Resenhas IME-USP 2004, Vol. 6, No. 2/3, 291 - 307.

Sabinin Algebras: The Basis of a Nonassociative Lie Theory

José M. Pérez–Izquierdo[1]

Abstract: Certain famous concepts and results such as Universal enveloping algebras, Poincaré–Birkhoff–Witt Theorem and the Lie correspondence have been, up to some extend, synonymous of Lie algebras. Other nonassociative algebras seemed not to fit in that context. This paper presents some recent results on what might be called "nonassociative Lie Theory": the Lie correspondence between loops and Sabinin algebras, the existence of nonassociative universal enveloping algebras, the nonassociative Poincaré–Birkhoff–Witt Theorem, the Milnor–Moore Theorem for nonassociative Hopf algebras and the equational logic behind the new kind of nonassociative identities that arise from these new nonassociative Hopf algebras[2]. Connections in the spirit of Magnus with some central series on loops recently introduced by J. Mostovoy are also discussed.

1 Sabinin algebras and local loops

Unless explicitly stated, all over this paper we will assume that the field F has characteristic zero.

Definition 1 *A nonempty set Q with three maps $(a, b) \mapsto ab$ (multiplication), $(a, b) \mapsto a\backslash b$ (left division) and $(a, b) \mapsto a/b$ (right division) is called a loop if the following identities hold:*

$$a\backslash(ab) = b = a(a\backslash b), \ \ (ab)/b = a = (a/b)b \text{ and } \ a\backslash a = b/b.$$

In the sight of the other identities, the last identity is equivalent to imposing that $e = a\backslash a$ is the identity or unit element for the product, that is, $ea = a = ae$. Usually loops are presented in this way. In the case that we do not impose the existence of a unit element then we obtain the definition of *quasigroups* [2, 15, 3].

Definition 2 *An analytic manifold M is called a local analytic loop if analytic maps $(a, b) \mapsto ab, a\backslash b, a/b$ are defined in a neighborhood of an element $e \in M$ and the identities*

$$a\backslash(ab) = b = a(a\backslash b), \ \ (ab)/b = a = (a/b)b, \ \ ea = a = ae$$

hold whenever the expressions are defined.

[1]Supported by MCYT (BFM 2001-3239-C03-02) and the Comunidad Autónoma de La Rioja (ANGI 2001/26). The author was also partially supported by FINEP.

[2]The proof of some of these results is available in [13].

One of the main tasks in loop theory was to find an infinitesimal object to locally classify the analytic local loop. Since Lie groups are the first examples of analytic local loops, namely, those for which $(ab)c = a(bc)$ holds, then this object should generalize the usual notion of Lie algebra.

The following identities are equivalent for a loop

$$a(b(ac)) = ((ab)a)c, \;\; ((ba)c)a = b(a(ca)) \text{ and } (ab)(ca) = (a(bc))a.$$

A loop is called a *Moufang loop* if it satisfies any of them. In 1955 Malcev proved that the tangent space on the identity of an analytic local Moufang loop inherits a bilinear product $[\, ,\,]$ with the following properties

$$[x, x] = 0 \quad \text{and} \quad [J(x, y, z), x] = J(x, y, [x, z]),$$

where $J(x, y, z) = [[x, y], z] - [[x, z], y] - [x, [y, z]]$ denotes the jacobian of x, y and z; that is, the tangent space on the identity becomes a Malcev algebra (Moufang–Lie in the work of Malcev [7]). Later, in 1971, Kuz'min proved the existence of an analytic local Moufang loop for any finite–dimensional real Malcev algebra [6]. Therefore, the Lie correspondence holds for Moufang loops and Malcev algebras. As a moral we learn that associativity is not essential for the Lie correspondence.

On the way to obtaining an infinitesimal object to locally classify the loop it became clear that a single binary product on the tangent space of the identity was not enough. A left Bol loop is a loop that satisfies the identity

$$a(b(ac)) = (a(ba))c.$$

The tangent space of an analytic local left Bol loop is a left Bol algebra, that is, a vector space V equipped with a bilinear and a trilinear product, denoted by $[\, ,\,]$ and $[\, ,\, ,\,]$ respectively, which satisfy the following conditions:

1. $(V, [\, ,\, ,\,])$ is a Lie triple system; that is,
   (a) $[a, a, b] = 0$,
   (b) $[a, b, c] + [b, c, a] + [c, a, b] = 0$ and
   (c) $[x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], c] + [a, b, [x, y, c]]$.
2. $[\, ,\,]$ is skewsymmetric and it is related with the trilinear operation by $[a, b, [c, d]] = [[a, b, c], d] + [c, [a, b, d]] + [c, d, [a, b]] + [[a, b], [c, d]]$.

Any Moufang loop is a left Bol loop. Correspondingly, any Malcev algebra is a left Bol algebra by setting $[a, b, c] = [[a, b], c] - \frac{1}{3}J(a, b, c)$. Sabinin and Mikheev [18] proved that the Lie correspondence holds for analytic local (left) Bol loops and (left) Bol algebras. Again the moral is that the associativity is not essential for the Lie correspondence.

The general case is more subtle and many multilinear operations are needed to locally classify the loop. Sabinin and Mikheev ([20]) proved that for any analytic local loop the tangent space on the identity inherits two families of multilinear operations

$$\langle x_1,\ldots,x_m;y,z\rangle \ m\geq 0 \text{ and } \Phi(x_1,\ldots,x_m;y_1,\ldots,y_n) \ m\geq 1, n\geq 2$$

that make it a *Sabinin algebra.*

Before giving the formal definition of Sabinin algebras, let us present the following three results to illustrate the goodness of this structure.

