	bypass route	217	125	342
	total	858	342	1200

The equilibrium that is obtained with the RDSUE model is independent of both the initial network conditions and the policy path. This is different from the results that has been obtained by Delle Site and Filippi (2011) for the same two-link network and policy. The equilibrium that they have found according to the model with exogenous reference points is dependent on the sequence of interventions, i.e. the equilibrium obtained with a single-stage policy (bypass opening and charging simultaneously) is different from the equilibrium obtained with a two-stage policy (bypass opened and equilibrium set, charge implemented in a later time).

Another difference lies in the equilibrium flows. In the RDSUE with endogenous reference points here the flow on the bypass (342 veh/h) is higher than the flows obtained in Delle Site and Filippi (2011) with a single-stage policy (321 veh/h) and a two-stage policy (333 veh/h). The reason lies in the reference points. In the model with endogenous reference points only part of the users found on the bypass have a reference point that implies the loss of the 1 EUR charge (these are the users having as reference the town centre route, the others having as reference the bypass do not suffer this loss), while in the model with exogenous reference points all the users found on the bypass have a reference point implying the loss of the 1 EUR charge.

3.2.1 Sensitivity to loss aversion in time attribute

We investigated in the case of absence of tolls the sensitivity of the equilibrium to the assumption on the degree of loss aversion with respect to the travel time attribute. The degree of loss aversion is defined as the absolute value of the ratio between the loss coefficient and the gain coefficient: $\left|\beta_{LT} / \beta_{GT}\right|$.

In the route choice model estimated the degree of loss aversion is 1.16. This value is low when compared with the results obtained by Hess et al. (2008). They found for the two demand segments considered a degree of loss aversion in the free-flow time attribute of 1.49 and 2.44. This comparison suggests that it is meaningful to explore the sensitivity to the degree of loss aversion. We explored the range from 1 to 3 (only β_{LT} is changed while β_{GT} is left unchanged). The case of degree equal to 1 is that where demand exhibits no loss aversion. The RDSUE collapses then to a conventional SUE and the solution is obtained by solving the equation:

$$F^1 = 1200 \cdot P^1\left(F^1, 1200 - F^1\right) \tag{23}$$

which in the case of a multinomial route choice model specializes to:

$$F^1 = 1200 \cdot \frac{\exp V^1(F^1)}{\exp V^1(F^1) + \exp V^2(1200 - F^1)} \tag{24}$$

where in the systematic utilities V^1 and V^2 the lower index is omitted because without loss aversion systematic utilities are reference independent.

Table 4 shows the results of the sensitivity analysis. As the degree of loss aversion increases, the flow on the town centre route, which is the route where travel time is higher, decreases. The variation in the flow values is not large (maximum of 30 veh/h out of a total flow of 1200 veh/h). The difference between the flow on the town centre route and the flow on the bypass route increases as the loss aversion increases. The Table also shows that the total time spent on the network decreases with loss aversion.

Table 4. RDSUE results with no toll: sensitivity to loss aversion in the time attribute

degree of loss aversion	town centre route		bypass route		time expenditure
	flow (veh/h)	time (minutes)	flow (veh/h)	time (minutes)	(h)
1	563	3.97	637	2.79	66.8
1.16	560	3.95	640	2.79	66.7
1.5	555	3.93	645	2.79	66.4
2	547	3.89	653	2.80	65.9
2.5	539	3.86	661	2.80	65.6
3	532	3.83	668	2.81	65.3

3.2.2 Sensitivity to dispersion parameter

In multinomial logit the dispersion parameter divides the systematic utilities and is directly proportional to the variance of the stochastic terms. The numerical results shown so far have implicitly assumed in the route choice model a dispersion parameter equal to unity. The results of a sensitivity analysis of RDSUE to the dispersion parameter in the case of the two-link network where no toll is charged are shown in Table 5. The dispersion parameter varies in the interval between 0.25 and 1.75. As the dispersion parameter increases, the difference between the flow on the town centre route and the flow on the bypass decreases. This is consistent with the intuition that with a higher dispersion parameter the OD flow is less concentrated and tends towards a uniform distribution across routes.

