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A generalization of the concept of differentiability

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Abstract. In this work we unify and generalize the existing definitions of derivatives of functions by presenting a new concept on differentiability.

1. Introduction

There are classic definitions for derivatives of functions from a Banach space to another one, due to Gateaux, Hadamard and Fréchet. The rate of change of a function restricted to straight lines through a given point is its Gateaux-derivative at that point; although a function may have this derivative at a point, it may not have the local rate of change which would correspond to its Fréchet-derivative. In this work (Theorem 3.1) we show that Hadamard derivative at a point is the rate of change of the function along each and every embedded C^1-curve that passes through that point; thus, the class of functions that has Hadamard-derivative contains that of Fréchet differentiable functions and is contained in Gateaux differentiable class of functions, since limits through straight lines are weaker than limits through embedded C^1- curves which are weaker than limits through neighbourhoods. It is important to note that in all these derivatives the type of sets which is used to calculate the rate of change is the fundamental element for distinguishing one from another. These facts lead us to imagine the use of other sets to calculate these limits, namely topological or differentiable manifolds. We have chosen manifolds that are embedded in affine spaces; for example, if we take C^1- curves embedded in straight lines we obtain Gateaux-differentiability and taking all embedded C^1- curves we obtain Hadamard differentiability. What would happen if we chose C^1-curves embedded in two-dimensional planes, C^1- curves embedded in n-dimensional spaces, m-dimensional manifolds embedded in n-dimensional affine spaces or infinite-dimensional manifolds? Differentiable manifolds

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are locally images of differentiable embeddings in the already known differentiabilities which are defined by the type of sets in which the rate of change is calculated. How can we generalize these ideas? In this work we propose a hierarchical choice of types of sets to define the new differentiabilities. By its turn, each of these sets, recursively, are C^1- differentiable manifolds according to smaller sets in its hierarchy till we arrive at the first differentiability performed through a topological manifold. This hierarchical choice is formally made by the concept of vias of Banach spaces and each via defines a differentiability. This definition of differentiablity will not only include the former ones but also generalize the concept of differentiability. As a matter of fact, Fréchet, Hadamard and Gateaux derivatives will have their own corresponding vias. The class of Fréchet differentiable functions, which is the one contained in the other ones, and that of Gateaux, which contains the other ones, will still remain as the extremes of an inclusion ordered set of "via-differentiability classes" but a wide spectrum of differentiabilities arises in between them. One of these differentiabilities is Hadamard-differentiablity. We wish you a good reading!

2. Some Basics

Let X, Y be real Banach spaces. We denote by $L(X,Y)$ the set of continuous linear functions from X to Y. For $x_0 \in U \subset X$, U open set of X and $f : U \to Y$, we say that

i) f is Gateaux differentiable at x_0 and write G-differentiable at x_0 if and only if there exists the G-derivative $\delta f(x_0, .) : X \to Y$ given by, for $v \in X$,

$$\delta f(x_0, v) = \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(x_0 + tv) - f(x_0)}{t}$$

ii) f is Hadamard differentiable at x_0 (H-differentiable at x_0) if and only if there exists the H-derivative ${}^H f'(x_0) \in L(X,Y)$, given by, for $v \in X$,

$${}^H f'(x_0)v = \lim_{\substack{(t,l) \to (0,0) \\ (t,l) \in \mathbb{R} \times X}} \frac{f(x_0 + tv + tl) - f(x_0)}{t}$$

Hadamard differentiability was originally defined for functions $f : U \subset X \to Y$, where X and Y are topological vector spaces (see [1, 3, 4, 6, 10]). Let us call, for instance, the original definition, in our context, $\hat{H}$-differentiabitity.

Let G be the class of all functions $g : I \to U$, $I \subset \mathbb{R}$ neighbourhood of 0, such that $g(0) = x_0$ and $\exists\ g'(0) = \lim_{t \to 0} \dfrac{g(t) - g(0)}{t} \in X$.

The function f is $\hat{H}$-differentiable at x_0 if and only if $\exists\ {}^{\hat{H}} f'(x_0) \in L(X,Y)$

such that $\forall g \in G,\ \exists (f\circ g)'(0) = \lim_{t\to 0} \frac{(f\circ g)(t)-(f\circ g)(0)}{t} \in Y$

and $(f\circ g)'(0) =^{\hat{H}} f'(x_0)g'(0)$.
We estabilish the equivalence between this Hadamard's original definition and the one given above in item ii) (which was presented in [9]), in Proposition 2.1.
iii) f is Fréchet differentiable at x_0 (F-differentiable at x_0) if and only if there exists the F-derivative $f'(x_0) \in L(X,Y)$ such that

$$\lim_{\substack{h\to 0\\ h\in X}} \frac{f(x_0+h)-f(x_0)-f'(x_0)h}{||h||} = 0 \in Y$$

Let us define the classes of differentiable functions at x_0, relative to these differentiability types by:

$$F_{x_0}(X,Y) = \{f : U \to Y \mid x_0 \in U \subset X,\ U \text{ open set},\ f \text{ is } F\text{-differentiable at } x_0\}$$

$$H_{x_0}(X,Y) = \{f : U \to Y | x_0 \in U \subset X,\ U \text{ open set},\ f \text{ is } H\text{-differentiable at } x_0\}$$

$$G_{x_0}(X,Y) = \{f : U \to Y | x_0 \in U \subset X,\ U \text{ open set},\ f \text{ is } G\text{-differentiable at } x_0 \text{ and } \delta f(x_0,.) \in L(X,Y)\}.$$

Many results concerning these derivatives are shown in [9] from which we select:

Proposition 2.1 (some results from [9]).
a) $\exists\, f'(x_0) \Rightarrow \exists^{H} f'(x_0) \Leftrightarrow \exists^{\hat{H}} f'(x_0) \Rightarrow \exists \delta f(x_0,.)$
and whenever some of them exist, they are equal.
Thus we have $F_{x_0}(X,Y) \subset H_{x_0}(X,Y) \subset G_{x_0}(X,Y)$.
b) If $f \in H_{x_0}(X,Y)$ *then* f *satisfies*

$$(\mathcal{O})_{x_0} \qquad ||f(x_0+h)-f(x_0)|| = \mathcal{O}(h)$$

(that is,

$$\exists M>0,\ \exists r>0 \mid \forall h\in X, ||h|| < r \Rightarrow ||f(x_0+h)-f(x_0)|| \le M||h||.)$$

c) If $\dim X < \infty$ *then* $F_{x_0}(X,Y) = H_{x_0}(X,Y)$.

Proof. a) If $f \in F_{x_0}(X,Y)$, taking $h = tv + tl$, since

$$\frac{f(x_0+tv+tl)-f(x_0)-t\,f'(x_0)v}{t} = \frac{f(x_0+h)-f(x_0)-f'(x_0)h}{||h||}.\frac{|t|\,||v+l||}{t} + f'(x_0)l$$

and the second member above goes to 0 when $(t,l)\to(0,0)\in\mathbb{R}\times X$

(for $h \to 0$, $f \in F_{x_0}(X,Y)$ and $\dfrac{|t|\,||v+l||}{t}$ is bounded as $t \to 0$),
then $f \in H_{x_0}(X,Y)$ and ${}^{H}f'(x_0) = f'(x_0)$.
Take now $f \in H_{x_0}(X,Y)$. If f is not $\hat{H}$-differentiable at x_0 we have that $\exists g \in G$, $\varepsilon_0 > 0$ and $t_n \neq 0$, $t_n \to 0$ in $\mathbb{R}$ as $n \to \infty$ in $\mathbb{N}$ such that

$$\left\| \frac{(f \circ g)(t_n) - f(x_0)}{t_n} - {}^{H}f'(x_0)g'(0) \right\| \geq \varepsilon_0 \qquad \forall n \in \mathbb{N}.$$

Taking $l_n = \dfrac{g(t_n) - g(0)}{t_n} - g'(0)$ and $v = g'(0)$ we have

$$\left\| \frac{f(x_0 + t_n v + t_n l_n) - f(x_0)}{t_n} - {}^{H}f'(x_0)v \right\| \geq \varepsilon_0 \qquad \forall n \in \mathbb{N},$$

a contradiction since $f \in H_{x_0}(X,Y)$.
Then f is $\hat{H}$-differentiable at x_0 and ${}^{\hat{H}}f'(x_0) = {}^{H}f'(x_0)$.
On the other hand, suppose f $\hat{H}$-differentiable at x_0. If $f \notin H_{x_0}(X,Y)$ we have that $\exists v \in X$, $\varepsilon_0 > 0$, $t_n \neq 0$, $t_n \to 0$ in $\mathbb{R}$ and $l_n \to 0$ in X as $n \to \infty$ in $\mathbb{N}$ such that $\left\| \dfrac{f(x_0 + t_n v + t_n l_n) - f(x_0)}{t_n} - {}^{\hat{H}}f'(x_0)v \right\| \geq \varepsilon_0 \qquad \forall n \in \mathbb{N}.$
Take $g :]-1,1[\to U$ given by $g(t_n) = x_0 + t_n v + t_n l_n \qquad \forall n \in \mathbb{N}$
and $g(t) = x_0 + tv$ otherwise.
We see that $g \in G$ since $\displaystyle\lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0}(v + l(t)) = v$ where
$\begin{cases} l(t_n) = l_n & \forall n \in \mathbb{N} \\ l(t) = 0 & \text{otherwise} \end{cases}$ and $\left\| \dfrac{(f \circ g)(t_n) - (f \circ g)(0)}{t_n} - {}^{\hat{H}}f'(x_0)g'(0) \right\| \geq \varepsilon_0$
$\forall n \in \mathbb{N}$, a contradiction since f is $\hat{H}$-differentiable at x_0.
Then $f \in H_{x_0}(X,Y)$ and ${}^{H}f'(x_0) = {}^{\hat{H}}f'(x_0)$.
Now, if f is $\hat{H}$-differentiable at x_0, given $v \in X$ it is sufficient to take $g(t) = x_0 + tv$ to see that $f \in G_{x_0}(X,Y)$ and $\delta f(x_0, .) = {}^{\hat{H}}f'(x_0)$.
b) If $f \in H_{x_0}(X,Y)$, taking $v = 0 \in X$ we have $\displaystyle\lim_{\substack{(t,l) \to (0,0) \\ (t,l) \in \mathbb{R} \times X}} \frac{f(x_0 + tl) - f(x_0)}{t} = 0.$
Then fixing $\varepsilon > 0$ ($\varepsilon = 1$, for example), $\exists\ \delta_1 > 0$, $\delta > 0$ with $\delta < \delta_1 |$
$0 \neq |t| < \delta$, $||l|| < \delta_1 \Rightarrow\ ||f(x_0 + tl) - f(x_0)|| < \varepsilon|t|$.
Set $r = \delta^2$ and $M = \dfrac{\varepsilon}{\delta}$. For $h \in X$ with $0 < ||h|| < r$ we have that $h = tl$ where $t = \dfrac{||h||}{\delta}$, $l = \dfrac{h}{||h||}\delta$, and $|t| < \dfrac{r}{\delta} = \delta$, $||l|| = \delta < \delta_1$, then

$$||f(x_0 + h) - f(x_0)|| < \varepsilon \frac{||h||}{\delta} = M||h||. \quad \text{So } f \text{ satisfies } (\mathcal{O})_{x_0}.$$

c) Suppose $\dim X < \infty$ and $f \in H_{x_0}(X,Y)$. If f is not F- differentiable at x_0 we have that $\exists \varepsilon_0 > 0$ and $h_n \neq 0$, $h_n \to 0$ in X as $n \to \infty$ in $\mathbb{N}$ such that $\left\| \dfrac{f(x_0 + h_n) - f(x_0) - {}^{H}f'(x_0)h_n}{||h_n||} \right\| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$. Set $v_n = \dfrac{h_n}{||h_n||}$,

so $||v_n|| = 1, \ \forall n \in \mathbb{N}$. Since dim $X < \infty$, taking a subsequence, if necessary, we will have $v_n \to v$ for some $v \in X$ with $||v|| = 1$. Then, for $t_n = ||h_n||$ and $l_n = v_n - v$, we have $h_n = t_n v_n = t_n v + t_n l_n$ and
$\varepsilon_0 \leq ||\dfrac{f(x_0 + t_n v + t_n l_n) - f(x_0)}{t_n} -^H f'(x_0)v -^H f'(x_0)l_n|| \to 0$ as $n \to \infty$,
since $f \in H_{x_0}(X, Y)$, a contradiction! Thus $f \in F_{x_0}(X, Y)$.

□

Gateaux differentiability encompasses a greater class of functions than the other ones. Even for the simple case $X = \mathbb{R}^2$ and $Y = \mathbb{R}$ there are examples of functions f such that $f \in G_{x_0}(X, Y)$ and $f \notin H_{x_0}(X, Y) = F_{x_0}(X, Y)$. From now on, X and Y are real Banach spaces, $X \neq \{0\} \neq Y$ and $x_0 \in X$. Whenever it is necessary, functions $f : U_1 \subset X \to Y$ and $g : U_2 \subset X \to Y$ are identified when they coincide on U, open set of X, $x_0 \in U \subset U_1 \cap U_2$.

Definition 2.1 (Limit through a set). *Let $U \subset X$, $S \subset X$ and x_0 be an accumulation point of $S \cap U$.*
The limit through S of the function g:$U \to Y$ at x_0 will be denoted by $\lim\limits_{\substack{x \to x_0 \\ x \in S}} g(x)$ and is defined by $\lim\limits_{\substack{x \to x_0 \\ x \in S}} g(x) = L \in Y \Leftrightarrow$
$\forall \varepsilon > 0, \ \exists \delta > 0 \mid (x \in S \cap U \wedge 0 < ||x - x_0|| < \delta) \Rightarrow ||g(x) - L|| < \varepsilon$.

Observe that when $S \subset U$ is a neighbourhood of x_0 the definition above reduces to the ordinary limit of a function.
Example: Take $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $U = (\mathbb{R}^*_+ - \{1\}) \times \mathbb{R}^*_+$, where
$\mathbb{R}^*_+ = \{t \in \mathbb{R} | t > 0\}$, and $g : U \to Y$ written $g(x) = \log_a b$ where $x = (a, b)$. Let $x_0 = (1, 1)$ and for each $r \in \mathbb{R}$ define $S_r = \{(a, b) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ | b = a^r\}$. So for all $r \in \mathbb{R}$ we have $\lim\limits_{\substack{x \to x_0 \\ x \in S_r}} g(x) = r$. This exemplifies the important feature of limits through sets: A function may not have limit, in the ordinary sense, at x_0 but still can exhibit infinitely many limits through sets at x_0.