Theorem 1 ([20]) *Analytic local loops are locally isomorphic if and only if their corresponding Sabinin algebras are isomorphic.*

Theorem 2 ([20]) *Any finite–dimensional Sabinin algebra over the real numbers whose structure constants satisfy certain convergence conditions (see [20]) is the Sabinin algebra of some analytic local loop.*

Theorem 3 ([20]) *Let $(Q,\cdot,\backslash,/,e)$ be an analytic local loop and $\mathfrak{q}$ the corresponding Sabinin algebra. If R is a local subloop of Q then $\mathfrak{r}=T_eR$ is a Sabinin subalgebra of $\mathfrak{q}$. Conversely, for any subalgebra $\mathfrak{r}$ of $\mathfrak{q}$ there exists a unique local subloop R in Q such that $T_e(R)=\mathfrak{r}$. The subloop R is normal if and only if $\mathfrak{r}$ is an ideal of $\mathfrak{q}$.*

The definition of Sabinin algebras implicitly used by Shestakov and Umirbaev in [22] is formulated in terms of a comultiplication $\Delta\colon T(V)\to T(V)\otimes T(V)$ on the tensor algebra $T(V)$ of the vector space V. The map

$$a\mapsto a\otimes 1+1\otimes a$$

from V to $T(V)\otimes T(V)$ extends to a homomorphism of unital algebras, the comultiplication,

$$\begin{array}{rcl}\Delta\colon T(V) & \to & T(V)\otimes T(V)\\ x & \mapsto & \sum x_{(1)}\otimes x_{(2)}\end{array}$$

where we have used Sweedler's notation [23]. With this notation the definition of Sabinin algebras is as follows:

Definition 3 *A Sabinin algebra $(V,\langle\,;\,,\,\rangle,\Phi)$ is a vector space together with two families of multilinear maps $\langle\,;\,,\,\rangle : T(V)\otimes V\otimes V\to V$ and $\Phi(x_1,\ldots,x_m;y_1,\ldots,y_n)$ $(m\geq 1, n\geq 2)$ satisfying*

1. $\langle x;a,b\rangle=-\langle x;b,a\rangle,$
2. $\langle xaby;c,e\rangle=\langle xbay;c,e\rangle-\sum\langle x_{(1)}\langle x_{(2)};a,b\rangle y;c,e\rangle,$

3. $\sigma_{abc}\left(\langle xc; a, b\rangle + \sum\langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\rangle\right) = 0$ *where* σ_{abc} *denotes the cyclic summation with respect to* a, b *and* c*, and*

4. $\Phi(x_1, \ldots, x_m; y_1, \ldots, y_n) = \Phi(x_{\tau_1}, \ldots, x_{\tau_m}; y_{\delta_1}, \ldots, y_{\delta_n})$ $\forall m \geq 1$, $n \geq 2$ *and* $\tau \in S_m, \delta \in S_n$, *where* S_k *stands for the symmetric group on* k *symbols,*

for all $x, y \in T(V)$ *and* $a, b, c, e \in V$.

To appreciate the importance of Sweedler's notation compare the previous definition with the original definition as formulated in [22]:

Definition 4 *A Sabinin algebra* $(V, \langle\, ;\, ,\, \rangle, \Phi)$ *is a vector space together with two families of multilinear operations* $\langle\, x_1, \ \ldots\ , x_m\ ;\ a, b\,\rangle$, $m = 0, 1, \ldots$ *and* $\Phi(x_1, \ldots, x_m; y_1, \ldots, y_n)$ $(m \geq 1, n \geq 2)$ *with the properties*

1. $\langle x_1, x_2, \ldots, x_m; y, z\rangle = -\langle x_1, x_2, \ldots, x_m; z, y\rangle$,

2. $$\begin{aligned}&\langle x_1, x_2, \ldots, x_r, a, b, x_{r+1}, \ldots, x_m; y, z\rangle \\ &\quad -\langle x_1, x_2, \ldots, x_r, b, a, x_{r+1}, \ldots, x_m; y, z\rangle \\ &\quad +\sum_{k=0}^{r}\sum_{\alpha}\langle x_{\alpha_1}, \ldots, x_{\alpha_k}, \langle x_{\alpha_{k+1}}, \ldots, x_{\alpha_r}; a, b\rangle, \ldots, x_m; y, z\rangle = 0,\end{aligned}$$

3. $$\sigma_{x,y,z}\left(\langle x_1, \ldots, x_r, x; y, z\rangle + \sum_{k=0}^{r}\sum_{\alpha}\langle x_{\alpha_1},, \ldots, x_{\alpha_k}; \langle x_{\alpha_{k+1}}, \ldots, x_{\alpha_r}; y, z\rangle, x\rangle\right) = 0,$$

4. $\Phi(x_1, \ldots, x_m; y_1, \ldots, y_n) = \Phi(x_{\tau_1}, \ldots, x_{\tau_m}; y_{\delta_1}, \ldots, y_{\delta_n})$,

where α *runs the set of all bijections* $\alpha\colon \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}$ *of the type* $i \mapsto \alpha_i$, $\alpha_1 < \alpha_2 < \cdots < \alpha_k, \alpha_{k+1} < \cdots < \alpha_r$, $k = 0, 1, \ldots, r$, $r \geq 0$, $\sigma_{x,y,z}$ *denotes the cyclic sum by* x, y, z; $\tau \in S_m$, $\delta \in S_n$ *and* S_l *stand for the symmetric group on* l *symbols.*

Example 1 Lie algebras are the simplest example of Sabinin algebras. Given any Lie algebra $(L, [\,,\,])$ the multilinear operations

$$\langle x_1, \ldots, x_m; a, b\rangle = \begin{cases} 0 & n \geq 1 \\ -[a, b] & n = 0 \end{cases} \quad \text{and } \Phi = 0$$

provide a structure of Sabinin algebra on L.

Example 2 Any left Bol algebra $(V, [\,,\,], [\,,\,,\,])$ becomes a Sabinin algebra with

$$\begin{aligned}\langle 1; a, b\rangle &= [a,b],\\ \langle c; a, b\rangle &= [a,b,c] - [[a,b],c],\\ \langle cx; a, b\rangle &= -\sum \langle x_{(1)}; c, \langle x_{(2)}; a, b\rangle\rangle \text{ if } |x| \geq 1\end{aligned}$$

and $\Phi = 0$, where $|x|$ denotes the degree of x. We note that this formula differs from the formula given in [13] since there right Bol algebras are considered instead.