Table 5. RDSUE results with no toll: sensitivity to dispersion parameter

dispersion parameter	town centre route		bypass route		time expenditure
	flow (veh/h)	time (minutes)	flow (veh/h)	time (minutes)	(h)
0.25	486	3.7	713	2.85	63.7
0.5	530	3.8	669	2.81	65.2
0.75	549	3.9	650	2.8	66.1
1	560	3.95	640	2.79	66.7
1.25	567	4	633	2.79	67.1
1.5	572	4	628	2.78	67.4
1.75	575	4	625	2.78	67.7

3.3 Nguyen-Dupuis network

In the second example, the Nguyen-Dupuis network (Nguyen and Dupuis, 1984) is used. The network, which includes 13 nodes, 19 directed links and 4 OD pairs, is shown in Figure 3. The link-path incidence relationship is shown in Table 6.

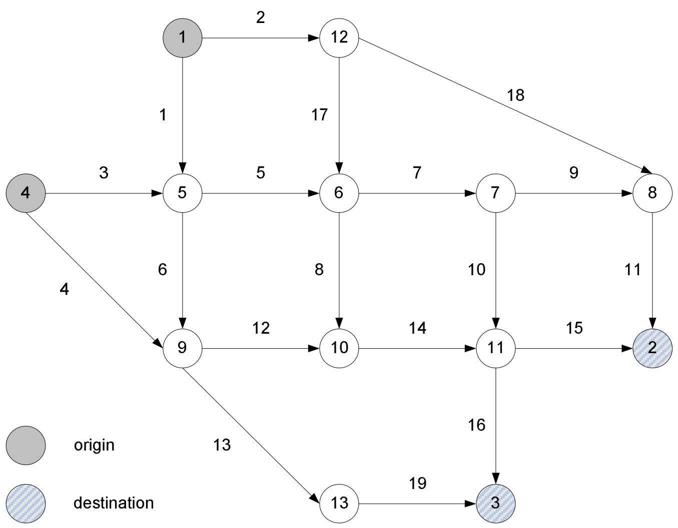

Figure 3. Nguyen-Dupuis network

Table 6. Link-path incidence relationship for the example network

OD pair	path	link sequence	OD pair	path	link sequence
(1,2)	1	2-18-11	(1,3)	9	2-17-8-14-16
	2	2-17-8-14-15		10	2-17-7-10-16
	3	2-17-7-10-15		11	1-6-13-19
	4	2-17-7-9-11		12	1-6-12-14-16
	5	1-6-12-14-15		13	1-5-8-14-16
	6	1-5-8-14-15		14	1-5-7-10-16
	7	1-5-7-10-15			
	8	1-5-7-9-11			
(4,2)	15	4-12-14-15	(4,3)	20	4-13-19
	16	3-6-12-14-15		21	4-12-14-16
	17	3-5-8-14-15		22	3-6-13-19
	18	3-5-7-10-15		23	3-6-12-14-16
	19	3-5-7-9-11		24	3-5-8-14-16
				25	3-5-7-10-16

There is a total of 25 paths. The OD demand flows are $q_{1,2}$=660, $q_{1,3}$=495, $q_{4,2}$=412.5, $q_{4,3}$=495 (as assumed in Xu et al., 2011). The following BPR time-flow functions are used: $t_a = t_a^0 \cdot [1 + 0.15(z_a / c_a)^4]$, where the free-flow travel time t_a^0 and the capacity c_a are given, for each link, in Table 7 (the values in the Table are from Xu et al., 2011). At the route choice level, travel time is the only attribute considered in the utilities. The RDSUE is found using the MSA algorithm of section 2.4.