The following Lemmas will be useful.

Lemma 2.1. *Let U_1, U_2 and S be subsets of X such that x_0 is an accumulation point of $U_1 \cap U_2 \cap S$, and let $g_i : U_i \to Y$, $i = 1, 2$, be functions. If there exists the limits $\lim\limits_{\substack{x \to x_0 \\ x \in S}} g_1(x) = L_1$ and $\lim\limits_{\substack{x \to x_0 \\ x \in S}} g_2(x) = L_2$ then there also exists the limits $\lim\limits_{\substack{x \to x_0 \\ x \in S}} [g_1 + g_2](x) = L_1 + L_2$ and $\lim\limits_{\substack{x \to x_0 \\ x \in S}} t g_1(x) = t L_1$ for all $t \in \mathbb{R}$.*

Proof. Immediate. □

Lemma 2.2. *i) Assuming the same notation in Definition 2.1, let also $\hat{S} \subset S$ such that x_0 is still an accumulation point of $\hat{S} \cap U$. Then, the* *follwing implication holds:*

$$\left(\lim_{\substack{x\to x_0\\ x\in S}} g(x) = L\right) \Rightarrow \left(\lim_{\substack{x\to x_0\\ x\in \hat{S}}} g(x) = L\right)$$

ii) If $\hat{S} = S \cap V$, $x_0 \in V$ open set of X, then we may replace the implication above by an equivalence.

Proof. Immediate. □

3. Differentiability in types of sets and Hadamard derivative

Let $f : U \subset X \to Y$, $x_0 \in U$ be as in the previous section.
Our goal here is to prove that the differentiability of f at x_0, be it in the sense of Gateaux, Fréchet or Hadamard, is equivalent to the existence of a bounded linear operator $T \in L(X, Y)$ such that the relation

$$\lim_{\substack{h\to 0\\ h\in S}} \frac{f(x_0+h) - f(x_0) - T(h)}{||h||} = 0 \in Y \tag{1}$$

is fullfilled for all sets $S \in \mathcal{S} \subset \mathcal{P}(X)$ where $\mathcal{S}$ is a special class of subsets of X.
More specifically:

I) For G-differentiability (with bounded linear derivative), $\mathcal{S}$ should be the class of all open neighbourhoods of 0 in all unidimensional subspaces of X.
II) For F-differentiability, $\mathcal{S}$ should be the class of all open neighbourhoods of 0 in X.
III) For H-differentiability, assuming that f satisfies $(\mathcal{O})_{x_0}$, $\mathcal{S}$ should be the class of all embedded C^1-curves passing through 0.

We note that each one of these classes is constituted by a type of embedded submanifold passing through 0 which is characterized by the dimension of the submanifold and by the dimension of the subspace which the manifold is embedded. We indicate these classes by

I) $\mathcal{S}_{(1,1)} = \{V \subset E \subset X | E \text{ subspace of } X, \dim E = 1,\ V \text{ open neighbourhood of 0 in } E\}$.
II) $\mathcal{S}_{(X,X)} = \{V \subset X | V \text{ open neighbourhood of } 0\}$.
III) $\mathcal{S}^1_{(1,X)} = \{\gamma(V) \subset X | \gamma : V \subset \mathbb{R} \to X\ \ C^1\text{-embedding}, V \text{ open neighbourhood of 0 in } \mathbb{R},\ \gamma(0) = 0\}$.

This leads us to define other types of classes of subsets of X. Set $N_X = \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ if $\dim X = \infty$ and set $N_X = \{n \in \mathbb{N}^* \mid n \leq \dim X\}$ if $\dim X < \infty$.

Definition 3.1. *For $m, n \in N_X$, $m \leq n$, we define the following classes of embedded submanifolds of X:*
$\mathcal{S}_{(m,n)} = \{\gamma(V) \subset X | \gamma : V \subset \mathbb{R}^m \to E$ *topological embedding, E subspace of X,* $\dim\ E = n$, *V open neighbourhood of* 0 *in* $\mathbb{R}^m$, $\gamma(0) = 0\}$.
$\mathcal{S}^1_{(m,n)} = \{\gamma(V) \subset X | \gamma : V \subset \mathbb{R}^m \to E$ *C^1-embedding, E subspace of X,* $\dim\ E = n$, *V open neighbourhood of* 0 *in* $\mathbb{R}^m$, $\gamma(0) = 0\}$.
$\mathcal{S}_{(m,X)} = \{\gamma(V) \subset X | \gamma : V \subset \mathbb{R}^m \to X$ *topological embedding, V open neighbourhood of* 0 *in* $\mathbb{R}^m$, $\gamma(0) = 0\}$.
$\mathcal{S}^1_{(m,X)} = \{\gamma(V) \subset X | \gamma : V \subset \mathbb{R}^m \to X$ *C^1-embedding, V open neighbourhood of* 0 *in* $\mathbb{R}^m$, $\gamma(0) = 0\}$.

By topological embedding we mean a map γ which is a homeomorphism onto its image and by C^1-embedding we mean a topological embedding of class C^1 whose differential is injective at every point.
Since in the definition above the domain of γ is always finite dimensional, the differentiability of γ is to be understood, as usual, in the sense of Fréchet.
Note that $\mathcal{S}^1_{(m,n)} \subset \mathcal{S}_{(m,n)} \subset \mathcal{S}_{(m,X)}$ and $\mathcal{S}^1_{(m,n)} \subset \mathcal{S}^1_{(m,X)} \subset \mathcal{S}_{(m,X)}$ also $\mathcal{S}_{(n,n)} = \mathcal{S}^1_{(n,n)} = \{V \subset E \subset X | E$ subspace of X, $\dim\ E = n$, V open neighbourhood of 0 in $E\}$, $\quad \forall m, n \in N_X,\ m \leq n$.
In the next section we generalize these classes of embedded submanifolds using the notion of "vias of Banach spaces", so that we do not restrict ourselves only to the case where the domain of γ is finite dimensional, neither only to Fréchet differentiability. Each "via" will determine a class $\mathcal{S}$ of embedded submanifolds of X and a type of differentiability for f at x_0, which will be precisely the existence of a bounded linear operator $T \in L(X, Y)$ such that (1) is fullfilled for all sets $S \in \mathcal{S}$. We will see that the classes given in Definitions 3.1 are associated to particular "vias".
Let us, for now, define the type of differentiability according to a class $\mathcal{S} \subset \mathcal{P}(X)$.

Definition 3.2. *Given $\mathcal{S} \subset \mathcal{P}(X)$ such that 0 is an accumulation point of every $S \in \mathcal{S}$, we define the class $\mathcal{S}_{x_0}(X, Y)$ of $\mathcal{S}$-differentiable functions at x_0, by:*

$$\mathcal{S}_{x_0}(X,Y) = \{f : U \to Y | U \text{ open neighbourhood of } x_0 \text{ in } X,\ \exists T \in L(X,Y) \text{ such that (1) is fullfilled } \forall S \in \mathcal{S}\}.$$

For the classes of Definition 3.1, more specifically,
when $\mathcal{S} = \mathcal{S}_{(m,n)}$ *we put* $\mathcal{S}_{x_0}(X,Y) = (m,n)_{x_0}(X,Y)$
when $\mathcal{S} = \mathcal{S}^1_{(m,n)}$ *we put* $\mathcal{S}_{x_0}(X,Y) = (m,n)^1_{x_0}(X,Y)$

when $\mathcal{S} = \mathcal{S}_{(m,X)}$ *we put* $\mathcal{S}_{x_0}(X,Y) = (m,X)_{x_0}(X,Y)$
when $\mathcal{S} = \mathcal{S}^1_{(m,X)}$ *we put* $\mathcal{S}_{x_0}(X,Y) = (m,X)^1_{x_0}(X,Y) \forall m,n \in N_X,\ m \leq n$.
Let us put also $\mathcal{S}_{x_0}(X,Y) = (X,X)_{x_0}(X,Y)$ *if* $\mathcal{S} = \mathcal{S}_{(X,X)}$.

From what we have observed above about the inclusion of classes we can see that $(m,X)_{x_0}(X,Y) \subset (m,n)_{x_0}(X,Y) \subset (m,n)^1_{x_0}(X,Y)$ and $(m,X)_{x_0}(X,Y) \subset (m,X)^1_{x_0}(X,Y) \subset (m,n)^1_{x_0}(X,Y)$, also $(n,n)_{x_0}(X,Y) = (n,n)^1_{x_0}(X,Y) \qquad \forall m,n \in N_X,\ m \leq n$.
The unicity of operator $T \in L(X,Y)$ when (1) is satisfied $\forall S \in \mathcal{S}$, for $\mathcal{S}$ given in Definition 3.1, will be shown in the next proposition, where we will see that $T = \delta f(x_0,.)$, the G-derivative of f at x_0. When f is $\mathcal{S}$-differentiable at x_0 we call $T = {}^{\mathcal{S}}df(x_0) = {}^{\mathcal{S}}f'(x_0)$, the $\mathcal{S}$-derivative of f at x_0.
The next proposition characterizes Gateaux and Fréchet differentiabilities.

Proposition 3.1. *a)* $G_{x_0}(X,Y) = (1,1)_{x_0}(X,Y)$
b) $F_{x_0}(X,Y) = (X,X)_{x_0}(X,Y)$
c) Let $\mathcal{S}$ *be one of the classes given in Definition 3.1, then* $F_{x_0}(X,Y) \subset \mathcal{S}_{x_0}(X,Y) \subset G_{x_0}(X,Y)$ *and if* $f \in \mathcal{S}_{x_0}(X,Y)$ *and* $T \in L(X,Y)$ *is such that (1) is fulfilled* $\forall S \in \mathcal{S}$, *then* $T = \delta f(x_0,.)$, *the* G-*derivative of* f *at* x_0.

Proof. a) For $S \in \mathcal{S}_{(1,1)}$ we have $\exists v \in X,\ ||v|| = 1,\ \exists V \subset \mathbb{R}$ open neighbourhood of $0 \in \mathbb{R}$ such that $S = \{tv \in X | t \in V\}$. Then we see that:
$\exists T \in L(X,Y)|$ (1) is true for f through S, $\forall S \in \mathcal{S}_{(1,1)} \Leftrightarrow$
$\exists T \in L(X,Y)| \lim\limits_{t\to 0} \dfrac{f(x_0+tv) - f(x_0) - tTv}{t} = 0,\ \forall v \in X,\ ||v|| = 1 \Leftrightarrow$
$\delta f(x_0,.) = T \in L(X,Y)$. So, $f \in (1,1)_{x_0}(X,Y) \Leftrightarrow f \in G_{x_0}(X,Y)$.
b) If $S \in \mathcal{S}_{(X,X)}$ then S is an open neighbourhood of 0, thus:
$f \in (X,X)_{x_0}(X,Y) \Leftrightarrow \exists T \in L(X,Y)|$ (1) is true for f through S, $\forall S$ open neighbourhood of $0 \Leftrightarrow f \in F_{x_0}(X,Y)$.
c) If $f \in F_{x_0}(X,Y)$ then (1) is satisfied for f with $T = f'(x_0) \in L(X,Y)$, through S, $\forall S$ open neighbourhood of 0.
Since, for $\hat{S} \in \mathcal{S}$, we have $\hat{S} \subset X$ and X is an open neighbourhood of 0, by Lemma 2.2, we see that (1) is satisfied for f with $T = f'(x_0)$, through $\hat{S}$, $\forall \hat{S} \in \mathcal{S}$. Then $f \in \mathcal{S}_{x_0}(X,Y)$.
To see that $\mathcal{S}_{x_0}(X,Y) \subset G_{x_0}(X,Y)$ it is sufficient to show that $(m,n)^1_{x_0}(X,Y) \subset G_{x_0}(X,Y) \quad \forall m,n \in N_X,\ m \leq n$, taking in view the inclusions mencioned after Definition 3.2.
We will show first that given $\hat{S} \in \mathcal{S}_{(1,1)}$, $\exists S \in \mathcal{S}^1_{(m,n)}$ such that $\hat{S} \subset S$.
For $\hat{S} \in \mathcal{S}_{(1,1)}$ we have $\exists v \in X,\ ||v|| = 1,\ \exists V \subset \mathbb{R}$ open neighbourhood of 0 in $\mathbb{R}$ such that $\hat{S} = \{tv \in X | t \in V\}$.

Let $E_1 =< \{v\} >$ be the one dimensional subspace of X generated by v, so $\hat{S} \subset E_1$. We take then E_m, E_n subspaces of X, $\dim E_m = m$, $\dim E_n = n$, such that $E_1 \subset E_m \subset E_n \subset X$ and define $\gamma : E_m \to E_n$ as the inclusion (which is C^1-embedding) and set $S = \gamma(E_m) = E_m$. Then $\hat{S} \subset S$ and $S \in \mathcal{S}^1_{(m,n)}$. Now if $f \in (m,n)^1_{x_0}(X,Y)$ we have that (1) is fullfilled for f, for some $T \in L(X,Y)$, $\forall S \in \mathcal{S}^1_{(m,n)}$. In particular, by Lemma 2.2, (1) is fulfilled also through $\hat{S}$, $\forall \hat{S} \in \mathcal{S}_{(1,1)}$. Then $f \in G_{x_0}(X,Y)$ and $T = \delta f(x_0,.)$.