2 Sabinin algebras and Lie algebras

Sabinin algebras and Lie algebras are very much connected. Given any Lie algebra L, a subalgebra H and a vector subspace $V \leq L$ with $L = H \oplus V$, one may induce multilinear operations on V

$$\langle\,;\,,\,\rangle \colon T(V) \otimes V \otimes V \to V$$

by means of the following recurrence

$$\{xab\} + \sum \{x_{(1)} \langle x_{(2)}; a, b\rangle\} = 0,$$

where $\{d_1, \ldots, d_n\} = \pi_V([d_1, [\cdots, [d_{n-1}, d_n]])$ with π_V the parallel projection onto V, and $\{d_1\} = \pi_V(d_1)$. We have

Theorem 4 ([20]) *$(V, \langle\,;\,,\,\rangle)$ is a Sabinin algebra, and any Sabinin algebra appears in this way.*

The Lie algebra L in the previous construction is called a *Lie envelope* of $(V, \langle\,;\,,\,\rangle)$.

Example 3 Let L be a Lie algebra, $H = 0$, $V = L$ and $\{d_1, \ldots, d_n\} = [d_1, \ldots, [d_{n-1}, d_n]]$. By the recurrence, $\langle 1; a, b\rangle = \{\langle 1; a, b\rangle\} = -\{ab\} = -[a,b]$. If the degree $|x|$ of x is ≥ 1 then $0 = \{xab\} + \sum\{x_{(1)}\langle x_{(2)}; a, b\rangle\} = \{xab\} + \{x\langle 1; a, b\rangle\} + \sum_{x_{(2)} \neq 1}\{x_{(1)}\langle x_{(2)}; a, b\rangle\}$. Since $\{xab\} + \{x\langle 1; a, b\rangle\} = \{xab\} - \{x[a,b]\} = 0$ then $\langle x; a, b\rangle = 0$ if $|x| \geq 1$ is the solution in this case. Hence we obtain the usual structure of Sabinin algebra on L.

Example 4 Given any nonassociative unital algebra C consider the Lie algebra L generated by the right multiplication operators by elements of C. Let H be the subalgebra of L that kills the unit element. Then $L = H \oplus \{R_x \mid x \in C\}$, so $\{R_x \mid x \in C\}$ inherits a structure of Sabinin algebra that moves to C. In particular, we obtain multilinear operations (in terms of the product on C) so that C endowed with these operations becomes a Sabinin algebra. This construction generalizes the usual functor from associative algebras to Lie algebras.

3 Sabinin algebras and nonassociative algebras

In [22] Sabinin algebras broke into a different context. In the important study that Shestakov and Umirbaev carried out on primitive elements $\mathrm{Prim}(C)$ of a nonassociative bialgebra C they defined some operations under which $\mathrm{Prim}(C)$ is closed. As Shestakov and Umirbaev realized, a modification of those operations make $\mathrm{Prim}(C)$ a Sabinin algebra.

The construction of Shestakov and Umirbaev works as follows. Let B a free nonassociative algebra on $X\cup Y\cup\{z\}$ with $X=\{x_1,x_2,\ldots\}$ and $Y=\{y_1,y_2,\ldots\}$. There exists a homomorphism (the comultiplication) of unital algebras

$$\begin{array}{rcl}\Delta\colon B & \to & B\otimes B\\ x & \mapsto & \sum x_{(1)}\otimes x_{(2)}\end{array}$$

induced by $X\cup Y\cup\{z\}\subseteq \mathrm{Prim}(B)$; that is, $\Delta(w)=w\otimes 1+1\otimes w\;\forall w\in X\cup Y\cup\{z\}$. Fix $u=((x_1x_2)\cdots)x_m$ and $v=((y_1y_2)\cdots)y_n\in B$ and define nonassociative polynomials[3]

$$q(x_1,\ldots,x_m;y_1,\ldots,y_n;z)=q(u,v,z)$$

by the recurrence

$$(uv)z-u(vz)=\sum u_{(1)}q(u_{(2)},v_{(2)},z)\cdot v_{(1)}$$

with initial conditions $q(1,1,z)=q(1,v,z)=q(u,1,z)=0$. These polynomials can be evaluated on any nonassociative algebra C inducing multilinear operations in the obvious way.

Theorem 5 **([22])** *Let C be an algebra with unit 1 and $\Delta\colon C\to C\otimes_F C$ be a nontrivial homomorphism of algebras. Then the set $Prim(C)$ of primitive elements of C is closed under $[\,,\,]$ and $q(\,;\,;\,)$.*

Another important results about primitive elements of nonassociative bialgebras are:

Theorem 6 **([22])** *Let C be an algebra with unit 1 over a field of characteristic zero, and $\Delta\colon C\to C\otimes_F C$ be a nontrivial homomorphism of algebras. Suppose that the algebra C is generated by a set M of primitive elements, and let $P(M)$ be the minimal subspace of C that contains M and is closed with respect to the primitive operations $[\,,\,]$ and $q(\,;\,;\,)$. Let $e_1,e_2,\ldots,e_\alpha,\ldots$ be a basis of $P(M)$. Then the set of right–normed words of the type*

$$((e_{i_1}e_{i_2})\cdots)e_{i_k}$$

where $i_1\le i_2\le\cdots\le i_k$, $k\ge 0$ forms a basis of the algebra C.

[3]These polynomials are a minor modification of the polynomials introduced by Shestakov and Umirbaev.

Corollary 1 ([22]) *Under the assumptions of the theorem, the set $P(M)$ coincides with* $\mathrm{Prim}(C)$. *In other words, any set of primitive elements which generates C generates also the set* $\mathrm{Prim}(C)$ *by the operations* $[\,,\,]$ *and* $q(\,;\,;\,)$.