Table 7. Link characteristics of the example network

link	free-flow travel time	capacity	link	free-flow travel time	capacity
1	7	300	11	9	500
2	9	200	12	10	550
3	9	200	13	9	200
4	12	200	14	6	400
5	3	350	15	9	300
6	9	400	16	8	300
7	5	500	17	7	200
8	13	250	18	14	300
9	5	250	19	11	200
10	9	300			

Table 8 shows the RDSUE path flows together with the class-specific flows for OD pair (1-3). The path flows are found in the bottom line. The cells in the centre of the table provide the class-specific flows, each class being identified by the pair constituted by the reference path and by the chosen path. The last column on the right provides the flows having each a given reference path.
Table 8 serves to illustrate the property of RDSUE: if the users of the OD pair update the reference point to the path chosen, the path flows do not change. In fact, the matrix of the class-specific flows remains unchanged with updating. This is consequence of the property that the sum of the elements on a row equals the sum of the elements on the corresponding column. The sum of the elements on the row equals the flow having as reference a given path, while the sum of the elements on the corresponding column equals the flow using that path. These two flows are equal by definition of RDSUE. As an example, the sum of the elements on row 1 and on column 1, which refer to path 9, are both equal to 29.1. There can be small differences between the row sum and the column sum for other paths due to the approximate convergence of the computations. In all cases, these deviations are less than unity because the algorithm used a convergence tolerance $\delta = 1$.
Figure 4 shows the convergence of the MSA algorithm. Initially, the infinity norm changes non-monotonically as the number of iterations increases, then it decreases monotonically and with a decreasing rate. The number of iterations required for convergence is 1323 and the computation time is around 100 milliseconds with an Intel i5-760 processor (2.80 GHz, 8 GB RAM).

Tests have been carried out to assess the impact on the RDSUE computed by the algorithm of a change in the initial values of the path flows. The results presented above are obtained with initial flows computed in the initialisation step on the basis of a reference path equal to the first path of the OD pair according to the order in the link-path incidence relationship of Table 6. The algorithm was re-run by considering as reference path in the initialisation step the path with minimum time in free-flow conditions and the path with maximum time in free-flow conditions. The algorithm resulted to be robust, as no change in the RDSUE path flows has been found.

Table 8. RDSUE class-specific and total path flows for OD pair (1,3)

		chosen path						total
		9	10	11	12	13	14	
reference path	9	1.9	3.6	7.3	5.5	3.7	6.9	29.1
	10	3.6	7.5	15.2	11.5	7.6	14.2	59.8
	11	7.3	15.2	34.4	24.9	15.4	31.8	129.2
	12	5.5	11.5	24.8	18.8	11.6	23.2	95.8
	13	3.6	7.6	15.3	11.6	7.7	14.4	60.5
	14	6.8	14.1	31.5	23.1	14.3	29.5	119.5
total		29.1	59.7	128.9	95.7	60.4	120.2	≈ 495