□

In Proposition 4.5 of the next section we will relate Gateaux-differentiability with some "vias" whose class of embedded submanifolds is $\mathcal{S}_{(1,1)}$ and we will relate Fréchet-differentiability with some "vias" whose class of embedded submanifolds is $\mathcal{S}_{(X,X)}$. We show also there that the class of embedded submanifolds $\mathcal{S}_{(1,X)}$ will give Fréchet-differentiability too.
Let us do now the Hadamard-differentiability characterization:

Theorem 3.1. *$f \in H_{x_0}(X,Y) \Leftrightarrow f$ satisfies $(\mathcal{O})_{x_0}$ and $f \in (1,X)^1_{x_0}(X,Y)$.*

Proof. There is no loss of generality in considering $x_0 = 0$ and $f(x_0) = 0$.
a) Suppose $f \in (1,X)^1_{x_0}(X,Y)$ with $(\mathcal{O})_{x_0}$ satisfied by f and $f \notin H_{x_0}(X,Y)$.
Now, $f \notin H_{x_0}(X,Y) \Rightarrow \forall T \in L(X,Y)$, $\exists v \in X$, $\exists \varepsilon_0 > 0 \mid \forall \delta > 0$, $\exists t_\delta \in \mathbb{R}$, $0 < |t_\delta| < \delta$, $\exists l_\delta \in X$, $0 \leq ||l_\delta|| < \delta$ such that we have

$$\frac{||f(t_\delta v + t_\delta l_\delta) - T(t_\delta v)||}{|t_\delta|} \geq \varepsilon_0 \tag{2}$$

For $T =^{\mathcal{S}^1_{(1,X)}} df(0)$, take $v \in X$ and $\varepsilon_0 > 0$ as the ones given by (2).
We can construct sequences $\{\delta_i\}_{i\in\mathbb{N}}, \{t_{\delta_i}\}_{i\in\mathbb{N}}$ and $\{l_{\delta_i}\}_{i\in\mathbb{N}}, t_{\delta_i}, l_{\delta_i}$, given by (2) where $\delta = \delta_i$, that obey $0 < \delta_{i+1} < \dfrac{|t_{\delta_i}|}{2} < \dfrac{\delta_i}{2}$.
Now, by Proposition 3.1 (c), f is Gateaux differentiable at 0 and, consequently, there are at most finitely many indexes i for which $l_{\delta_i} = 0$.
From these sequences we can choose subsequences $\{\delta_{i_j}\}_{j\in\mathbb{N}}$ and $\{l_{\delta_{i_j}}\}_{j\in\mathbb{N}}$ such that all $t_{\delta_{i_j}}$ have same sign, say, all of them are positive, and $l_{\delta_{i_j}} \neq 0$.
Rename these subsequences and call them simply $\{t_n\}_{n\in\mathbb{N}}$, $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{l_n\}_{n\in\mathbb{N}}$. Clearly, $\{\delta_n\} \downarrow 0$, $\{t_n\} \downarrow 0$ with $t_{n+1} < \dfrac{t_n}{2}$, and $\{l_n\} \to 0$ as $n \to \infty$.
Observe that $v \neq 0$ since $v = 0$ implies $\dfrac{||f(t_n l_n)||}{||t_n l_n||} \geq \dfrac{\varepsilon_0}{||l_n||} \to \infty$ as $n \to \infty$, a contradiction with the fact that f obeys $(\mathcal{O})_0$.
Define $h_j = t_j v + t_j l_j$ and $v_{j+1} = \dfrac{(h_j - h_{j+1})}{t_j - t_{j+1}}$ for all $j \in \mathbb{N}$.

Let $\{z_j\}_{j\in\mathbb{N}^*}$ be a sequence such that $0 < z_j \le t_{j-1} - t_j$ for all $j \in \mathbb{N}^*$.
Now $v_j = v + \dfrac{t_jl_j - t_{j-1}l_{j-1}}{t_j - t_{j-1}}$ and

$$\frac{||t_jl_j - t_{j-1}l_{j-1}||}{|t_j - t_{j-1}|} \le \frac{t_j||l_j||}{t_{j-1} - t_j} + \frac{||l_{j-1}||}{1 - \dfrac{t_j}{t_{j-1}}} \le ||l_j|| + 2||l_{j-1}|| \to 0$$

as $j \to \infty$, i.e., $v_j \to v$ as $j \to \infty$.
Construct the polygonal $\tilde{\gamma}(t) = \dfrac{t_j - t}{t_j - t_{j+1}}h_{j+1} + \dfrac{t - t_{j+1}}{t_j - t_{j+1}}h_j$ for $t_{j+1} \le t \le t_j$, $j \in \mathbb{N}$ and $\tilde{\gamma}(t) = tv$ for $t \le 0$.
Since $\dfrac{\tilde{\gamma}(t)}{t} - v = \dfrac{t_jt_{j+1}(l_{j+1} - l_j) + t(t_jl_j - t_{j+1}l_{j+1})}{t(t_j - t_{j+1})}$,

$$||\frac{\tilde{\gamma}(t)}{t} - v|| \le \frac{(t_jt_{j+1})(||l_{j+1}|| + ||l_j||)}{t_{j+1}\left(\dfrac{t_j}{2}\right)} + \frac{t_j(||l_j|| + ||l_{j+1}||)}{(\frac{t_j}{2})} = 4(||l_j|| + ||l_{j+1}||) \to 0$$

as $j \to \infty$ and we have that $\lim\limits_{t\to 0^+} \dfrac{\tilde{\gamma}(t)}{t} = v$.
Now, to avoid corners at $\tilde{\gamma}(t_j) = h_j$ for $j \in \mathbb{N}^*$ we "smooth" the polygonal $\tilde{\gamma}(t)$, i.e., we replace $\tilde{\gamma}$ by γ of class C^1 such that $\gamma(t_j) = \tilde{\gamma}(t_j)$ for all $j \in \mathbb{N}$.
Let $\tilde{\gamma}_j(t) = \dfrac{t_j - t}{t_j - t_{j+1}}h_{j+1} + \dfrac{t - t_{j+1}}{t_j - t_{j+1}}h_j$;
$\gamma_j(t) = \tilde{\gamma}_j(t) + (t - t_j)\left(\dfrac{t - t_j - z_j}{z_j}\right)^2 (v_{j+1} - v_j)$ for all $t \in \mathbb{R}$ and take

$$\gamma(t) = \begin{cases} \gamma_j(t) & t \in [t_j, t_j + z_j] \\ \tilde{\gamma}(t) & \text{otherwise.} \end{cases}$$

We have

$$\gamma'(t) = \begin{cases} v_j + \left(\dfrac{t - t_j - z_j}{z_j}\right)\left(\dfrac{3(t - t_j) - z_j}{z_j}\right)(v_{j+1} - v_j) & \text{if } t \in [t_j, t_j + z_j] \\ \tilde{\gamma}'(t) & \text{otherwise.} \end{cases}$$

Now, it is easy to verify that $\gamma'(t_j) = \lim\limits_{t\to t_j} \gamma'(t) = v_{j+1}$ and $\gamma'(t_j + z_j) = \lim\limits_{t\to t_j+z_j} \gamma'(t) = v_j$, whatever the choice of $\{z_j\}_{j\in\mathbb{N}^*}$ is.
Consequenlty, γ is C^1 on $(0, t_1)$.
Observe that

$$||\gamma(t) - \tilde{\gamma}(t)|| \le |t - t_j|\left(\frac{t - t_j - z_j}{z_j}\right)^2 ||v_{j+1} - v_j|| \le |t - t_j|\ ||v_{j+1} - v_j||$$

for all $t \in [t_j, t_j + z_j]$ and $||\gamma(t) - \tilde{\gamma}(t)|| = 0$ for all $t \in (z_j + t_j, t_{j-1}]$. We also have

$$||\gamma'(t)-\tilde{\gamma}'(t)|| \leq |\frac{t-t_j-z_j}{z_j}|\,|\frac{3(t-t_j)-z_j}{z_j}|\ ||v_{j+1}-v_j|| \leq 2||v_{j+1}-v_j||$$

for all $t \in (t_j, t_j+z_j]-\{t_{j-1}\}$ and $||\gamma'(t)-\tilde{\gamma}'(t)|| = 0$ for all $t \in (t_j+z_j, t_{j-1})$.
In this way, we have $||\gamma(t)-\tilde{\gamma}(t)|| \leq |t-t_j|\,||v_{j+1}-v_j|| \to 0$ on $[t_j, t_j+z_j]$ and $||\gamma'(t)-\tilde{\gamma}'(t)|| \leq 2||v_{j+1}-v_j|| \to 0$ on $[t_j, t_j+z_j]-\{t_j, t_{j-1}\}$ since $||v_{j+1}-v_j|| \to 0$ as $j \to \infty$.
So, we conclude that $||\gamma(t)-\tilde{\gamma}(t)|| \to 0$ as $t \to 0^+$, i.e., $\gamma(t) \to 0$ as $t \to 0^+$ and $||\gamma'(t)-\tilde{\gamma}'(t)|| \to 0$ on $(0, t_1)-\{t_j | j \in \mathbb{N}\}$. Now, $\tilde{\gamma}'(t) = v_j$ on (t_j, t_{j-1}) and $v_j \to v$ which implies that $\gamma'(t) \to v$ as $t \to 0$ through $(0, t_1)-\{t_j | j \in \mathbb{N}\}$.
Since γ is C^1 on $(0, t_1)$ we have $\lim\limits_{\substack{t \to 0^+ \\ t \in \mathbb{R}}} \gamma'(t) = v$.
Note that $\gamma(0) = 0$, $\lim\limits_{t \to 0^-} \gamma(t) = \gamma(0) = \lim\limits_{t \to 0^+} \gamma(t)$.
We also have $\lim\limits_{t \to 0^-} \gamma'(t) = v = \lim\limits_{t \to 0^+} \gamma'(t)$.
Now, $\gamma'(0) = v$, since $\lim\limits_{t \to 0^-} \dfrac{\gamma(t)-\gamma(0)}{t} = v$ and

$$\lim_{t \to 0^+} \frac{\gamma(t)-\gamma(0)}{t} = \lim_{t \to 0^+} \frac{\gamma(t)-\tilde{\gamma}(t)}{t} + \lim_{t \to 0^+} \frac{\tilde{\gamma}(t)}{t} = v.$$

The latter equality is due to the fact that $\lim\limits_{t \to 0^+} \dfrac{\tilde{\gamma}(t)}{t} = v$ and,
$\left\| \dfrac{\gamma(t)-\tilde{\gamma}(t)}{t} \right\| \leq \|v_{j+1}-v_j\| \to 0$ as $j \to \infty$.
Thus, we have obtained $\gamma'(0) = v$ and γ is C^1 on $(-\infty,\ t_1)$.

Observe that if we were obliged to choose negative t_j to form the initial sequence then we would have constructed, similarly, $\gamma : (t_1, \infty) \to X$.
Let $X = \langle\{v\}\rangle \oplus W$ and write $x = x_1 v + w, w \in W$, the decomposition of x. Define $\hat{\pi}_1 : X \to \mathbb{R}$ by $\hat{\pi}_1(x) = x_1$.
Since $\gamma'(t) \to v$ as $t \to 0$ we have $\exists s_1, s_2 \in \mathbb{R}, s_1 < 0 < s_2$ such that $\hat{\pi}_1(\gamma'(t)) > \dfrac{1}{2}$ $\forall t \in (s_1, s_2)$. Now, for all $t_1, t_2 \in (s_1, s_2), t_2 > t_1$ we have

$$\hat{\pi}_1(\gamma(t_2)-\gamma(t_1)) = \hat{\pi}_1\gamma(t_2) - \hat{\pi}_1\gamma(t_1) = \int_{t_1}^{t_2} (\hat{\pi}_1\gamma)'(t)dt =$$

$$= \int_{t_1}^{t_2} \hat{\pi}_1\gamma'(t)dt > \int_{t_1}^{t_2} \left(\frac{1}{2}\right) dt = \left(\frac{t_2-t_1}{2}\right) > 0$$

and, consequently, $\gamma(t_2) \neq \gamma(t_1)$, that is, γ is injective on (s_1, s_2).
Let $L \in \mathbb{N}$ be such that $t_L < s_2$. Clearly $\gamma : [-1, t_L] \to X$ is a continuous bijection over its image.

To prove that $\gamma^{-1} : \gamma([-1,t_L]) \to [-1,t_L]$ is continuous just observe that all closed set $F \subset [-1,t_L]$ is compact and that its inverse image by (γ^{-1}), i.e., $\gamma(F)$ is also a closed set, since it is the continuous image of a compact set.

So, we have obtained $\gamma : (-1,t_L) \to X$, a C^1-embedding such that $\gamma(t_j) = h_j$ for all $j > L$. Then $S = \gamma((-1,t_L)) \in \mathcal{S}^1_{(1,X)}$. Since f is $\mathcal{S}^1_{(1,X)}$-differentiable at $0,(1)$ is valid for f through S. Now, $\{h_j | j > L\} \subset S$ and $h_j \to 0$ as $j \to \infty$, so,

$$0 = \lim_{j\to\infty} \frac{f(h_j) - T(h_j)}{||h_j||} = \lim_{j\to\infty} \left\{ \left[\frac{f(t_jv + t_jl_j) - t_jT(v)}{t_j} \right] \frac{1}{||v + l_j||} - \frac{T(l_j)}{||v+l_j||} \right\}.$$

Thus, $\lim\limits_{j\to\infty} \dfrac{f(t_jv + t_jl_j) - t_jT(v)}{t_j} = 0$, a contradiction with (2).

Thus $f \in H_0(X,Y)$.

b) Suppose $f \in H_0(X,Y)$ and let $T =^H f'(0)$.

Let $S = \gamma(V) \in \mathcal{S}^1_{(1,X)}$ and $v = \gamma'(0) \neq 0$. Define $l(t) = \dfrac{\gamma(t)}{t} - v$ so that $\gamma(t) = tv + tl(t)$.

Now, $\lim\limits_{t\to 0} \dfrac{\gamma(t)}{t} = \gamma'(0) = v$ and we have $\lim\limits_{t\to 0} l(t) = 0$. So,

$$\lim_{\substack{h\to 0\\ h\in S}} \frac{f(h) - T(h)}{\|h\|} = \lim_{\substack{t\to 0\\ t\in V}} \frac{f(\gamma(t)) - T(\gamma(t))}{\|\gamma(t)\|} =$$

$$\lim_{\substack{t\to 0\\ t\in \mathbb{R}}} \frac{f(tv + tl(t)) - T(tv) - tT(l(t))}{t} \frac{t}{|t|\|v + l(t)\|} =$$

$$\lim_{\substack{t\to 0\\ t\in \mathbb{R}}} \left[\frac{f(tv + tl(t)) - tT(v)}{t} \frac{t}{|t|\|v + l(t)\|} - T(l(t)) \frac{t}{|t| \ \|v + l(t)\|} \right] = 0,$$

due to the existence of a neighbourhood of $0 \in \mathbb{R}$ for which $1/||v + l(t)||$ is limited, as a consequence of $v \neq 0$ and $l(t) \to 0$ as $t \to 0$, and also to the facts that $T(l(t)) \to 0$ and $\dfrac{f(tv + tl(t)) - tT(v)}{t} \to 0$, by virtue of $f \in H_0(X,Y)$,

as $t \to 0$.