Given any nonassociative algebra C, Shestakov and Umirbaev consider

$$\langle 1; a, b\rangle = -[a, b],$$
$$\langle x_1, \dots, x_m; y, z\rangle = -q(x_1, \dots, x_m; y, z) + q(x_1, \dots, x_m; z, y) \text{ and}$$
$$\Phi(x_1, \dots, x_m; y_1, \dots, y_n) = \tfrac{1}{m!}\tfrac{1}{n!}\textstyle\sum_{\tau,\delta} \Phi(x_{\tau_1}, \dots, x_{\tau_m}; y_{\delta_1}, \dots, y_{\delta_n}),$$

where τ and δ run the symmetric groups S_m and S_n respectively. The vector space C equipped with these new multilinear operations will be denoted by $\text{УШ}(C)$.

Theorem 7 ([22]) *For any nonassociative algebra C over a field of characteristic zero,* $\text{УШ}(C) = (C, \langle\,;\,,\,\rangle, \Phi)$ *is a Sabinin algebra.*

Remark 1 This structure of Sabinin algebra is the same as the one obtained in Example 4.

We have arrived to an interesting point:

$$\boxed{\begin{array}{l} A \text{ associative} \Rightarrow A^- = (A, [\,,\,]) \text{ is a Lie algebra} \\ C \text{ nonassociative} \Rightarrow \text{УШ}(C) = (C, \langle\,;\,,\,\rangle, \Phi) \text{ is a Sabinin algebra} \end{array}}$$

and we know that any Lie algebra arises as a subalgebra of U^- for some associative algebra U. So it is natural to ask the following question[22]:

> *Does any Sabinin algebra arise as a subalgebra of* $\text{УШ}(U)$ *for some nonassociative algebra U?*

The answer is provided by the following result.

Theorem 8 (Poincaré–Birkhoff–Witt) *Let $(V, \langle\,;\,,\,\rangle, \Phi)$ be a Sabinin algebra over a field of characteristic zero. There exist a unital algebra $U = U(V, \langle\,;\,,\,\rangle, \Phi)$ and a monomorphism*

$$\iota\colon V \hookrightarrow \text{УШ}(U)$$

of Sabinin algebras such that (U, ι) satisfies the following universal property:

> *For any unital algebra C and any homomorphism of Sabinin algebras $\varphi\colon V \to \text{УШ}(C)$ there exists a unique homomorphism of unital algebra $\bar\varphi\colon U \to C$ such that $\varphi = \bar\varphi \circ \iota$.*

Moreover, if $\{a_i \mid i \in \Lambda\}$ is an ordered basis of V then

$$\{((a_{i_1} a_{i_2}) \cdots) a_{i_n} \mid i_1 \leq \cdots \leq i_n \text{ and } n \geq 0\}$$

is a basis of U.

Example 5 Let L be a Lie algebra over a field of characteristic zero and $U'(L)$ its usual universal enveloping algebra. The Lie map $L \to U'(L)^-$ is in fact a homomorphism $L \to G(U'(L))$ of Sabinin algebras, so it extends to a homomorphism $U(L) \to U'(L)$ that sends a basis of $U(L)$ to the corresponding basis of $U'(L)$; therefore, $U(L)$ is isomorphic to the usual universal enveloping algebra of L.

Finally, let us mention that $U(V, \langle ; , \rangle, \Phi)$ is a nonassociative bialgebra and, by the Friedrichs criterion,

$$\operatorname{Prim}(U) = V.$$

Form this point of view Sabinin algebras over fields of characteristic zero are exactly the primitive elements of nonassociative bialgebras. For instance, the free Sabinin algebra on X is recovered as the primitive elements of the free nonassociative algebra on X. An interesting extension to free nonassociative algebras of Witt dimension formula for the Lie elements in the free associative algebra has been obtained in [1].

4 Sabinin algebras and nonassociative Hopf algebras

An algebra A is called alternative if it satisfies the alternative laws

$$x(xy) = x^2y \quad \text{and} \quad (yx)x = yx^2.$$

Over fields of characteristic $\neq 2$ these identities are equivalent to

$$(x, y, z) = -(y, x, z) = (y, z, x),$$

where $(x, y, z) = (xy)z - x(yz)$ stands for the associator.

In the same way that A^- is a Lie algebra for any associative algebra A, when starting with an alternative algebra A, A^- is a Malcev algebra. However, it remains an open problem whether any Malcev algebra appears as a subalgebra of A^- for some alternative algebra A.

Since any Malcev algebra is a Sabinin algebra, in the sight of Theorem 8 a natural question to ask is whether $U(M)$ is alternative or not. Unfortunately the answer turns out to be negative; thus, one is led to study the identities, if any, satisfied by $U(M)$.

Let us sketch a philosophical approach to this problem. To start with, consider Q an affine algebraic Moufang loop, whatever it might mean. Let $\mathcal{P}(Q)$ be the algebra of polynomial functions and $\mathcal{P}(Q)^\circ$ its Hopf dual. In the case of affine algebraic groups the tangent space is a Lie algebra and the universal enveloping algebra lives inside $\mathcal{P}(Q)^\circ$. A comultiplication is induced on $\mathcal{P}(Q)$ by $\Delta(f) = \sum f_{(1)} \otimes f_{(2)}$ with $f(ab) = \sum f_{(1)}(a) f_{(2)}(b)$. The product on $\mathcal{P}(Q)^\circ$ is determined by $(ab)(f) = \sum a(f_{(1)}) b(f_{(2)})$, and a comultiplication by $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ with $a(fg) = \sum a_{(1)}(f) a_{(2)}(g)$.