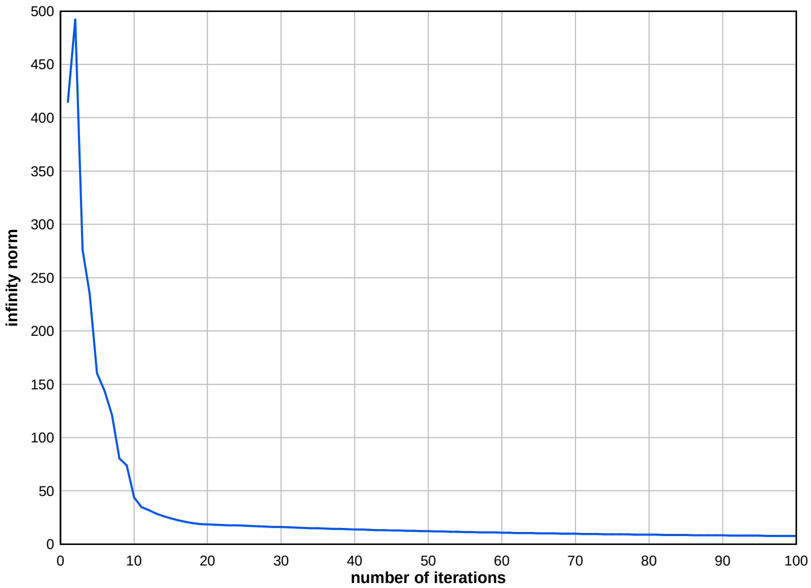

Figure 4. Convergence of the algorithm

3.3.1 Sensitivity to loss aversion in time attribute

The results of the sensitivity analysis with respect to the loss aversion in the time attribute are shown in Table 9. The degree of loss aversion in the time attribute is denoted by $\gamma = |\beta_{LT} / \beta_{GT}|$. The Table shows the RDSUE path and link flows. In the Table, three values for the degree of loss aversion are considered: the case of equal absolute values of the gain and loss coefficient ($\gamma = 1$) which is the conventional SUE, i.e. no loss aversion; the base case with the estimation values $\gamma = 1.16$; and a case with a marked loss aversion ($\gamma = 3$). The latter value is justified by literature as explained in the two-link network example. Again, as in the two-link network example, only β_{LT} is changed while β_{GT} is left unchanged.

It is possible to detect the following pattern when the distribution of path flows for a given OD pair is considered. If a standard deviation of path flows is computed for each OD pair, the standard deviation increases with the degree of loss aversion. This means that, as the degree of loss aversion increases, the OD flow is distributed across paths with higher variability. This pattern is shown in Figure 5, where the standard deviation of path flows is computed after having normalised path flows as percentage values of the corresponding OD flows. The same pattern had been found in the two-link network example (Table 4). Another pattern which can be detected relates to the total travel time spent on the network. This quantity decreases for each OD pair as the degree of loss aversion increases. This is shown in Figure 6. The same pattern had been found in the two-link network example (Table 4).

Table 9. Sensitivity of RDSUE path and link flows to loss aversion in the time attribute

path flows				**link flows**			
path	$\gamma = 1$	$\gamma = 1.16$	$\gamma = 3$	**link**	$\gamma = 1$	$\gamma = 1.16$	$\gamma = 3$
OD pair (1,2)				1	694.0	694.5	697.3
1	244.8	252.9	314.8	2	460.8	460.5	457.7
2	16.1	14.3	5.2	3	473.1	471.8	465.6
3	31.3	29.5	17.6	4	434.6	435.8	442.0
4	76.4	74.6	55.3	5	741.4	740.0	730.5
5	48.8	47.9	41.4	6	425.7	426.3	432.3
6	31.5	29.9	18.9	7	757.7	756.6	742.4
7	61.2	60.7	56.9	8	199.7	190.9	131.0
8	150.3	150.7	150.5	9	369.6	369.5	359.9
OD pair (1,3)				10	388.0	387.0	382.4
9	31.2	29.1	15.3	11	614.5	622.4	674.8
10	60.7	59.8	49.3	12	496.0	497.8	510.7
11	128.7	129.2	134.9	13	364.3	364.3	363.6
12	94.4	95.8	106.0	14	695.7	688.8	641.7
13	61.0	60.5	52.3	15	458.0	449.9	397.9
14	117.7	119.5	136.1	16	625.8	625.9	626.2
OD pair (4,2)				17	215.9	207.5	142.9
15	132.8	133.5	137.4	18	244.8	252.9	314.8
16	46.8	46.3	42.6	19	364.3	364.3	363.6
17	30.3	28.8	18.9				
18	58.8	58.7	58.6				
19	142.8	144.2	154.0				
OD pair (4,3)							
20	174.1	173.2	167.4				
21	127.6	128.9	137.1				
22	61.4	61.7	61.3				
23	45.3	45.2	45.8				
24	29.3	28.1	20.1				
25	58.0	58.5	63.8				