So we proved that $f \in (1,X)^1_0(X,Y)$ with ${}^{S^1_{(1,X)}}f'(0) =^H f'(0)$ and since $f \in H_0(X,Y)$, we have that f satisfies $(\mathcal{O})_0$. $\square$

Now we present some results that lead us to conclude that

$(n,X)^1_{x_0}(X,Y) = (1,X)^1_{x_0}(X,Y)\ \forall n \in N_X$, and then, by the previous theorem, this set, together with property $(\mathcal{O})_{x_0}$, will characterize Hadamard-differentiability.

Theorem 3.2. *For $m,n \in N_X$, $m \le n$, we have $(n,X)^1_{x_0}(X,Y) \subset (m,X)^1_{x_0}(X,Y)$.*

Proof. Let $\hat{S} \in \mathcal{S}^1_{(m,X)}$, so $\hat{S}$ is written as $\hat{S} = \gamma(V)$, $\gamma : V \subset \mathbb{R}^m \to X$ C^1-embedding, V open neighbourhood of 0 in $\mathbb{R}^m$, $\gamma(0) = 0$. Let $M = \gamma'(0)(\mathbb{R}^m)$ the tangent space of $\hat{S}$ at 0 and write $X = M \oplus W$ (this is possible by Hahn-Banach Theorem, since $\dim M = m < \infty$). So $\forall x \in X$, $\exists!$ $(x_1, x_2) \in M \times W \mid x = x_1 + x_2$. Let $\pi_i(x) = x_i$, $i = 1,2$.
Define $\phi : U \subset X \to X$, where $U = \gamma'(0)(V) + W$ by $\phi(x) = \gamma(\Gamma x_1) + x_2$, where $\Gamma = [\gamma'(0)]^{-1} \in L(M, \mathbb{R}^m)$. Now ϕ is of class C^1 on U in Fréchet sense, since it is a composition of functions of this kind. Moreover
$\phi'(x) = \gamma'(\Gamma x_1) \circ \Gamma \circ \pi_1 + \pi_2$, thus $\phi'(0) = \gamma'(0) \circ \Gamma \circ \pi_1 + \pi_2 = \pi_1 + \pi_2 = I$, the identity on X.
Inverse Function Theorem, for Banach spaces, guaranties the existence of $\tilde{U} \subset U$, $0 \in \tilde{U}$ open set of X such that $\phi : \tilde{U} \to \phi(\tilde{U})$ is a C^1-diffeomorphism. Now, $\phi(U \cap M) = \phi(\gamma'(0)V) = \gamma(\Gamma(\gamma'(0)V)) = \gamma(V) = \hat{S} \subset \phi(U)$; and if $N \supset M$ is a subspace of X, $\dim N = n$, we have $\phi(U \cap N) \supset \hat{S}$. Letting $\tilde{V} = \Gamma(\tilde{U} \cap M)$ we have $0 \in \tilde{V} \subset V \subset \mathbb{R}^m$ and $\gamma(\tilde{V}) = \phi(\tilde{U} \cap M) \subset \phi(\tilde{U} \cap N)$.
The set $S = \phi(\tilde{U} \cap N) \in \mathcal{S}^1_{(n,X)}$ since $\tilde{U} \cap N$ is an open set of N, $0 \in \tilde{U} \cap N$ and $\phi|_{\tilde{U} \cap N}$ is a C^1-embedding.
So, if $f \in (n,X)^1_{x_0}(X,Y)$ then (1) is true for f through S and, consequently, (1) is true through $\gamma(\tilde{V}) \subset S$. Now since $\gamma(\tilde{V})$ is an open set in $\hat{S} = \gamma(V)$, by Lemma 2.2, (1) is valid through $\hat{S}$, too. This means that $f \in (m,X)^1_{x_0}(X,Y)$. □

The next definition will give us a new characterization for $\mathcal{S}$-differentiability. Let us fix $\mathcal{S} \subset \mathcal{P}(X)$ one of the classes of embedded manifolds given in Definition 3.1.

Definition 3.3. *We will call a set $W \subset X$ an $\mathcal{S}$-type union if and only if $W = \bigcup_{S \in \mathcal{S}} V_s$, where, for each $S \in \mathcal{S}$, V_s is a neighbourhood of 0 in S.*

Using this definition, $\mathcal{S}$-differentiability at x_0 may be interpreted in the following way:

Proposition 3.2. *$f \in \mathcal{S}_{x_0}(X,Y)$ with ${}^{\mathcal{S}}df(x_0) = T$ if and only if we have*

(3) $$\forall \varepsilon > 0,\ \exists W \subset X, \text{ an } \mathcal{S}\text{-type union} \mid \forall h \in W,$$

$$||f(x_0+h) - f(x_0) - Th|| \leq \varepsilon ||h||$$

Proof. Since the limit (1) through $S \in \mathcal{S}$ means $\forall\varepsilon > 0$, $\exists V_s$ neighbourhood of 0 in S such that $(h \in V_s,\ h \neq 0) \Longrightarrow ||f(x_0+h) - f(x_0) - T.h|| < \varepsilon||h||$, the proof is immediate. □

Proposition 3.3. *(a) The statements (i) and (ii) are equivalent*
(i) If $W \subset X$ is an $\mathcal{S}$-type union then W is a neighbourhood of 0 *in X.*
(ii) $\mathcal{S}_{x_0}(X,Y) = F_{x_0}(X,Y)$.
(b) If $\mathcal{S}_{x_0}(X,Y) \neq F_{x_0}(X,Y)$ we can find functions f, g both in $\mathcal{S}_{x_0}(X,Y)$ but not in $F_{x_0}(X,Y)$ such that f satisfies $(\mathcal{O})_{x_0}$ and g doesn't satisfy $(\mathcal{O})_{x_0}$.

Proof. To see that $(i) \Longrightarrow (ii)$, let $f \in \mathcal{S}_{x_0}(X,Y)$. Then, $\forall\varepsilon > 0$, $\exists W \subset X$ an $\mathcal{S}$-type union such that (3) is fulfilled with $T = {}^{\mathcal{S}}df(x_0)$. Now, by (i), W is a neighbourhood of 0 in X which implies that (1) is valid for $h \to 0,\ h \in X$. Then $f \in F_{x_0}(X,Y)$.

To see now that (ii)$\Longrightarrow$ (i) and also (b), let us suppose that there is an $\mathcal{S}$-type union $W \subset X$ which is not a neighbourhood of 0 in X. We can choose a sequence $\{h_n\}_{n\in\mathbb{N}}, h_n \in X - W$ with $h_n \to 0$ when $n \to \infty$. Let $y, y_n \in Y$ be such that $||y|| = 1,\ ||y_n|| = ||h_n||,\ \ \forall n \in \mathbb{N}$, and let us define the functions $f, g : X \to Y$ by $f(x_0 + h_n) = y_n,\ g(x_0 + h_n) = y,\ \forall n \in \mathbb{N}$, and $f(x) = g(x) = 0$ otherwise. We have $f, g \in \mathcal{S}_{x_0}(X,Y)$ since $f = g = 0$ on $x_0 + W$ and then, for all $\varepsilon > 0$, W is an $\mathcal{S}$-type union for which (3) is valid with $T = 0$ for both f and g. Now we see that f satisfies $(\mathcal{O})_{x_0}$ and g doesn't satisfy $(\mathcal{O})_{x_0}$ and both f and g are not Fréchet-differentiable at x_0 (g is not even continuous at x_0). □

The next two corollaries of Theorem 3.1 are, by their own, interesting results of differential topology.

Corollary 3.1. *If* $\dim X < \infty$ *and W is an $\mathcal{S}^1_{(1,X)}$ type union then W is a neigbourhood of* 0 *in X.*

Proof. If W is an $\mathcal{S}^1_{(1,X)}$ type union which is not a neighbourhood of 0 in X, we can find $f \in (1,X)^1_{x_0}(X,Y), f$ satisfies $(\mathcal{O})_{x_0}$, with $f \notin F_{x_0}(X,Y)$ (Proposition 3.3). But, by Theorem 3.1, we have $f \in H_{x_0}(X,Y)$ and with the fact that $\dim X < \infty$ we have $f \in F_{x_0}(X,Y)$ (Proposition 2.1(c)), a contradiction. □

Corollary 3.2. *Given a sequence $\{h_j\}_{j\in\mathbb{N}},\ h_j \in \mathbb{R}^n,\ h_j \to 0$ as $j \to \infty$, we can choose from $\{h_j\}_{j\in\mathbb{N}}$ a subsequence that is contained in the image S of a C^1-embedding curve, that is, contained in some $S \in \mathcal{S}^1_{(1,n)}$.*

Proof. It is sufficient to give the proof for the case when $\{j \in \mathbb{N} | h_j = 0\}$ is finite and we can suppose, taking a subsequence, that $h_j \neq 0\quad \forall j \in \mathbb{N}$.

If each and every $S \in \mathcal{S}^1_{(1,n)}$ contains only a finite number of points h_j then we can choose V_S, neighbourhood of 0 in S, such that V_S does not contain any of the points of the sequence $\{h_j\}_{j\in\mathbb{N}}$. Form W the union of all such V_S, an $\mathcal{S}^1_{(1,n)}$ type union. Thus we arrive at the following contradiction: $W \cap \{h_j | j \in \mathbb{N}\} = \emptyset, h_j \to 0$, and, by corollary 3.1, W is a neighbourhood of 0 in $\mathbb{R}^n$. So, there must exist at least one $S, S \cap \{h_j | j \in \mathbb{N}\}$ is inifnite. □

Theorem 3.3. *For each* $n \in N_X$ *we have* $(n, X)^1_{x_0}(X, Y) = (1, X)^1_{x_0}(X, Y)$.

Proof. We already know that $(n, X)^1_{x_0}(X, Y) \subset (1, X)^1_{x_0}(X, Y)$ (Theorem 3.2).
Let $S \in \mathcal{S}^1_{(n,X)}$, so $S = \gamma(V)$, V neighbourhood of 0 in $\mathbb{R}^n$, $\gamma : V \to X$ is a C^1-embedding in Fréchet sense.
Thus, there is a bijective correspondence between embedded C^1 curves through 0 in $V \subset \mathbb{R}^n$ and embedded C^1 curves through 0 in $S \subset X$. Since, by corollary 3.1, the intersection of an $\mathcal{S}^1_{(1,n)}$-type union in $\mathbb{R}^n$ with V is a neighbourhood of 0 in $\mathbb{R}^n$, we have that an $\mathcal{S}^1_{(1,X)}$-type union in X will contain a neighbourhood of 0 in S. This is so because γ is an homeomorphism from V on S.
Now let $f \in (1, X)^1_{x_0}(X, Y)$. Then, for all $\varepsilon > 0$, relation (3) is valid for f with $T = {}^{\mathcal{S}^1_{(1,X)}}df(x_0)$ for some W, $\mathcal{S}^1_{(1,X)}$-type union. Since W contains a neighbourhood of 0 in S, we have that (1) is valid for f, through S. Thus $f \in (n, X)^1_{x_0}(X, Y)$. □

As, by Proposition 3.3, if $\mathcal{S}_{x_0}(X, Y) \neq F_{x_0}(X, Y)$ we have examples of $f \in \mathcal{S}_{x_0}(X, Y)$ that satisfies $(\mathcal{O})_{x_0}$ and $g \in \mathcal{S}_{x_0}(X, Y)$ which doens't satisfy $(\mathcal{O})_{x_0}$, with both f and g not F-differentiable at x_0; we could emphasize the property $(\mathcal{O})_{x_0}$ in any $\mathcal{S}$-differentiability by:

Definition 3.4. *We will say that a function* $f \in \mathcal{S}_{x_0}(X, Y)$ *is strongly* $\mathcal{S}$*-differentiable at* x_0*, and write* $f \in \underline{\mathcal{S}}_{x_0}(X, Y)$ *if* f *satisfies* $(\mathcal{O})_{x_0}$.

Let us note that if $f \in \mathcal{S}_{x_0}(X, Y)$, and we take any $\mathcal{S}$-type union W given in (3), for any $\varepsilon > 0$, then the inequality $||f(x_0 + h) - f(x_0)|| \leq M||h||$ is satisfied for $h \in W$.
Proposition 3.3 says that if $\mathcal{S}_{x_0}(X, Y) \neq F_{x_0}(X, Y)$, then
$F_{x_0}(X, Y) \subsetneq \underline{\mathcal{S}}_{x_0}(X, Y) \subsetneq \mathcal{S}_{x_0}(X, Y)$.
We could also say that $\underline{F}_{x_0}(X, Y) = F_{x_0}(X, Y)$ and $\underline{H}_{x_0}(X, Y) = H_{x_0}(X, Y)$ (Proposition 2.1(b)).
In this notation we have already proved:

Corollary 3.3. *For each* $n \in N_X$ *we have* $\underline{(n, X)^1_{x_0}}(X, Y) = H_{x_0}(X, Y)$.

Corollary 3.4. *If* $\dim X < \infty$, *for each* $n \in N_X$ *we have* $(n,X)^1_{x_0}(X,Y) = \underline{(n,X)}^1_{x_0}(X,Y) = H_{x_0}(X,Y)$.