Since $a(b(ac)) = ((ab)a)c$ then for any function $f \in \mathcal{P}(Q)$ we have that

$$\begin{aligned} f\left(a(b(ac))\right) &= \sum f_{(1)}(a)f_{(2)}(b(ac)) = \sum f_{(1)}(a)f_{(2)(1)}(b)f_{(2)(2)}(ac) \\ &= \sum f_{(1)}f_{(2)(2)(1)}(a)f_{(2)(1)}(b)f_{(2)(2)(2)}(c) \\ &= \left(\sum f_{(1)}f_{(2)(2)(1)} \otimes f_{(2)(1)} \otimes f_{(2)(2)(2)}\right)(a \otimes b \otimes c) \end{aligned}$$

and

$$\begin{aligned} f\left(((ab)a)c\right) &= \sum f_{(1)}((ab)a)f_{(2)}(c) = \sum f_{(1)(1)}(ab)f_{(1)(2)}(a)f_{(2)}(c) \\ &= \sum f_{(1)(1)(1)}(a)f_{(1)(1)(2)}(b)f_{(1)(2)}(a)f_{(2)}(c) \\ &= \left(\sum f_{(1)(1)(1)}f_{(1)(2)} \otimes f_{(1)(1)(2)} \otimes f_{(2)}\right)(a \otimes b \otimes c), \end{aligned}$$

so

$$\sum f_{(1)}f_{(2)(2)(1)} \otimes f_{(2)(1)} \otimes f_{(2)(2)(2)} = \sum f_{(1)(1)(1)}f_{(1)(2)} \otimes f_{(1)(1)(2)} \otimes f_{(2)}.$$

Since we are interested in the identities satisfied by $\mathcal{P}(Q)^\circ$ then we take $a, b, c \in \mathcal{P}(Q)^\circ$, $f \in \mathcal{P}(Q)$ and we observe that

$$\begin{aligned} &(a \otimes b \otimes c)(\sum f_{(1)}f_{(2)(2)(1)} \otimes f_{(2)(1)} \otimes f_{(2)(2)(2)}) \\ &\quad= \sum a(f_{(1)}f_{(2)(2)(1)})b(f_{(2)(1)})c(f_{(2)(2)(2)}) \\ &\quad= \sum a_{(1)}(f_{(1)})a_{(2)}(f_{(2)(2)(1)})b(f_{(2)(1)})c(f_{(2)(2)(2)}) \\ &\quad= \sum a_{(1)}(f_{(1)})b(f_{(2)(1)})(a_{(2)}c)(f_{(2)(2)}) \\ &\quad= \sum a_{(1)}(f_{(1)})(b(a_{(2)}c))(f_{(2)}) \\ &\quad= \left(\sum a_{(1)}(b(a_{(2)}c))\right)(f) \end{aligned}$$

and

$$(a \otimes b \otimes c)(\sum f_{(1)(1)(1)}f_{(1)(2)} \otimes f_{(1)(1)(2)} \otimes f_{(2)}) = (\sum((a_{(1)}b)a_{(2)})c)(f).$$

Therefore, $\sum a_{(1)}(b(a_{(2)}c)) = \sum((a_{(1)}b)a_{(2)})c$. When starting with the right Moufang identity $((ba)c)a = b(a(ca))$ we obtain $\sum((ba_{(1)})c)a_{(2)} = \sum b(a_{(1)}(ca_{(2)}))$ instead. Hence the expected identities of $U(M)$ are

$$\boxed{\begin{gathered} \sum a_{(1)}(b(a_{(2)}c)) = \sum((a_{(1)}b)a_{(2)})c \\ \sum((ba_{(1)})c)a_{(2)} = \sum b(a_{(1)}(ca_{(2)})) \end{gathered}}$$

While these considerations provide us with natural candidates for identities, however one misses the alternative law because after all, the known examples of Malcev algebras all arise from alternative algebras. To conclude our philosophical approach, observe that in the classical case the tangent space of the algebraic group lives inside the primitive elements of $\mathcal{P}(Q)^\circ$ and that when a primitive element a is plugged into the identities $\sum a_{(1)}(y(a_{(2)}z)) = \sum((a_{(1)}y)a_{(2)})z$ and $\sum((ya_{(1)})z)a_{(2)} = \sum y(a_{(1)}(za_{(2)}))$ then we obtain that

$$(a, y, z) = -(y, a, z) = (y, z, a),$$

thus recovering the spirit of the alternative laws.

Let us show now how these ideas have been developed.

Definition 5 *Given any algebra A, the generalized alternative nucleus of A is*

$$\mathrm{N}_{\mathrm{alt}}(A) = \{a \in A \mid (a, y, z) = -(y, a, z) = (y, z, a)\}.$$

Proposition 1 ([9]) *For any algebra A, $\mathrm{N}_{\mathrm{alt}}(A)$ is closed under the commutator product $[\,,\,]$. Moreover, $(\mathrm{N}_{\mathrm{alt}}(A), [\,,\,])$ is a Malcev algebra.*

In the case of A being alternative then we recover the usual construction of Malcev algebras from alternative algebras.

Theorem 9 ([14]) *Given a Malcev algebra M over a field of characteristic $\neq 2, 3$ there exist a unital algebra $U(M)$ and a monomorphism of Malcev algebras $\iota\colon M \hookrightarrow \mathrm{N}_{\mathrm{alt}}(U(M))$ such that $(U(M), \iota)$ satisfies the following universal property:*

> *For any unital algebra C and any homomorphism of Malcev algebras $\varphi\colon M \to \mathrm{N}_{\mathrm{alt}}(C)$ there exists a unique homomorphism $\bar{\varphi}\colon U(M) \to C$ of unital algebra such that $\varphi = \bar{\varphi} \circ \iota$.*

Theorem 10 *Let $(M, [\,,\,])$ be a Malcev algebra over a field of characteristic $\neq 2, 3$, and $U(M)$ its universal enveloping algebra. Then $U(M)$ is a bialgebra that satisfies the identities $\sum a_{(1)}(b(a_{(2)}c)) = \sum((a_{(1)}b)a_{(2)})c$ and $\sum((ba_{(1)})c)a_{(2)} = \sum b(a_{(1)}(ca_{(2)}))$.*

This universal enveloping algebra is isomorphic, over fields of characteristic zero, to the universal enveloping algebra of M as a Sabinin algebra.