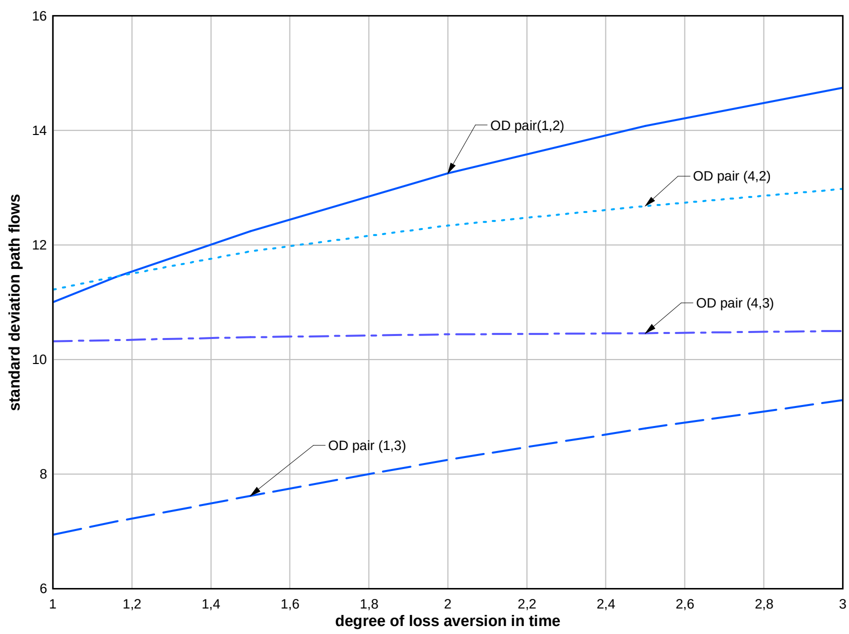

Figure 5. Variation of standard deviation of path flows with degree of loss aversion in the time attribute

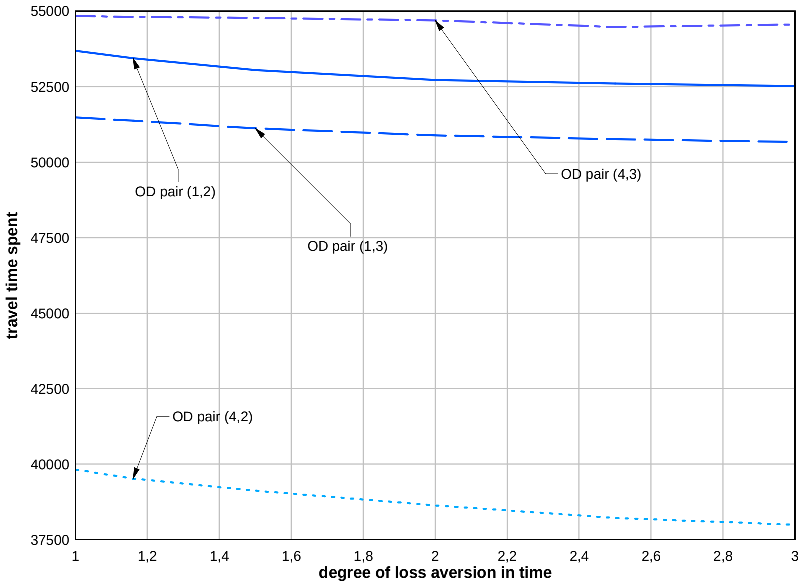

Figure 6. Variation of total travel time spent on the network with degree of loss aversion in the time attribute

3.3.2 Sensitivity to dispersion parameter

The results shown so far have implicitly assumed a dispersion parameter of the multinomial logit route choice model equal to unity. When the dispersion parameter is changed, the OD flow tends to be distributed across paths with lower variability as the dispersion parameter increases. This pattern can be detected in Figure 7 which shows the variation of the standard deviation of normalised path flows with the dispersion parameter by OD pair. The result is consistent with intuition, based on the meaning of the dispersion parameter, and is similar to what has been found in the two-link network example (Table 5). The total travel time spent on the network has an increasing trend as the dispersion parameter increases (Figure 8), similarly to the case of the two-link network (Table 5).