Proof. If $f \in (n,X)^1_{x_0}(X,Y)$ then $f \in (1,X)^1_{x_0}(X,Y)$ (Theorem 3.2) and then there is some W, $\mathcal{S}^1_{(1,X)}$-type union, where condition $(\mathcal{O})_{x_0}$ is valid if we replace $h \in X$ by $h \in W$. But since $\dim X < \infty$, W is a neighbourhood of 0 in X (Corollary 3.1), and then, for convenient $r > 0$, $h \in W$ with $\|h\| < r$ is the same as $h \in X$ with $\|h\| < r$; then f satisfies $(\mathcal{O})_{x_0}$. Thus, $f \in \underline{(n,X)}^1_{x_0}(X,Y) = H_{x_0}(X,Y)$. □

We want to show, as the last result of this section, that

$$(4) \qquad (1,X)^1_{x_0}(X,Y) \subset (n,n)_{x_0}(X,Y) \subset (m,m)_{x_0}(X,Y) \subset (1,1)_{x_0}(X,Y) \quad \forall m,n \in N_X,\ \ m \leq n.$$

In the case $\dim X = n < \infty$ we have already seen that

$$(1,X)^1_{x_0}(X,Y) = H_{x_0}(X,Y) = F_{x_0}(X,Y) = (n,n)_{x_0}(X,Y)$$

We will see that if $m < n$ we have $(n,n)_{x_0}(X,Y) \subsetneq (m,m)_{x_0}(X,Y)$, and if $\dim X = \infty$ all inclusions in (4) are, actually, proper inclusions.

Proposition 3.4. *(a) Let* $m,n \in N_X$, $m \leq n$, *then we have* $(n,n)_{x_0}(X,Y) \subset (m,m)_{x_0}(X,Y)$ *and if* $m < n$ *we have* $(n,n)_{x_0}(X,Y) \neq (m,m)_{x_0}(X,Y)$.
(b) If $\dim X = \infty$ *then* $(1,X)^1_{x_0}(X,Y) \subset (n,n)_{x_0}(X,Y)$ $\forall n \in \mathbb{N}^*$.

Proof. Let $f \in (n,n)_{x_0}(X,Y)$. If $\hat{S} \in \mathcal{S}_{(m,m)}$ then $\hat{S}$ is an open neighbourhood of 0 in an m-dimensional subspace of X. Let $\{v_1,...,v_m\}$ be a base for this subspace and let $\{v_{m+1},...,v_n\}$ be such that $\{v_1,...,v_n\}$ is linearly independent and write $S = \langle\{v_1,...,v_n\}\rangle$ the subspace generated by $\{v_1,...,v_n\}$. We have $\hat{S} \subset S$ and $S \in \mathcal{S}_{(n,n)}$. So (1) holds for f through S and, by Lemma 2.2, through $\hat{S}$ too.
Thus $f \in (m,m)_{x_0}(X,Y)$.
Assume now $m < n$. Let $A \equiv \mathbb{R}^n$, $A \subset X$ and $\phi : \mathbb{R}^n \to A$ be an isomorphism.
Define $\gamma : \mathbb{R} \to \mathbb{R}^n$ by $\gamma(t) = (t,t^2,t^3,...,t^n)$ and let $\varphi : X \to Y$ be such that $\|\varphi(x_0 + \phi(\gamma(t)))\| = \|\phi(\gamma(t))\|$, $\forall t \in \mathbb{R}$, and $\varphi(x) = 0$ otherwise.
For all subspace $S \subset A \subset X$ such that $\dim S = m < n$ let $L = \phi^{-1}(S) \subset \mathbb{R}^n$. Clearly, $\dim L = m < n$ and we can choose $u = (u_1,...,u_n) \in \mathbb{R}^n - \{0\}$ such that $\langle u,v\rangle = 0$ for all $v \in L$. Calling j the first integer such that $u_j \neq 0$, we write
$\langle\gamma'(t),u\rangle = (u_1+2tu_2+...+nt^{n-1}u_n) = t^{j-1}(ju_j+(j+1)u_{j+1}t+...+nt^{n-j}u_n)$
and choose $\varepsilon > 0$ such that the sign of $\langle\gamma'(t),u\rangle$ is that of u_jt^{j-1} on $(-\varepsilon,\varepsilon)$. Thus, 0 is the only solution for $\gamma(t) \in L$, $t \in (-\varepsilon,\varepsilon)$. This implies that

there exists U_L, neighbourhood of 0 in L such that $U_L \cap \gamma(\mathbb{R}) = \{0\}$ and, consequently, $U = \phi(U_L)$ is a neighbourhood of 0 in S such that $U \cap \phi(\gamma(\mathbb{R})) = \{0\}$. Taking $T = 0 \in L(X,Y)$, since $\varphi = 0$ on U, (1) is fulfilled for φ through S. Now, since for all subspace $S \subset X$ with $\dim S = m$ we have $\dim(S \cap A) \leq m$, we conclude that $\varphi \in (m,m)_{x_0}(X,Y)$, with ${}^{\mathcal{S}_{(m,m)}}d\varphi(x_0) = 0$.
Now, if $\varphi \in (n,n)_{x_0}(X,Y)$, by Proposition 3.1 (c), we would have ${}^{\mathcal{S}_{(n,n)}}d\varphi(x_0) = {}^{\mathcal{S}_{(m,m)}}d\varphi(x_0) = 0$. But (1) is not true for φ with $T = 0$ through $S = A \in \mathcal{S}_{(n,n)}$, since $\phi(\gamma(\mathbb{R})) \subset A$. Then $\varphi \notin (n,n)_{x_0}(X,Y)$.

(b) For $n \in N_X$, we have $\mathcal{S}_{(n,n)} = \mathcal{S}^1_{(n,n)} \subset \mathcal{S}^1_{(n,X)}$ and then $(n,X)^1_{x_0}(X,Y) \subset (n,n)_{x_0}(X,Y)$ but $(n,X)^1_{x_0}(X,Y) = (1,X)^1_{x_0}(X,Y)$ (Theorem 3.3).
So, if $\dim X = \infty$, we will have $(1,X)^1_{x_0}(X,Y) \subset (n,n)_{x_0}(X,Y)$ $\forall n \in \mathbb{N}^*$. □

We observe that the function φ given in the proof of previous proposition satisfies conditions $(\mathcal{O})_{x_0}$, since $\|\varphi(x_0 + \phi(\gamma(t))\| = \|\phi(\gamma(t))\|$ for $t \in \mathbb{R}$ and $\varphi(x) = 0$ otherwise. We could give $g \in (m,m)_{x_0}(X,Y)$ with $g \notin (n,n)_{x_0}(X,Y)$ such that g doesn't satisty $(\mathcal{O})_{x_0}$ simply putting $\|g(x_0 + \phi(\gamma(t))\| = 1$ $\forall t \in \mathbb{R}$ and $g(x) = 0$ otherwise, for example. Then we see that, if $m < n$, $\underline{(n,n)}_{x_0}(X,Y) \subsetneq \underline{(m,m)}_{x_0}(X,Y)$ and also $(n,n)_{x_0}(X,Y) - \underline{(n,n)}_{x_0}(X,Y) \subsetneq (m,m)_{x_0}(X,Y) - \underline{(m,m)}_{x_0}(X,Y)$.

4. The concept of "vias" and "via-differentiability"

We will extend, now, the ideas of previous section, by taking more general classes of embedded submanifolds of X, not restricted only to finite dimensions neither only to Fréchet-differentiability in the C^1 case. These classes will appear through the concept of "vias of Banach-space". Each via, let us call one by π, for example, will generate a class ${}^{\pi}\mathcal{S}_X$ of embedded submanifolds of X, or simply $\mathcal{S}_\pi$, when X is fixed, and a class $\pi_{x_0}(X,Y)$ of functions $f : U \subset X \to Y$, U neighbourhood of x_0, which we will call the class of π-differentiable functions at x_0, exactly as in Definition 3.2, for $\mathcal{S} = \mathcal{S}_\pi$. In this case, $\mathcal{S}_{x_0}(X,Y) = \pi_{x_0}(X,Y)$. The classes of embedded submanifolds of the previous section (Definition 3.1) will be the classes determinated by some particular "vias".

We begin presenting a way of comparing Banach spaces. Let A and B be Banach spaces.

Definition 4.1. *i) (Transport from A into B). A linear continuous mapping $\psi : A \to B$ is a transport from A into B if and only if ψ is injective*

and $\psi(A)$ is a closed set in B. Whenever ψ is a transport we may also say that A is transported into B by ψ.
The Open Mapping Theorem guaranties that ψ is an isomorphism, i.e., linear homeomorphism, betweem the Banach spaces A and $\psi(A) \subset B$. Observe that the composition of transports is still a transport.
ii) ($\tilde{\leq}$). We will write $A\tilde{\leq}B$ if and only if A can be transported into B, i.e., there is $\psi : A \to B$, ψ transport.
Since $\tilde{\leq}$ is reflexive and transitive, $\tilde{\equiv}$ defined by $A\tilde{\equiv}B \Leftrightarrow (A\tilde{\leq}B \wedge B\tilde{\leq}A)$ is an equivalence relation. We will denote by $[\tilde{A}]$ the equivalence class modulo $\tilde{\equiv}$ of A. Note that $\tilde{\leq}$ induces an order, $\leq$, in the set of classes defined by $[\tilde{A}] \leq [\tilde{B}] \Leftrightarrow A\tilde{\leq}B$. It is easily checked that $\leq$ is well defined and, it is an order relation. For simplicity of notation we will write $\leq$ instead of $\tilde{\leq}$ for Banach spaces too. The equivalence relation that will be used in this work is contained in $\tilde{\equiv}$ and is given by:
iii) $A \equiv B \Leftrightarrow A$ and B are isomorphic, i.e., there exists ψ subjective transport from A onto B.

Clearly, $A \equiv B \Rightarrow A\tilde{\equiv}B$. We observe that if A and B are Hilbert spaces then we have $A\tilde{\equiv}B \Leftrightarrow A \equiv B$. We will denote $[A]$ the equivalence class of A modulo $\equiv$. Observe that if $A \leq B$ and ψ is a transport from A into B then $A \equiv \psi(A)$.

Proposition 4.1. *Let be given non zero vectors $u, v \in X$. Then there is a transport $\psi : X \to X$ such that $\psi(u) = v$.*

Proof. If $v = tu$ for some $t \in \mathbb{R}^*$ then take $\psi(x) = tx$. If u and v are linearly independent then, using Hahn-Banach Theorem, it is possible to choose W a closed subspace of X with co-dimension 2 such that $X = \langle\{u, v\}\rangle \oplus W$. Let g be a transport from $\langle\{u, v\}\rangle$ onto $\langle\{u, v\}\rangle$ such that $g(u) = v$, a multiple of rotation for example. Now, take $\psi \in L(X, Y)$ given by
$\psi(x) = g(x_1) + x_2$ where $x = x_1 + x_2$, $x_1 \in \langle\{u, v\}\rangle$ and $x_2 \in W$. □

Now we define what we will call a "via of Banach spaces". This definition is solely dependent on the equivalence classes of Banach spaces.

Definition 4.2. *Let $A_1, B_1, ..., A_k, B_k, k \in \mathbb{N}^*$, be Banach spaces. The k-tuple of ordered pairs $(([A_1], [B_1]), ..., ([A_k], [B_k]))$ will be called a* **via** *of Banach spaces with length k if and only if $A_1 \leq B_1 \leq ... \leq A_k \leq B_k$.*
We will use the notation $(A_1, B_1)\_...\_(A_k, B_k)$ for the via and will often denote vias by greek letters. Let π be a via. Its length will be denoted by $l(\pi)$, its first element, $[A_1]$, by $\alpha(\pi)$ and its last element $[B_k]$ by $w(\pi)$.

Definition 4.3. *Given a Banach space A, we say that a via π is A-admissible if and only if $w(\pi) \leq A$.*
Observe that, given vias $\pi_1, ..., \pi_m$, such that π_i is $\alpha(\pi_{i+1})$ admissible, $1 \leq i \leq m-1$, we can form the via $\pi = \pi_1\_\pi_2\_...\_\pi_m$ in a natural way and we will have $l(\pi) = \sum_{i=1}^{m} l(\pi_i)$.

Now we are in position to define the concept of π-differentiability of a function $f : U \to Y$, at $x_0 \in U, U$ open set in X for an X-admissible via π. In the same way of the previous section, we are interested in the existence of a continuous linear operator $T \in L(X,Y)$, that we will call π-derivative (π-differential) of f at x_0, such that the relation (1) is fulfilled for f through all sets $S \subset X$ that are characterized by the via π as follows: If $\pi = (A,B)$ or $\pi = \sigma\_(A,B), \sigma$ an A-admissible via, the set S is the transport into X of the image of some embedding from A to B. These embeddings are topological when $\pi = (A,B)$ or of class C^1, in the σ-differentiability sense, when $\pi = \sigma\_(A,B)$. These sets S are said to be of type π.
The notions of π-differentiability on π-type sets are made precise in the following definition, where we used recursion over $k = l(\pi)$.

Definition 4.4. *Let $f : U \to Y, x_0 \in U$ open set of X be given and π be an X-admissible via with $l(\pi) = k$.*
If $k = 1$, $\pi = (A,B)$, we will say that $S \subset X$ is a set of type π (or a π-type set) if and only if statement (i) holds and that f is π-differentiable at x_0 if and only if statement (ii) holds.
(i) $\exists \psi : B \to X$ transport $\exists \gamma : U_\gamma \subset A \to B$ topological embedding, $0 \in U_\gamma$ open set of $A, \gamma(0) = 0$, such that $S = \psi(\gamma(U_\gamma))$.
(ii) $\exists T \in L(X,Y)$ such that relation (1) if fulfilled for f with T through S, for all $S \subset X$, S a π-type set.
We will call the linear operator, T, a π-derivative of f at x_0 or, equivalently, a π-differential of f at x_0.
If $k > 1$ let $\pi = \sigma\_(A,B)$, $l(\sigma) = k-1$. By recursion we assume that all sets $\tilde{S} \subset \tilde{X}$ of type λ, for all Banach spaces $\tilde{X}$ and all $\tilde{X}$-admissible vias λ with $l(\lambda) < k$, are already defined. Analogously, for all Banach spaces $\tilde{Y}$, we assume that the λ-differentiability of functions $g : U_g \subset \tilde{X} \to \tilde{Y}$ at $\tilde{x} \in U_g$ open set of $\tilde{X}$, are also already defined.
We will say that $S \subset X$ is of type π (or a π-type set) if and only if relation (iii) holds.
(iii) item (i) with the additional conditions that $\forall u \in U_\gamma$, γ is σ-differentiable at u and that there is a continuous function $\Gamma : U_\gamma \to L(A,B)$ such that for all $u \in U_\gamma$, $\Gamma(u)$ is not only a σ-derivative of γ at u but also a transport from A into B.

Having defined π-type sets, we define π-differentiability of f at x_0 by (ii).