All these considerations move to left Bol loops and left Bol algebras. The identity that the universal enveloping algebra of a left Bol algebra should satisfy is

$$\sum a_{(1)}(y(a_{(2)}z)) = \sum(a_{(1)}(ya_{(2)}))z.$$

In case that a is chosen from the primitive elements then this identity becomes $(a, y, z) = -(y, a, z)$.

Definition 6 *Given any algebra A, the left generalized alternative nucleus of A is*

$$\mathrm{LN}_{\mathrm{alt}}(A) = \{a \in A \mid (a,y,z) = -(y,a,z)\ \forall y,z \in A\}.$$

The left generalized alternative nucleus is closed under the triple product

$$[a,b,c] = a(bc) - b(ac) - c(ab - ba)$$

and in fact $(\mathrm{LN}_{\mathrm{alt}}(A), [\,,\,,\,])$ is a Lie triple system [12].

Proposition 2 ([12]) *Let A be an algebra and V a subspace of $\mathrm{LN}_{\mathrm{alt}}(A)$ closed under $[\,,\,,\,]$ and $[\,,\,]$. Then $(V, [\,,\,,\,], [\,,\,])$ is a left Bol algebra.*

Theorem 11 ([12]) *Let $(V, [\,,\,,\,], [\,,\,])$ be a left Bol algebra, then there exist a unital algebra $U(V)$ and a linear injective map $\iota: V \hookrightarrow \mathrm{LN}_{\mathrm{alt}}(U(V))$ $a \mapsto a$ such that*

$$\iota([a,b]) = ab - ba \quad \textit{and} \quad \iota([a,b,c]) = a(bc) - b(ac) - c(ab - ba)$$

and the following universal property holds:

> *For any unital algebra A and any linear map $\iota': V \to \mathrm{LN}_{\mathrm{alt}}(A)$ $a \mapsto a'$ with $\iota'([a,b]) = a'b' - b'a'$ and $\iota'([a,b,c]) = a'(b'c') - b'(a'c') - c'(a'b' - b'a')$ there exists a homomorphism of unital algebras $\varphi: U(V) \to A$ satisfying $\iota' = \varphi \circ \iota$.*

Over fields of characteristic zero this universal enveloping algebra is isomorphic to the universal enveloping algebra of V as a Sabinin algebra.

At this point it is clear that a *linearizing process* for linearizing identities of quasigroups on coassociative (i.e. $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$) and cocommutative (i.e. $\Delta = \tau \circ \Delta$ with $\tau(x \otimes y) = y \otimes x$) bialgebras has emerged. The way it works is quite simple: *In any side of the identity $p \approx q$ of the quasigroup substitute any repeated occurrence of a variable, let say a, by $a_{(1)}, a_{(2)}, \ldots$ In case that a occurs only in one side, then multiply the other side by $\epsilon(a)$, where ϵ denotes the counit, to keep the linearity of both sides of the linearized identity in that variable.*

Example 6 The natural operations on a group are the multiplication and the inverse map $S: a \mapsto a^{-1}$. The identities satisfied by these two operations are: $(ab)c = a(bc)$ (associativity), $S(a)(ab) = b = a(S(a)b)$ (the left multiplication operator is bijective) and $(ba)S(a) = b = (bS(a))a$ (the right multiplication operator is bijective). To linearize these identities on a bialgebra H we need to assume that the bialgebra is endowed with an extra linear operation $S: H \to H$. Now the linearizing process gives: $(ab)c = a(bc)$ (the bialgebra must be associative), $\sum S(a_{(1)})(a_{(2)}b) = \epsilon(a)b = \sum a_{(1)}(S(a_{(2)})b)$ and $\sum (ba_{(1)})S(a_{(2)}) = \epsilon(a)b = \sum (bS(a_{(1)}))a_{(2)}$. In the presence of the associativity the last four identities are equivalent to saying that the bialgebra has unit element 1 and $\sum S(a_{(1)})a_{(2)} = \epsilon(a)1 = \sum a_{(1)}S(a_{(2)})$; that is, the linearization of the identities that define a group originates the definition of Hopf algebras (S is called the *antipode*).

This example motivates that a natural definition of the notion of nonassociative (nonunital) Hopf algebras will come from the linearization of the identities defining the structure of quasigroup.

Definition 7 *An H–bialgebra $(H, \Delta, \epsilon, \cdot, \backslash, /)$ is a bialgebra $(H, \Delta, \epsilon, \cdot)$ with two extra bilinear operations, the left and right division,*

$$\begin{array}{rcl} \backslash : H \times H & \to & H \\ (x, y) & \mapsto & x \backslash y \end{array} \qquad\qquad \begin{array}{rcl} / : H \times H & \to & H \\ (x, y) & \mapsto & x / y \end{array}$$

such that

$$\sum x_{(1)} \backslash (x_{(2)} y) = \epsilon(x) y = \sum x_{(1)} (x_{(2)} \backslash y)$$

and

$$\sum (y x_{(1)}) / x_{(2)} = \epsilon(x) y = \sum (y / x_{(1)}) x_{(2)}.$$

Example 7 Given a loop Q and a field F, the *loop algebra* of Q is defined as $F[Q] = \bigoplus_{a \in Q} Fa$ with the multiplication, left division and right division induced by those of Q, and Δ and ϵ induced by $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$ for any $a \in Q$. The loop algebra of any loop is an H–bialgebra and it is obvious that if Q satisfies some identity then $F[Q]$ will satisfy its linearization.

Example 8 The universal enveloping algebras of Sabinin algebras are H–bialgebras.

The linearizing process gives some insights in the study of identities on H–bialgebras.