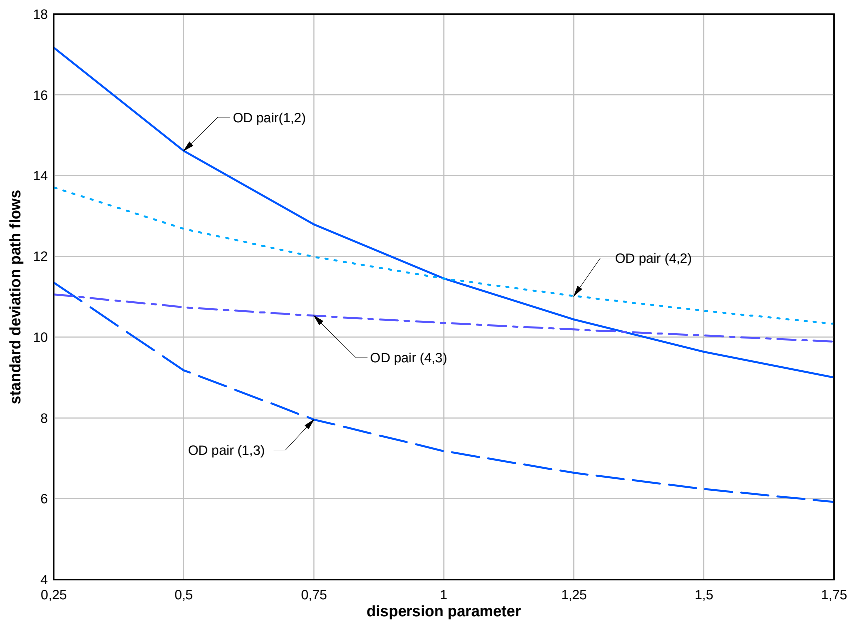

Figure 7. Variation of standard deviation of path flows with dispersion parameter

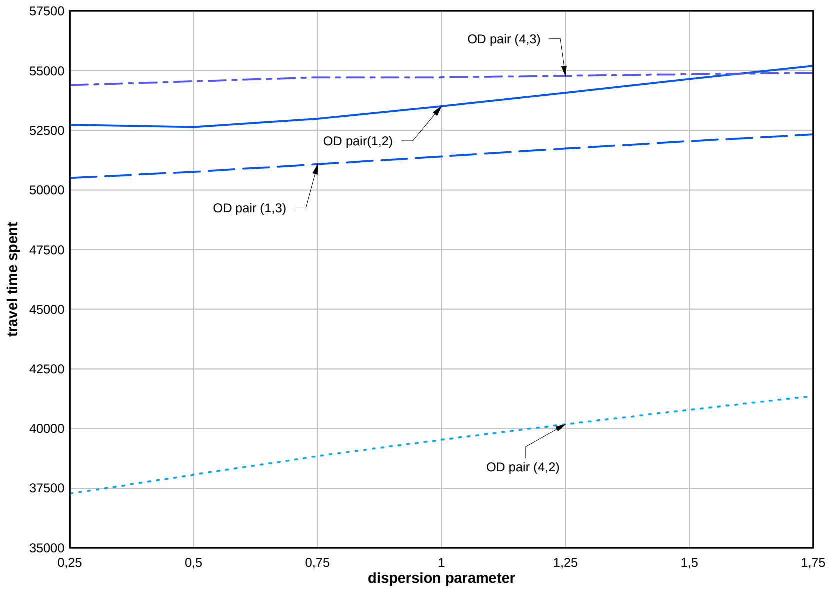

Figure 8. Variation of total travel time spent on the network with dispersion parameter

3.3.3 Sensitivity to size of path choice set

The results shown so far have considered for each OD pair the full set of paths. It is interesting to examine the impact of the size of the path choice set on RDSUE. To this aim, we varied the maximum number of paths allowed for each OD pair. The paths in each OD pair have been ordered according to increasing free-flow travel time. The first paths in the ordered set have been considered for each RDSUE computation. As a measure of the deviation of the equilibrium solution with respect to the equilibrium that is obtained with the full path choice set we considered the root mean square error (RMSE) of the link flows defined by:

$$RMSE = \sqrt{\sum_{a \in A} \frac{(z_a - z_a{*})^2}{|A|}} \tag{25}$$

where $z_a{*}$ denotes the flow on link *a* in the RDSUE with the full path choice set.

Figure 9 shows that the RMSE decreases, at an increasing rate, with the maximum number of paths per OD pair. This is an encouraging result because, in case it were confirmed in networks of larger size, it suggests that the RDSUE may be performed without a need for complete path enumeration. A similar result has been found by Bekhor and Toledo (2005) in the conventional SUE model.