Observe that the definition of π-type sets for a given via π does not depend on the particular Banach spaces that form the via but only on their equivalence classes. We will denote ${}^{\pi}\mathcal{S}_X$ the class of π-type sets in X, and when X is fixed, we can write simply $\mathcal{S}_\pi$.

Proposition 4.2 (Uniqueness). *Under the same notation as in Definition 4.4, if f is π-differentiable at x_0, the operator T that appears in relation (ii) is* **unique**.

Proof. Consequence of the following facts:
Every continuous linear function f is π-differentiable at all points of its domain whatever the X-admissible via π is, since $T = f$ is its π-differential.
Given a via, $\pi = (A, B)$ or $\pi = \sigma_-(A, B)$, and a vector $v \in X,\ v \neq 0$, the straight line $\langle\{v\}\rangle$ is contained in a set $S = \psi(\gamma(A))$ for some transport $\gamma \in L(A, B)$ and ψ from B into X. Observe that γ is σ-differentiable at u, $\forall u \in A$, and that ψ is chosen, if necessary, as a composition of a transport from B into X and another transport from X into X where we use Proposition 4.1 to obtain $\langle\{v\}\rangle \subset S$.
Letting $T_1, T_2 \in L(X, Y)$ satisfy (1), given an arbitrary $v \in X, v \neq 0$, take a π-type set S such that $\langle\{v\}\rangle \subset S$. Then, by Lemma 2.2, we have, for $i = 1, 2$

$$\lim_{\substack{h \to 0 \\ h \in \langle\{v\}\rangle}} \frac{f(x_0 + h) - f(x_0) - T_i(h)}{\|h\|} = \lim_{\substack{h \to 0 \\ h \in S}} \frac{f(x_0 + h) - f(x_0) - T_i(h)}{\|h\|} = 0.$$

Now, $h \to 0,\ h \in \langle\{v\}\rangle$ means $h = tv, t \in \mathbb{R}, t \to 0$ which implies $T_1(v) = T_2(v)$ and $T_1 = T_2$. $\square$

We will denote the π-derivative of f at x_0 by ${}^{\pi}f'(x_0)$ or ${}^{\pi}df(x_0)$.

Definition 4.5. *Let π be an X-admissible via. The class of π-differentiable functions at x_0 is the set*

$$\pi_{x_0}(X, Y) = \{f : U \to Y | x_0 \in U \textit{ open set of } X \textit{ and } f \textit{ is } \pi\textit{-differentiable at } x_0\}.$$

Proposition 4.3 (Operation rules). *With pointwise addition and scalar multiplication, $\pi_{x_0}(X, Y)$ is a real vector space and the operator D_{x_0} defined by*

$$D_{x_0} : \pi_{x_0}(X, Y) \to L(X, Y),\ D_{x_0}(f) =^{\pi} f'(x_0), \textit{is linear.}$$

Proposition 4.4 (Leibniz rule.). *Let Y be a Banach Algebra and define the product of f and g in $\pi_{x_0}(X, Y)$ as the pointwise product. If f or g is continuous at x_0 then $fg \in \pi_{x_0}(X, Y)$ and*

$${}^{\pi}d(fg)(x_0) =^{\pi} df(x_0)g(x_0) + f(x_0)^{\pi}dg(x_0) \in L(X, Y).$$

The proof of both propositions above follow the steps of the proofs of analogous propositions for Fréchet-differentiable functions, using the uniqueness of via derivate and Lemmas 2.1 and 2.2 for calculating limits through sets. For simplicity of notation when $\dim A = m < \infty$, that is $A \equiv \mathbb{R}^m$, we will denote the via (A, B) by (m, B). If $B \equiv \mathbb{R}^n$, $m \leq n$, we will simply write (m, n).
Taking $\pi = (m, n)$ an X-admissible via (that is, $m, n \in N_X$, $m \leq n$), we see that $\mathcal{S}_{(m,n)}$ is exactly the same given in Definition 3.1. We can say the same for vias (m, X) and (X, X), that is, $\mathcal{S}_{(m,X)}$ and $\mathcal{S}_{(X,X)}$ are the same as the ones given in Definition 3.1.
Then the sets $(m, n)_{x_0}(X, Y)$, $(m, X)_{x_0}(X, Y)$ and $(X, X)_{x_0}(X, Y)$ each one seen as the set of π-differentiable functions at x_0 for $\pi = (m, n)$, $\pi = (m, X)$ and $\pi = (X, X)$, respectively, are the same sets given in Definition 3.2.
To see that the other classes $\mathcal{S}$ given in Definition 3.1 are also the type sets of particular "vias" and to identify the corresponding vias, let us first show:

Proposition 4.5. *For every Banach space Y the following are valid:*
i) The vias $U_k = (1,1)\_\ldots\_(1,1)$ with $l(U_k) = k, k \in \mathbb{N}^*$, are all equivalent in the sense that $(U_k)_{x_0}(X, Y) = G_{x_0}(X, Y)$ with ${}^{U_k}f'(x_0) = \delta f(x_0, .)$ for all $f \in (U_k)_{x_0}(X, Y)$. Moreover, $\pi_{x_0}(X, Y) \subset G_{x_0}(X, Y)$ for all X-admissible via π and ${}^{\pi}f'(x_0) = \delta f(x_0, .)$ for all $f \in \pi_{x_0}(X, Y)$.
ii) If $f \in (\pi_1)_{x_0}(X, Y) \cap (\pi_2)_{x_0}(X, Y)$ where π_1 and π_2 are X-admissible vias, then ${}^{\pi_1}f'(x_0) = {}^{\pi_2} f'(x_0)$.
iii) The vias (X, X), $\sigma\_(X, X)$, where σ is any X-admissible via, and $(1, X)$ are all equivalent, in the sense that $(1, X)_{x_0}(X, Y) = \sigma\_(X, X)_{x_0}(X, Y) = (X, X)_{x_0}(X, Y) = F_{x_0}(X, Y)$, with ${}^{(X,X)}f'(x_0) = f'(x_0)$ for all $f \in (X, X)_{x_0}(X, Y)$. Moreover $F_{x_0}(X, Y) \subset \pi_{x_0}(X, Y)$ for all X-admissible via π.

Proof. i) $\forall k \in \mathbb{N}^*$, $S \subset X$ is a set of type U_k if and only if S is an open neighbourhood of 0 in some one-dimensional subspace of X, that is, $\mathcal{S}_{U_k} = \mathcal{S}_{(1,1)}$.
Then, from Proposition 3.1(a) we have $(U_k)_{x_0}(X, Y) = (1, 1)_{x_0}(X, Y) = G_{x_0}(X, Y)$. From uniqueness of ${}^{U_k}f'(x_0)$ we have ${}^{U_k}f'(x_0) = \delta f(x_0, .)$.
Given π an X-admissible via and $v \in X$, $v \neq 0$, choose a π-type set $S \subset X$ such that $\langle\{v\}\rangle \subset S$. Let $f \in \pi_{x_0}(X, Y)$ and $T =^{\pi} f'(x_0)$. Then

$$\lim_{\substack{h \to 0 \\ h \in S}} \frac{f(x_0 + h) - f(x_0) - T(h)}{\|h\|} = 0 \Longrightarrow \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(x_0 + tv) - f(x_0) - tT(v)}{t} = 0$$

by Lemma 2.2, and so, $f \in G_{x_0}(X, Y)$ and $T =^{\pi} f'(x_0) = \delta f(x_0, .)$.

This completes the proof of (i) and it also proves (ii) since, under the hypothesis of item (ii), we have ${}^{\pi_1}f'(x_0) ={}^{\pi_2} f'(x_0) = \delta f(x_0,.)$.

iii) Since $\mathcal{S}_{(X,X)} = \mathcal{S}_{\sigma_-(X,X)}$ is the class of all open neighbourhood of 0 in X, we have $\sigma\_(X,X)_{x_0}(X,Y) = (X,X)_{x_0}(X,Y)$ and from Proposition 3.1 (b) we have $(X,X)_{x_0}(X,Y) = F_{x_0}(X,Y)$, and ${}^{(X,X)}f'(x_0) = f'(x_0)$ $\forall f \in (X,X)_{x_0}(X,Y)$.

Now, given an X-admissible via π, using $\mathcal{S} = \mathcal{S}_\pi$ in the proof of Proposition 3.1 (c), we conclude that $F_{x_0}(X,Y) \subset \pi_{x_0}(X,Y)$.

Let us prove now that $(1,X)_{x_0}(X,Y) = F_{x_0}(X,Y)$.

We already have $F_{x_0}(X,Y) \subset (1,X)_{x_0}(X,Y)$.

Suppose that there is a function $f \in (1,X)_{x_0}(X,Y)$ which is not in $F_{x_0}(X,Y)$. Since f is not Fréchet differentiable at x_0 we have $\forall T \in L(X,Y)$ $\exists \epsilon_0 > 0$ such that $\forall \delta > 0$, $\exists h_\delta \in X,\ 0 < ||h_\delta|| < \delta$ with

$$\frac{\|f(x_0 + h_\delta) - f(x_0) - T(h_\delta)\|}{\|h_\delta\|} \geq \epsilon_0 \tag{5}$$

and, in particular, this statement is true for $T ={}^{(1,X)} f'(x_0)$.

Construct $\{\delta_i\}_{i\in\mathbb{N}}$ a sequence of positive real numbers, $\delta_i \downarrow 0$, and a sequence of points $\{h_i\}_{i\in\mathbb{N}}, h_i = h_{\delta_i}$ in such a way that (5) is valid for all $i \in \mathbb{N}$ and $0 < \delta_{i+1} < \|h_{\delta_i}\|/2 < \delta_i/2$. This guaranties the monotonicity of the sequence $\{h_i\}_{i\in\mathbb{N}}$, i.e. $\|h_i\| \downarrow 0$.

Now observe that there can not be any vector $v \in X,\ v \neq 0$, such that the line $\langle\{v\}\rangle$ contains infinitely many points h_i, since the subspace $\langle\{v\}\rangle$ is of type $(1,X)$ and, being $f \in (1,X)_{x_0}(X,Y)$, (1) is valid for f through $S = \langle\{v\}\rangle$ for all $v \in X, v \neq 0$. Take $u \in X,\ u \neq 0$ and $W \subset X$ such that $X = \langle\{u\}\rangle \oplus W$. Let

$$X_+ = \{tu + w | w \in W, t \in \mathbb{R}_+\} \text{ and } X_- = \{tu + w | w \in W,\ t \in \mathbb{R}_-\}.$$

Now, infinitely many h_i must belong to either X_+ or X_-. Without loss of generality suppose they are in X_+ and rename them h_j so that we write the sequence $\{h_j\}_{j\in\mathbb{N}} \subset X_+$.

Given $x, y \in X$. Let us denote the set $\{px + (1-p)y | p \in [0,1]\}$ by $[x,y]$ and the open ball centered at x with radius r by $B(x,r)$.

Choose a subsequence $\{h_{j_k}\}_{k\in\mathbb{N}} \subset \{h_j\}_{j\in\mathbb{N}}$ such that $j_1 = 1$ and, for all $k \in \mathbb{N}^*$, j_{k+1} is such that $0 \notin [h_{j_{k+1}}, h_{j_k}]$ and $B(0, \delta_{j_{k+1}}) \bigcap (\bigcup_{l=1}^{k-1} [h_{j_{l+1}}, h_{j_l}]) = \emptyset$. This can be done because $\langle\{h_{j_k}\}\rangle$ contains at most finitely many point h_j and, by construction, $\delta_i \downarrow 0$ and $\|h_i\| \downarrow 0$. For simplicity of notation, call $\{h_{j_k}\}_{k\in\mathbb{N}}$ simply $\{h_m\}_{m\in\mathbb{N}}$.

Now we can construct a polygonal $\gamma : (-1,1) \to X$ which image is the set $(-u, 0] \bigcup (\bigcup_{m=1}^{\infty} [h_m, h_{m+1}])$ by writting $\gamma(t) = tu$ for $-1 < t \leq 0$, and for all $m \in \mathbb{N}^*$, $\gamma(t) = p_m(t)h_{m+1} + (1 - p_m(t))h_m$

where $p_m(t) = \dfrac{\dfrac{1}{m} - t}{\dfrac{1}{m} - \dfrac{1}{m+1}}$ for $\dfrac{1}{m+1} \leq t \leq \dfrac{1}{m}$.

Observe that, by construction, $(-u, 0] \subset X_-$; $[h_{m+1}, h_m] \subset X_+$ for all $m \in \mathbb{N}^*$ and, for all $m, l \in \mathbb{N}^*$, $m \neq l$, $[h_{m+1}, h_m] \cap [h_{l+1}, h_l] \subset \{h_{m+1}, h_m\}$, so that γ has no self crossings.

Clearly, $\gamma(t) \to 0$ as $t \to 0$. Now, γ is an homeomorphism over its image and, as we can take $\psi : X \to X$ the identity map, $S = \gamma((-1, 1))$ is a set of type $(1, X)$.

On the other hand, (1) can not be true for f through S since $\{h_m\} \subset S$ and (5) holds for h_m. This is a contradiction with $f \in (1, X)_{x_0}(X, X)$.

So, f is Fréchet differentiable at x_0 and we have $(1, X)_{x_0}(X, Y) = F_{x_0}(X, Y)$.

□

Taking in view Proposition 4.5 (iii), we see that $(A, A)_-(A, B)$ and $\sigma_-(A, A)_-(A, B)$, with $B \leq X$, have the same class of embedded manifolds, precisely:

$\mathcal{S}_{(A,A)_-(A,B)} = \mathcal{S}_{\sigma_-(A,A)_-(A,B)} = \mathcal{S}^1_{(A,B)} = \{\psi(\gamma(U_\gamma)) \subset X | \psi$ is a transport from B into $X, \gamma : U_\gamma \subset A \to B$, C^1-embedding in Fréchet sense, U_γ open neighbourhood of 0 in A, $\gamma(0) = 0\}$.

Then, in particular, we have that the classes $\mathcal{S}^1_{(m,n)}$ and $\mathcal{S}^1_{(m,X)}$ given in Definition 3.1 (with $m, n \in N_X \quad m \leq n$), are the classes for the vias $(m, m)_-(m, n)$ and $(m, m)_-(m, X)$ respectively.