Theorem 12 *Let Σ be a set of identities for quasigroups and $p \approx q$ a consequence of Σ. If C is a cocommutative and coassociative H–bialgebra that satisfies the linearization of the identities in Σ then C satisfies the linearization of $p \approx q$.*

Example 9 In any quasigroup the Moufang identities are equivalent. Moreover, it is a not obvious result in the theory of quasigroups that if a quasigroup satisfies any of them then it is a loop [21, 5]. As a consequence, in any cocommutative and coassociative H–bialgebra the linearizations of the Moufang identities are equivalent, and any of them implies that there exists unit element.

The definition of H–bialgebra and the universal enveloping algebras of Sabinin algebras allow us to formulate the nonassociative version of the well–known Milnor–Moore Theorem [8].

Theorem 13 (Milnor–Moore) *Let H be an H–bialgebra over a field of characteristic zero. If H is generated by* $\mathrm{Prim}(H)$ *then*

$$H \cong U(\mathrm{Prim}(H)).$$

5 Sabinin algebras and central series

Free groups were studied by Magnus by embedding them in formal power series rings on noncommuting variables. Given a free group G on $\{x_i \mid i \in \Lambda\}$ and the ring $\mathbb{Z}[[x'_i \mid i \in \Lambda]]$ of formal power series on $\{x'_i \mid i \in \Lambda\}$, the map $x_i \mapsto 1 + x'_i$ induces an embedding $\mathcal{M}: G \to \mathbb{Z}[[x_i \mid i \in \Lambda]]$ of G in the group of units of $\mathbb{Z}[[x_i \mid i \in \Lambda]]$, so one may identify G with $\mathcal{M}(G)$. Through this embedding some interesting questions about G become rather obvious. For instance, the lower central series of a group G is defined as

$$G = \gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \cdots$$

with $\gamma_{n+1}(G) = [G, \gamma_n(G)]$ and $[a,b] = a^{-1}b^{-1}ab$. In the case that G is free then $\gamma_n(G) \subseteq G \cap (1 + I^n)$, where I denotes the ideal of all formal power series with zero constant term, so it is evident that $\cap_{n=1}^{\infty} \gamma_n(G) = 1$. In fact, $\gamma_n(G)$ is fully recovered as $\gamma_n(G) = G \cap (1 + I^n)$.

In general, for an arbitrary group G no such embedding is available. So, one has to choose a candidate to play the role of the formal power series ring: *the group algebra*. From $F[G]$ and the augmentation ideal I the *dimension subgroups* are defined by

$$D_n(G) = \{a \in G \mid a \in 1 + I^n\}.$$

These subgroups provide another central series

$$G = D_1(G) \trianglerighteq D_2(G) \trianglerighteq \cdots$$

As before, $\gamma_n(G) \subseteq D_n(G)$ although the equality in this case does not always hold. The precise relationship between these subgroups is beautifully explained in [4].

From the group algebra $F[G]$ and the augmentation ideal I one defines another cocommutative Hopf algebra

$$\mathcal{G}(G) = \bigoplus_{n=0}^{\infty} I^n / I^{n+1}.$$

Since $\Delta(a-1) = (a-1) \otimes 1 + 1 \otimes (a-1) + (a-1) \otimes (a-1)$ then $a-1$ induces a primitive element in $\mathcal{G}(G)$. Thus $\mathcal{G}(G)$ is generated by $\mathrm{Prim}(\mathcal{G}(G))$, and by the Milnor–Moore Theorem it must be isomorphic to the universal enveloping algebra of the Lie algebra $\mathrm{Prim}(\mathcal{G}(G))$.

The map

$$\begin{array}{rcl} D_n(G)/D_{n+1}(G) & \to & I^n/I^{n+1} \\ aD_{n+1}(G) & \mapsto & (a-1) + I^{n+1} \end{array}$$

defines a monomorphism of abelian groups

$$\mathcal{D}(G) = \bigoplus_{n=1}^{\infty} D_n(G)/D_{n+1}(G) \to \mathcal{G}(G).$$

Moreover, given $a-1 \in I^n$ and $b-1 \in I^m$ then $[a-1,b-1]+I^{n+m+1} = (a-1)(b-1)-(b-1)(a-1)+I^{n+m+1} = ab-ba+I^{n+m+1} = ba(a^{-1}b^{-1}ab-1)+I^{n+m+1}$. Since $ba \in 1+I$ then $[a-1,b-1]+I^{n+m+1} = (a^{-1}b^{-1}ab-1)+I^{n+m+1}$. In other words, through this monomorphism $\mathcal{D}(G)$ inherits the structure of a Lie ring with operations

$$aD_{n+1}(G) + bD_{n+1}(G) = abD_{n+1}(G)$$

and

$$[aD_{n+1}(G), bD_{m+1}(G)] = a^{-1}b^{-1}abD_{n+m+1}(G),$$

and we have a monomorphism of Lie algebras

$$F \otimes_{\mathbb{Z}} \mathcal{D}(G) \to \mathcal{G}(G).$$

Over fields of characteristic zero the image of $F \otimes_{\mathbb{Z}} \mathcal{D}(G)$ coincides with $\mathrm{Prim}(\mathcal{G}(G))$.

Theorem 14 ([16]) *Given any group G and a field F of characteristic zero then $\mathcal{G}(G) \cong U(F \otimes_{\mathbb{Z}} \mathcal{D}(G))$.*

The Lie ring structure defined in $\mathcal{D}(G)$ was well known for group theorists. In fact, the abelian group

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} \gamma_n(G)/\gamma_{n+1}(G)$$

becomes a Lie ring with virtually the same operations. This construction allowed the fruitful introduction of Lie methods in the theory of finite and infinite groups.