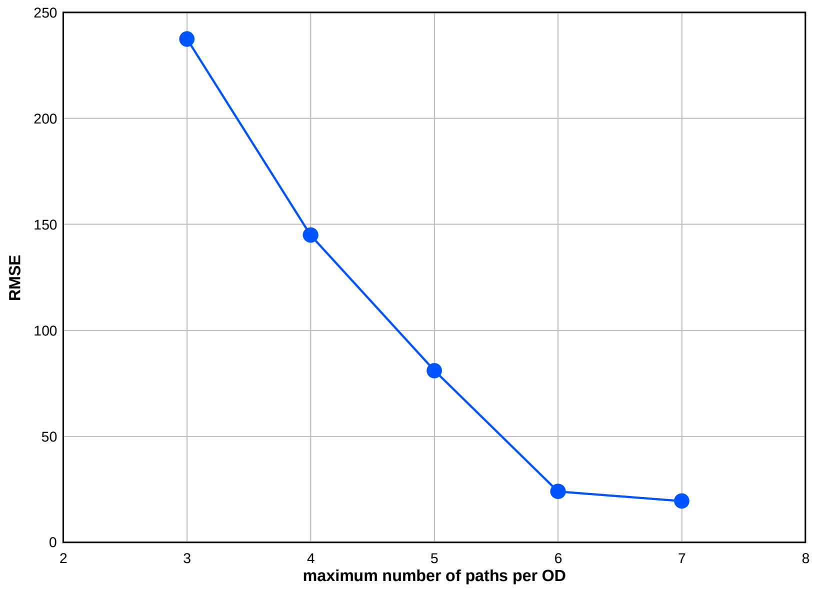

Figure 9. Variation of RMSE of link flows with the maximum number of paths per OD

4. Conclusion

In reference-dependent models of route choice the reference point of a traveler needs to be determined in terms of travel time and money expenditure attributes. When the route choice model is applied to an equilibrium setting, the reference point can be exogenously given or treated as endogenous. The reference point can be regarded as exogenous when it is assumed to be independent of the current travel conditions over the network. This occurs if the ideal travel conditions, typically the free-flow travel time and money expenditure on the shortest route, are considered as reference point. In contrast, the reference point needs to be treated as endogenous when users are assumed to take as reference the status quo, i.e. the current choices. In this case, the reference point of each user is given by the currently chosen path with the related travel time and money expenditure attributes.

The paper has formulated a reference-dependent stochastic user equilibrium model, in short RDSUE, with endogenously determined reference points. Equilibrium conditions are defined where reference points are determined consistently with current flows and travel times. It is considered that users of each OD pair are subdivided into classes with each class having as reference one of the paths of the OD, and attendant travel time and money expenditure. The equilibrium is defined as the condition where the updating of the reference path to the currently used path leaves unchanged the resulting path flows. This is obtained by setting the number of users of a path equal to the number of users having that path as reference.

The equilibrium can be characterized from either a micro-level or a macro-level perspective. The micro level relates to the individual user according to the usual interpretation of the stochastic terms as individual specific. The macro level relates to the path flows observed by the modeller. RDSUE is fundamentally different from the conventional SUE. In a RDSUE, each user updates the reference point, i.e. changes class, and therefore, may change path as well. In the special case where the stochastic terms are unchanged with the updating no change of path would occur (this was proved in Delle Site and Filippi, 2011), changes of path may occur if the stochastic terms are assumed to change with the updating. RDSUE is a condition of potential micro-shifts while at the macro level the flows do not change. In contrast, in a conventional SUE each user is in a condition of “rest” because she is assigned to a path and she sticks to this.

The RDSUE is formulated as a fixed point in the path flows. It has been proved that the solution exists under conditions usually satisfied in practice. The RDSUE model is a generalization to the case of asymmetric preferences of the conventional SUE model, since RDSUE reduces to SUE when the absolute values of the loss and gain coefficients of the travel time and of the money expenditure attributes are equal. In such special case, the algorithm proposed to solve RDSUE reduces to the "classic" path flow-based MSA for SUE. The enhanced behavioral realism which is gained with RDSUE is paid at the price of the loss of the uniqueness conditions, which are left for future research. The formulation of RDSUE in the paper can be extended to reference-dependent route choice models with diminishing sensitivity.