Then the classes of functions $(m, n)^1_{x_0}(X, Y)$ and $(m, X)^1_{x_0}(X, Y)$ given in Definition 3.2 are the classes of functions $(m, m)_-(m, n)$-differentiable at x_0 and $(m, m)_-(m, X)$-differentiable at x_0, respectively, that is:

$(m, n)^1_{x_0}(X, Y) = (m, m)_-(m, n)_{x_0}(X, Y)$ and

$(m, X)^1_{x_0}(X, Y) = (m, m)_-(m, X)_{x_0}(X, Y)$.

Now we see that all classes of differentiabilities of previous section are included and generalized with the concept of via-differentiability.

The notions of type unions and strong-differentiability are also generalized for vias following the same way of what was done in previous section.

We point out all this here.

Let π, an X-admissible via, be given:

Definition 4.6. *We will call $W \subset X$ a π-type union if and only if W is an $\mathcal{S}$-type union (see Definition 3.3) for $\mathcal{S} = \mathcal{S}_\pi$.*

Proposition 4.6. *$f \in \pi_{x_0}(X,Y)$ with ${}^{\pi}df(x_0) = T \in L(X,Y)$ if only if (3) is valid for f with $\mathcal{S} = \mathcal{S}_\pi$, that is,*
$\forall \epsilon > 0$, $\exists W \subset X$ a π-type union $|\forall h \in W$, $\|f(x_0+h)_-f(x_0)-Th\| \leq \epsilon\|h\|$.

Proof. Same proof of Proposition 3.2. □

Proposition 4.7. *For every Banach space Y we have:*
(a) the statements (i) and (ii) are equivalent:
(i) If $W \subset X$ is a π-type union, then W is a neighbourhood of 0 in X.
(ii) $\pi_{x_0}(X,Y) = F_{x_0}(X,Y)$.
(b) If $\pi_{x_0}(X,Y) \neq F_{x_0}(X,Y)$ we can find functions f, g in $\pi_{x_0}(X,Y)$ which are not in $F_{x_0}(X,Y)$ such that f satisfies $(\mathcal{O})_{x_0}$ and g doesn't satisfy $(\mathcal{O})_{x_0}$.

Proof. Same proof of Proposition 3.3, using $\mathcal{S} = \mathcal{S}_\pi$, $\mathcal{S}_{x_0}(X,Y) = \pi_{x_0}(X,Y)$, ${}^{\mathcal{S}}df(x_0) = {}^{\pi} df(x_0)$. □

Definition 4.7. *We will say that a function $f \in \pi_{x_0}(X,Y)$ is strongly π-differentiable at x_0, and write $f \in \underline{\pi}_{x_0}(X,Y)$, if f satifies $(\mathcal{O})_{x_0}$.*

By Proposition 4.7 (b), if $\pi_{x_0}(X,Y) \neq F_{x_0}(X,Y)$, then
$F_{x_0}(X,Y) \subsetneq \underline{\pi}_{x_0}(X,Y) \subsetneq \pi_{x_0}(X,Y)$.
Also, for $f \in \pi_{x_0}(X,Y)$, if we take any π-type union W given in (3), for any $\epsilon > 0$, then the inequality $||f(x_0+h) - f(x_0)|| \leq M||h||$ is satisfied for $h \in W$.
For what we saw in previous section, we know that
$\underline{(n,n)_-(n,X)}_{x_0}(X,Y) = H_{x_0}(X,Y), \quad \forall n \in N_X$.
Let us see now some general properties of π-differentiability.
Observe that there is a natural bijection between $\pi_{x_0}(X,Y)$ and $\pi_{x_1}(X,Y)$, x_0 and x_1 in X, since for every $f \in \pi_{x_0}(X,Y)$ we can define $\tilde{f} : \tilde{U} \to Y$ by $\tilde{f}(x - x_0 + x_1) = f(x)$, which yields $\tilde{f} \in \pi_{x_1}(X,Y)$.
This observation was made in order to make clear that the definitions that follow will not depend on the particular $x_0 \in X$, so that we can take it to be $0 \in X$.
In order to compare vias, let $\pi(X,Y) = \pi_0(X,Y)$.

Definition 4.8. *i) Given π_1, π_2, two X-admissible vias, we will say that π_1 is weaker than π_2, equivalently, π_2 is stronger than π_1, and write $\pi_1 \tilde{\leq} \pi_2$ if and only if $\pi_1(X,Y) \supset \pi_2(X,Y)$ for each Banach space Y.*
As the relation $\tilde{\leq}$ is reflexive and transitive we can obtain an equivalence relation over the vias defining
ii) $\pi_1 \equiv \pi_2 \Leftrightarrow \pi_1(X,Y) = \pi_2(X,Y)$ for each Banach space Y.
As previously done, $\tilde{\leq}$ induces an order relation $\leq$ in the set of equivalence classes of vias. We denote the equivalence class of π by $[\pi]$ and we will also use $\leq$ instead of $\tilde{\leq}$ for vias.

By Proposition 3.4 we have: $(m,m) \leq (n,n) \leq (1,1)_{-}(1,X)$ $\forall m,n \in N_X,\ m \leq n$.
If $\dim X = n < \infty$ we have $(n,n) \equiv (1,1)_{-}(1,X)$
If $\dim X = \infty$ we have $(1,1) \leq (2,2) \leq ... \leq (n,n) \leq ... \leq (1,1)_{-}(1,X)$ $\forall n \in \mathbb{N}^*$.
By Proposition 4.5 we have: $(1,1) \leq \pi \leq (X,X)$ $\forall \pi, X$-admissible via.
Let π_1, π_2 be X-admissible vias. The following results are immediate:

Proposition 4.8. *(a) If $\mathcal{S}_{\pi_1} \subset \mathcal{S}_{\pi_2}$ then $\pi_1 \leq \pi_2$.*
(b) If for each $\hat{S} \in \mathcal{S}_{\pi_1}$ we have $\exists S \in \mathcal{S}_{\pi_2}$ with $\hat{S} \subset S$, then $\pi_1 \leq \pi_2$.
(c) If for each $\hat{S} \in \mathcal{S}_{\pi_1}$ we have $\exists$ $S \in \mathcal{S}_{\pi_2}$ and $\tilde{S} \in \mathcal{S}_{\pi_1}$ with $\tilde{S} \subset \hat{S} \cap S$, then $\pi_1 \leq \pi_2$.

Proof. Given a Banach space Y, take $f \in \pi_2(X,Y)$. Then (1) is valid for f with $T =^{\pi_2} df(0)$, through S, $\forall S \in \mathcal{S}_{\pi_2}$.
(a) If $\mathcal{S}_{\pi_1} \subset \mathcal{S}_{\pi_2}$, given $\hat{S} \in \mathcal{S}_{\pi_1}$, (1) is valid for f through $\hat{S}$, since $\hat{S} \in \mathcal{S}_{\pi_2}$. Then $f \in \pi_1(X,Y)$.
(b) Given $\hat{S} \in \mathcal{S}_{\pi_1}$, take $S \in \mathcal{S}_{\pi_2}$ with $\hat{S} \subset S$. Since (1) is valid for f through S, by Lemma 2.2, (1) is valid for f through $\hat{S}$. Then $f \in \pi_1(X,Y)$.
(c) Given $\hat{S} \in \mathcal{S}_{\pi_1}$, take $S \in \mathcal{S}_{\pi_2}$, $\tilde{S} \in \mathcal{S}_{\pi_1}$ with $\tilde{S} \subset \hat{S} \cap S$. Since $\tilde{S} \subset \hat{S}$ and $\tilde{S} \in \mathcal{S}_{\pi_1}$ means that $\tilde{S}$ is an open neighbourhood of 0 in $\hat{S}$, and since (1) is valid for f through S, by Lemma 2.2, we have both, (1) is valid for f through $\tilde{S}$ and also through $\hat{S}$. Then $f \in \pi_1(X,Y)$.

□

Theorem 4.1. *If $A \leq B \leq X$ then*
(i) $(A,A) \equiv \sigma_{-}(A,A)$ for all A-admissible via σ.
(ii) $(A,B) \leq (B,B)$
(iii) If X is a Hilbert space then $(A,A) \leq (B,B)$.

Proof. (i) Since $\mathcal{S}_{\sigma_{-}(A,A)} = \mathcal{S}_{(A,A)} = \{V \subset E \subset X | E$ subspace of X isomorphic to A, V open neighbourhood of 0 in $E\}$, we have $\sigma_{-}(A,A)(X,Y) = (A,A)(X,Y)$ for each Banach space Y.
(ii) If $\hat{S} \in \mathcal{S}_{(A,B)}$ then $\exists$ E subspace of X isomorphic to B and S open neighbourhood of 0 in E, that is, $S \in \mathcal{S}_{(B,B)}$, such that $\hat{S} \subset S$. By Proposition 4.8 (b) we have $(A,B) \leq (B,B)$.
(iii) The finite dimensional case is already proved in Proposition 3.4.
If $\dim X = \infty$, write $X = l_2(\Gamma)$, for some infinite Γ. There is a family $\{v_s\}_{s\in\Gamma}$ of linearly independent vectors of X that form a Bessel base for X and $B \leq X \Leftrightarrow B \equiv l_2(\hat{\Gamma})$ for some $\hat{\Gamma} \subset \Gamma$.
If $\hat{S}$ is an (A,A) type set, $\hat{S}$ is an open set of a subspace of X that is isomorphic to A. Now, by its turn, a subspace of X isomorphic to A is

contained in a subspace S of X isomorphic to B, when $A \leq B$, since we can write $A = l_2(\overline{\Gamma}), B = l_2(\hat{\Gamma})$ with $\overline{\Gamma} \subset \hat{\Gamma} \subset \Gamma$ and we can complete $\overline{\Gamma}$ with linear independent vectors to form $\hat{\Gamma}$. Since $S \in \mathcal{S}_{(B,B)}$ and $\hat{S} \subset S$, by Proposition 4.8 (b) we have $(A, A) \leq (B, B)$.

□

Corollary 4.1. *Let σ be an X admissible via. Then $\sigma \leq (\omega(\sigma), \omega(\sigma))$.*

Proof. σ is either (A, B) or $\lambda\_(A, B)$ with $B \leq X$, and $(\omega(\sigma), \omega(\sigma)) = (B, B)$.
Since $\mathcal{S}_{\lambda\_(A,B)} \subset \mathcal{S}_{(A,B)}$, by Proposition 4.8 a) we have $\lambda\_(A, B) \leq (A, B)$ and the proof follows by Theorem 4.1 ii). □

Theorem 4.2. *If $\mathbb{R}^m \leq \mathbb{R}^n \leq A \leq X$, then $(m,m)_-(m,A) \leq (n,n)_-(n,A) \leq (A,A)$.*

Proof. If $X = A$ the proof is the same given in Theorem 3.2. In the general case, to see that $(n,n)_-(n,A)(X,Y) \subset (m,m)_-(m,A)(X,Y)$ we use the same arguments of Theorem 3.2 using A instead of X in the following way: If $\hat{S} \in \mathcal{S}_{(m,m)_-(m,A)}$ then $\hat{S}$ is written as $\hat{S} = \psi(\gamma(V))$, ψ a transport from A to X, and $\gamma : V \subset \mathbb{R}^m \to A$ as in the proof of Theorem 3.2. Thus we put $M = \gamma'(0)\mathbb{R}^m \subset A$, $A = M \oplus W$ and everything goes the same way of there to obtain $\tilde{V} \subset V$, open set in V, and S_1 an $(n,n)_-(n,A)$ type set in A such that $\gamma(\tilde{V}) \subset S_1$.
Then $\tilde{S} = \psi(\gamma(\tilde{V})) \subset S = \psi(S_1)$ which is an $(n,n)_-(n,A)$ type set in X, that is , $S \in \mathcal{S}_{(n,n)_-(n,A)}$. Since $\tilde{S}$ is an open neighbourhood of 0 in $\hat{S}$, by Proposition 4.8 (c), we have $(m,m)_-(m,A) \leq (n,n)_-(n,A)$.
Now, note that if $\hat{S} \in \mathcal{S}_{(m,m)_-(m,A)}$, we have $\hat{S} = \psi(\gamma(V))$ as before and then $\hat{S} \subset S$ for some open neighbourhood of 0 in $E = \psi(A)$, that is, for some $S \in \mathcal{S}_{(A,A)}$.
Thus, by proposition 4.8 (b), we have $(m,m)_-(m,A) \leq (A,A)$. □

Theorem 4.3. *If $\mathbb{R}^n \leq A \leq X$ then $(n,n)_-(n,A) \equiv (1,1)_-(1,A)$.*

Proof. The proof follows the steps of the proof of Theorem 3.3 using Corollary 3.1 and the fact that if $S \in \mathcal{S}_{(n,n)_-(n,A)}$, $S = \psi(\gamma(V))$, ψ a transport from A to X, V open neighbourhood 0 in $\mathbb{R}^n$, $\gamma : V \to A$ C^1-embedding in Fréchet-sense; then, in particular, $\psi \circ \gamma$ is an homeomorsphim from V on S. □

Corollary 4.2. *For all $n \in N_X$ we have $(n,n) \equiv (1,1)_-(1,n)$.*

Proof. Taking $A \equiv \mathbb{R}^n$ in Theorem 4.3 we have $(n,n)_-(n,n) \equiv (1,1)_-(1,n)$. Since $\mathcal{S}_{(n,n)_-(n,n)} = \mathcal{S}^1_{(n,n)} = \mathcal{S}_{(n,n)}$ we have $(n,n)_-(n,n) \equiv (n,n)$. □

Theorem 4.4. *If σ is a via such that $\omega(\sigma) \equiv \mathbb{R}^m \leq X$, then we have $\sigma \equiv (m,m)$.*

Proof. The via σ is written in one of the two forms, either $\sigma = (k,m)$ or $\sigma = \lambda_-(k,m)$, for $k = 1,2,...m$. Now we have $(1,1)_-(1,m) \equiv (k,k)_-(k,m)$ (Theorem 4.3). Since $\mathcal{S}_{(k,k)_-(k,m)} = \mathcal{S}^1_{(k,m)} \subset \mathcal{S}_{\lambda_-(k,m)} \subset \mathcal{S}_{(k,m)}$ we have $(k,k)_-(k,m) \leq \lambda_-(k,m) \leq (k,m)$ (Proposition 4.8(a)). Also we have $(k,m) \leq (m,m)$ (Theorem 4.1 (ii)) and $(m,m) \equiv (1,1)_-(1,m)$ (Corollary 4.2). Then $(1,1)_-(1,m) \leq (k,k)_-(k,m) \leq \lambda_-(k,m) \leq (k,m) \leq (m,m) \leq (1,1)_-(1,m)$, that is, these vias are all equivalent. $\square$