References

Bekhor S., Toledo T. (2005). Investigating path-based solution algorithms to the stochastic user equilibrium problem. *Transportation Research Part B*, 39(3), 279-295.

Ben-Akiva M., Bierlaire M. (1999). Discrete choice methods and their applications to short-term travel decisions. In: R. Hall (ed.) *Handbook of Transportation Science*, Kluwer Academic Publishers, Boston, pp. 5-34.

Cantarella G.E., Velonà P. (2010). *Assegnazioni a Reti di Trasporto. Modelli di Punto Fisso*. In Italian: Transport Network Assignment: Fixed-Point Models. Franco Angeli, Milano.

Cascetta E. (2009). *Transportation Systems Analysis. Models and Applications*. Springer, New York.

Cascetta E., Nuzzolo A., Russo F., Vitetta A. (1996). A modified logit route choice model overcoming path overlapping problems: specification and some calibration results for interurban networks. In: J.B. Lesort (ed.) *Proceedings of the International Symposium on Transportation and Traffic Theory*, Lyon, pp. 697-711.

Chen A., Pravinvongvuth S., Xu X., Ryu S., Chootinan P. (2012). Examining the scaling effect and overlapping problem in logit-based stochastic equilibrium models. *Transportation Research Part A*, 46(8), 1343-1358.

Chin A., Knetsch J.L. (2006). Values depend on the measure: are many transport project valuations seriously biased? Paper presented at the TRAIL Masterclass, 10 April 2006, Delft, The Netherlands.

De Borger B., Fosgerau M. (2008). The trade-off between money and travel time: a test of the theory of reference-dependent preferences. *Journal of Urban Economics*, 64(1), 101-115.

Delle Site P., Filippi F. (2011). Stochastic user equilibrium and value-of-time analysis with reference-dependent route choice. *European Journal of Transport and Infrastructure Research*, 11(2), 194-218.

Hess S., Rose J.M., Hensher D.A. (2008). Asymmetric preference formation in willingness to pay estimates in discrete choice models. *Transportation Research Part E*, 44(5), 847-863.

Kahneman D., Tversky A. (1979). Prospect theory: an analysis of decision under risk. *Econometrica*, 47(2), 263-291.

Knetsch J.L. (2007). Biased valuations, damage assessments, and policy choices: the choice of measure matters. *Ecological Economics*, 63(4), 684-689.

Munro A. (2009). *Bounded Rationality and Public Policy: a Perspective from Behavioural Economics*, Springer.

Munro A., Sugden R. (2003). On the theory of reference-dependent preferences. *Journal of Economic Behavior and Organization*, 50(4), 407-428.

Nguyen S., Dupuis C. (1984). An efficient method for computing traffic equilibria in networks with asymmetric transportation costs, *Transportation Science*, 18(2), 185-202.

Powell W.B., Sheffi Y. (1982). The convergence of equilibrium algorithms with predetermined step sizes. *Transportation Science*, 16(1), 45-55.

Prashker J.N., Bekhor S. (1998). Investigation of stochastic network loading procedures. *Transportation Research Record*, 1645, 94-102.

Stathopoulos A., Hess S. (2012). Revisiting reference point formation, gains-losses asymmetry and non-linear sensitivities with an emphasis on attribute specific treatment. *Transportation Research Part A*, 46(10), 1673-1689.

Suzuki Y., Tyworth J.E., Novack R.A. (2001). Airline market share and customer service quality: a reference-dependent model. *Transportation Research Part A*, 35(9), 773-788.

Tversky A., Kahneman D. (1991). Loss aversion in riskless choice: a reference-dependent model. *The Quarterly Journal of Economics*, 106(4), 1039-1061.

Vovsha P., Bekhor S. (1998). Link-nested logit model of route choice overcoming route overlapping problem. *Transportation Research Record*, 1645, 133-142.

Xu H., Lou Y., Yin Y., Zhou J. (2011). A prospect-based user equilibrium model with endogenous reference points and its application in congestion pricing. *Transportation Research Part B*, 45(2), 311-328.

Yai T., Iwakura S., Morichi S. (1997). Multinomial probit with structured covariance for route choice behaviour. *Transportation Research Part B*, 31(3), 195-207.