Corollary 4.3. *For $X = \mathbb{R}^n$ there are exactly n different types of via-differentiabilities.*

Proof. By Proposition 3.4 we have $(i \neq j) \Longrightarrow (i,i) \not\equiv (j,j)$ and the conclusion follows from Theorem 4.4. $\square$

Theorem 4.5. *If $\pi \equiv \sigma_-\theta$ where $\omega(\sigma) \equiv \mathbb{R}^m \leq \alpha(\theta)$, then $\pi \equiv (m,m)_-\theta$.*

Proof. By induction on $l(\theta) = k \in N_X$.
For $k = 1$, $\theta = (A,B)$ and given a Banach space Y, $f \in (m,m)_-(A,B)(X,Y)$ means that (1) is fulfilled for f through S, for all $S = \psi(\gamma(V))$ where, among other things, $\gamma : V \subset A \to B$ is (m,m)-differentiable. But from Theorem 4.4, γ is (m,m)-differentiable if and only if γ is σ-differentiable.
So $f \in (m,m)_-(A,B)(X,Y) \Leftrightarrow f \in \sigma_-(A,B)(X,Y)$ thus $(m,m)_-(A,B) \equiv \sigma_-(A,B)$.
Now, if $k > 1$, $\theta = \lambda_-(A,B)$, and $f \in (m,m)_-\lambda_-(A,B)(X,Y)$ means that (1) is fulfilled for f through S, for all $S = \psi(\gamma(V))$ where, among other things, $\gamma : V \subset A \to B$ is $(m,m)_-\lambda$ differentiable.
But from induction hypothesis, since $l(\lambda) = k-1$, we have that γ is $(m,m)_-\lambda$ differentiable if and only if γ is $\sigma_-\lambda$ differentiable.
So $(m,m)_-\lambda_-(A,B)(X,Y) = \sigma_-\lambda_-(A,B)(X,Y)$ and $\sigma_-\theta \equiv (m,m)_-\theta$. $\square$

Theorem 4.6. *For every X-admissible via π and all $\alpha(\pi)$-admissible via λ, the following "inequalities" are valid:*
(i) $\lambda_-\pi \leq \pi$ if $l(\pi)$ is odd.
(ii) $\pi \leq \lambda_-\pi$ if $l(\pi)$ is even.

Proof. We will prove (i) and (ii) together using induction over $k = l(\pi) \in N_X$. If $k = 1, \pi = (A,B)$, it is clear that if $S \subset X$ is of type $\lambda_-\pi$ then it is of π-type since condition (iii) of Definition 4.4 implies its condition (i). Thus, given a Banach space Y, if $f \in \pi(X,Y)$ and S is of type $\lambda_-\pi$ then relation (1) holds for f with $T =^\pi f'(0)$ through S, which implies that $f \in$

$\lambda_{-}\pi(X,Y)$ and ${}^{\lambda_{-}\pi}f'(0) ={}^{\pi} f'(0)$. So, for $k=1$, $\lambda_{-}\pi(X,Y) \supset \pi(X,Y)$, i.e., $\lambda_{-}\pi \leq \pi$.
For $k>1$, assuming that (i) and (ii) are true for all σ with $l(\sigma) < k$, we write $\pi = \sigma_{-}(A,B)$, $l(\sigma) = k-1$.
Suppose k is even.
Let $f \in \lambda_{-}\pi(X,Y) = \lambda_{-}\sigma_{-}(A,B)(X,Y)$. Thus (1) is valid for f with $T ={}^{\lambda_{-}\pi} f'(0)$, through S, for all S of type $\lambda_{-}\pi$. Since $l(\sigma)$ is odd, from induction hypothesis $\lambda_{-}\sigma \leq \sigma$, that is, $\sigma(A,B) \subset \lambda_{-}\sigma(A,B)$ which implies that $\sigma_u(A,B) \subset \lambda_{-}\sigma_u(A,B)$ for all $u \in A$. Now, if $\gamma : U_\gamma \subset A \to B$ satisfies condition (iii) of Definition 4,4, for the via σ, it will also satisfy this condition for the via $\lambda_{-}\sigma$, with the same function $\Gamma : U_\gamma \to L(A,B)$ where $\Gamma(u) ={}^{\sigma} \gamma'(u) ={}^{\lambda_{-}\sigma} \gamma'(u)$ is a transport from A into B for all $u \in U_\gamma$. This means that all S of type $\sigma_{-}(A,B) = \pi$ are of type $\lambda_{-}\sigma_{-}(A,B) = \lambda_{-}\pi$ and so, (1) will be true for f through every S of type π. Thus $f \in \pi(X,Y)$ with ${}^{\pi}f'(0) ={}^{\lambda_{-}\pi} f'(0)$. So $\pi \leq \lambda_{-}\pi$.
The argument for odd k is similar. □

Note that if $\pi = (A_k,B_k)_{-}\ldots{}_{-}(A_1,B_1)$, by Theorem 4.6, we can write the following comparison of vias:

$$\begin{aligned}(A_2,B_2)_{-}(A_1,B_1) &\leq (A_4,B_4) - \ldots - (A_1,B_1) \leq \ldots \leq \pi \leq \ldots \\ &\leq (A_3,B_3)_{-}(A_2,B_2)_{-}(A_1,B_1) \leq (A_1,B_1).\end{aligned}$$

Now we present vias which strong-differentiabilities lie between Hadamard and Fréchet ones.

Proposition 4.9. *For $n \in N_X$ and σ an $\mathbb{R}^n$-admissible via we have* $(n,n)_{-}(n,X) \leq \sigma_{-}(n,X) \leq (n,X)$.

Proof. Since $\mathcal{S}_{(n,n)_{-}(n,X)} = \mathcal{S}^1_{(n,X)} \subset \mathcal{S}_{\sigma_{-}(n,X)} \subset \mathcal{S}_{(n,X)}$, the proof follows from Proposition 4.8 (a). □

Proposition 4.10. *For $k,m,n \in N_X$ with $k \leq m \leq n$ we have*

$$(n,n)_{-}(n,X) \leq (m,m)_{-}(n,X) \leq (k,k)_{-}(n,X) \leq (n,X) \leq (X,X).$$

Proof. For every via σ with $w(\sigma) \equiv \mathbb{R}^m$ we already know that $(n,n)_{-}(n,X) \leq \sigma_{-}(n,X) \equiv (m,m)_{-}(n,X) \leq (n,X)$ (Proposition 4.9, Theorem 4.4). Now, since if $\gamma : V \subset \mathbb{R}^n \to X$ is a C^1-embedding in $(m,m)_{-}$ sense then it is a C^1-embedding in $(k,k)_{-}$sense; we have that $\mathcal{S}_{(m,m)_{-}(n,X)} \subset \mathcal{S}_{(k,k)_{-}(n,X)}$; and the proof follows from Proposition 4.8 (a). □

Since $(n,n)_{-}(n,X)$, in conjunction with condition $(\mathcal{O})_{x_0}$, is Hadamard differentiability, the proposition above shows that there are possibilities for via differentiabilities between Hadamard and Fréchet differentiabilities.

To end this section, let us define the set of all X-admissible vias,

$$\chi = \{\pi | \pi \text{ is an } X - \text{admissible via }\},$$

and let $\Omega \subset \chi$ be an arbitrary non void subset of χ.
Observe that, given a Banach space Y, we have $\emptyset \neq L(X,Y) \subset (X,X)(X,Y) \subset \bigcap_{\pi\in\Omega} \pi(X,Y)$. Now we can define a completion of χ by writing

$$\overline{\chi} = \{\bigcap_{\pi\in\Omega} \pi(X,Y) \;\; \emptyset \neq \Omega \subset \chi\},$$

i.e, an element of $\overline{\chi}$ is the set $\{f : U \subset X \to Y | 0 \in U \textit{ open set}, \exists \Omega \subset \chi, f$ is π differentiable at 0 for all $\pi \in \Omega\}$, and try to study this completion. Some questions naturally appear. If Ω is a chain of vias, that is, Ω has the property that for all π and σ in Ω either $\pi \leq \sigma$ or $\sigma \leq \pi$, then is it true that there is an element $\lambda \in \chi$ such that $\bigcap_{\pi\in\Omega} \pi(X,Y) = \lambda(X,Y)$? Under what conditions on Ω, X and Y we can solve the equation $\bigcap_{\pi\in\Omega} \pi(X,Y) = \lambda(X,Y)$ for some via $\lambda \in \chi$?
Remember that $(1,1)(X,Y) \supset (2,2)(X,Y) \supset ... \supset (n,n)(X,Y) \supset ...$
$\supset (1,1)_{-}(1,X)(X,Y)$. Is is true that there is π an X-admissible via such that $\bigcap_{n\in\mathbb{N}} (n,n)(X,Y) = \pi(X,Y)$ or does the limit via $\bigcap_{n\in\mathbb{N}} (n,n)(X,Y)$ lie outside χ?
Observe that for X such that $l_2(\mathbb{N}) \leq X$, $\bigcap_{n\in\mathbb{N}} (n,n)(X,Y)$ in conjunction with $(\mathcal{O})_0$ is less than and not equal to $\underline{(1,1)_{-}(1,X)}(X,Y) = H_0(X,Y)$. This can be verified, similarly to what was done in Proposition 3.4, by taking $f : l_2(\mathbb{N}) \to Y$ such that $\|f(\phi(t))\| = \|\phi(t)\|$ and $f(x) = 0$ otherwise where $\phi(t) = (t, t^2, ..., t^n, ...)$ for $t \in \mathbb{R}$, $|t| < 1$, and noting that for all $n \in \mathbb{N}^*, {}^{(n,n)}df(0) = 0$ and (1) is not fulfilled for f, with $T = 0$, through $S = \phi((-1,1))$ an $(1,1)_{-}(1,X)$ type set.

5. Applications

In the section we apply the concept of via differentiability in two different contexts to obtain interesting results.
The first one, a consequence of Corollary 3.1 is a reciprocal statement of the following well known elementary calculus result. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be such that there exists $\lim_{x\to x_0} f(x) = l$. Then for every embedded C^1-curve $c : (-\epsilon, \epsilon) \to \mathbb{R}^n$, such that $c(0) = x_0$, we have $\lim_{t\to 0} f(c(t)) = l$. We remark that we are assuming only embedded C^1-curves in the hypothesis

of the reciprocal theorem, since by Corollary 3.1, every $(1,1)_{-}(1,n)$ type union, translated from 0 to x_0, contains an open neighbourhood of x_0.
Now we apply this theory to linearize a discrete dynamical system in $\mathbb{R}^n$ around its fixed point where it is not Fréchet differentiable. We present one situation where there is local conjugacy between the system and its linearized form, that is, where a Hartman Grobman type theorem is valid; and another situation with no local conjugacy though the derivative at the fixed point is hyperbolic.
The first situation shows that there is the possibility of linearizing a non-Fréchet differentiable system with the newly defined derivatives and still have a good linear approximation on neighbourhoods of the fixed point.
Now we construct the dynamical system. Let $0 < a < b \leq 1$ and $\mathcal{R} \subset \mathbb{R}^n, n \geq 2$, be given by
$\mathcal{R} = \{(t, p_1 t^2, ..., p_{(n-1)} t^n) \in \mathbb{R}^n | t > 0,\ a \leq p_j \leq 2b - a,\ 1 \leq j \leq (n-1)\}$. Take the linear isomorphism $g : \mathbb{R}^n \to \mathbb{R}^n,\ g(x_1, ..., x_n) = (ax_1, a^2 x_2, ..., a^n x_n)$ and define the homeomorphism f by $f = g$ outside $\mathcal{R}$ and
$f(t, p_1 t^2, ..., p_{(n-1)} t^n) = ((b-z)t, (b-z)^2 p_1 t^2, ..., (b-z)^n p_{(n-1)} t^n)$ where $z = \max\{|b - p_j|\ |1 \leq j \leq (n-1)\}$. Note that for each $\Lambda = (p_1, ..., p_{(n-1)}) \in \mathbb{R}^{(n-1)}$ the curve $P_\Lambda : \mathbb{R} \to \mathbb{R}^n,\ P_\Lambda(t) = (t, p_1 t^2, ..., p_{(n-1)} t^n)$ is invariant under both actions of g and f and that we can write $\mathcal{R}$ as the union of the images of all curves P_Λ, for positive t and Λ in $[a, 2b-a]^{n-1}$. Since $f(P_\Lambda(t)) = P_\Lambda(at)$ when $P_\Lambda(t) \notin \mathcal{R}$ and $f(P_\Lambda(t)) = P_\Lambda((b-z)t)$ otherwise, the inverse f^{-1} is such that $f^{-1}(P_\Lambda(t)) = P_\Lambda(t/a)$ outside $\mathcal{R}$ and $f^{-1}(P_\Lambda(t)) = P_\Lambda(t/(b-z))$ in $\mathcal{R}$.
Clearly f is an homeomorphism and it is not Fréchet differentiable at 0, since 0 is in the closure of $\mathcal{R}$.
However, for every m dimensional subspace E of $\mathbb{R}^n, m < n$, we can find an open neighbourhood $V \subset E$ of 0 in E which is disjoint of $\mathcal{R}$ as can be seen in the proof of Proposition 3.4. Now $f = g$ in V and ${}^{(m,m)}f'(0) = g$ which is attractive, thus hyperbolic.
The situations mentioned above correspond respectively to the cases $b < 1$ and $b = 1$ as is shown as follows:
In case $b < 1, f$ is attractive and the fundamental domain $B - f(B)$, where B is some closed ball in $\mathbb{R}^n$ centered at 0, is homeomorphic to $B - g(B)$ which is a fundamental domain for g. This permits us to construct a global conjugation between the two systems. Finally, if $b = 1$, every point in the set $F = \{(t, t^2, ..., t^n) | t \geq 0\}$ is a fixed point of f and no conjugation with g is possible since 0 is an accumulation point of F.

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