Given a loop Q, by analogy we may consider the loop algebra $F[Q]$, the augmentation ideal I, the dimension subloops $D_n(Q) = Q \cap (1+I^n)$ and $\mathcal{D}(Q) = \bigoplus_{n=1}^{\infty} D_n(Q)/D_{n+1}(Q)$. The algebra

$$\mathcal{G}(Q) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$$

inherits the structure of H–bialgebra, and since it is generated by the primitive elements then by the Milnor–More Theorem it must be isomorphic to the universal enveloping algebra of the Sabinin algebra $\mathrm{Prim}(\mathcal{G}(Q))$. As before,

$$\begin{array}{rcl} D_n(Q)/D_{n+1}(Q) & \to & I^n/I^{n+1} \\ aD_{n+1}(G) & \mapsto & (a-1)+I^n \end{array}$$

is a monomorphism of abelian groups that induces $F \otimes_{\mathbb{Z}} \mathcal{D}(Q) \to \mathcal{G}(Q)$, and the image of $F \otimes_{\mathbb{Z}} \mathcal{D}(Q)$ coincides with $\mathrm{Prim}(\mathcal{G}(Q))$.

Theorem 15 ([11]) *Given any loop Q and a field F of characteristic zero then $\mathcal{G}(Q) \cong U(F \otimes_{\mathbb{Z}} \mathcal{D}(Q))$.*

Let us explain a little bit some details about the proof of this result since it involves a new important ingredient introduced by J. Mostovoy: *the linearizers.* To prove that the image of $F \otimes_{\mathbb{Z}} \mathcal{D}(Q)$ coincides with $\mathrm{Prim}(\mathcal{G}(Q))$ first one defines certain operations on Q and prove that they induce multilinear operations on the abelian group $\mathcal{D}(Q)$. Then one checks that in the image of $F \otimes_{\mathbb{Z}} \mathcal{D}(Q)$ these operations correspond to the operations defined by Shestakov and Umirbaev. At this point the result follows from Corollary 1.

As we will see in a moment, the dimension subloops are closed under too many operations. The main problem then is to discriminate which ones will serve. Given a function $f: Q \times \cdots \times Q \to Q$ the *linearizer* of f on the slot i is

$$\begin{aligned} f_i(\ldots, a_{i-1}, a, b, a_{i+2}, \ldots) &= \\ &f(\ldots, a_{i-1}, a, a_{i+2}, \ldots) f(\ldots, a_{i-1}, b, a_{i+2}, \ldots) \backslash f(\ldots, a_{i-1}, ab, a_{i+2}, \ldots). \end{aligned}$$

It is clear that if f is defined in terms of $\cdot, \backslash$ and $/$, and it satisfies the property

$$f(a_1, \ldots, a_n) = 1 \text{ if } a_i = 1 \text{ for some } i$$

then any linearizer of f also satisfies the same property. The linearization of f, in the sense of Section 4, extends f to a multilinear operation $\tilde{f}: F[Q] \otimes \cdots \otimes F[Q] \to F[Q]$ with $\tilde{f}(a_1, \ldots, a_n) = \epsilon(a_1) \cdots \epsilon(a_n) 1$ if $a_i \in F$ for some i. Since $I^i \cdot I^j$, $I^i \backslash I^j$, $I^i / I^j \subseteq I^{i+j}$, this property obviously implies that $f(D_{n_1}(Q), \ldots, D_{n_k}(Q)) \subseteq D_{n_1 + \cdots + n_k}(Q)$ (recall that $\tilde{f}$ extends f). Similarly $f_i(D_{n_1}(Q), \ldots, D_{n_{k+1}}(Q)) \subseteq D_{n_1 + \cdots + n_{k+1}}(Q)$. This behavior of f and f_i with respect to the dimension subloops and the very definition of f_i imply that the operation

$$f(a_1 D_{n_1+1}(Q), \ldots, a_k D_{n_k+1}(Q)) = f(a_1, \ldots, a_k) D_{n_1 + \cdots + n_k + 1}(Q)$$

is well defined and provides a multilinear operation

$$f: \mathcal{D}(Q) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathcal{D}(Q) \to \mathcal{D}(Q).$$

The iterated linearizers $f_{1, \cdots, 1, m+1, \ldots, m+1}(a_1, \ldots, a_m, b_1, \ldots, b_n, c)$ of the associator $f(a, b, c) = (a(bc)) \backslash ((ab)c)$ will correspond to the operations of Shestakov and Umirbaev [11].

By analogy with groups, given a normal subloop N of a loop Q there exists a unique smallest normal subloop $[N, Q]$ such that $N/[N, Q]$ lies in the center of $Q/[N, Q]$. The lower central series of Q is defined as

$$Q = \gamma_1(Q) \trianglerighteq \gamma_2(Q) \trianglerighteq \cdots$$

with $\gamma_{n+1}(Q) = [\gamma_n(Q), Q]$ and generalizes the usual lower central series for groups. It is natural to try to induce multilinear maps on $F \otimes_{\mathbb{Z}} \mathcal{L}(Q)$, with $\mathcal{L}(Q) = \bigoplus_{n=1}^{\infty} \gamma_n(Q)/\gamma_{n+1}(Q)$, so that it becomes a Sabinin algebra. However, contrary to what happens with the dimension subloops, even for the iterated linearizers of the associator it may occur that

$$f(\gamma_{n_1}(Q),\ldots,\gamma_{n_k}(Q)) \not\subseteq \gamma_{n_1+\cdots+n_k}(Q)$$

as shown in [10]. Therefore, it seems quite difficult to induce such an structure on $F \otimes_{\mathbb{Z}} \mathcal{L}(Q)$. All these considerations have motivated the definition of the *commutator–associator series* by J. Mostovoy as a substitute for the usual lower central series [10].

References

[1] M.R. Bremner, I.R. Hentzel and L.A. Peresi, Dimension formulas for the free nonassociative algebra, *preprint*.

[2] R.H. Bruck, *A survey of binary systems* (Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 20. Reihe: Gruppentheorie Springer Verlag, Berlin–Göttingen-Heidelberg 1958)

[3] O. Chein, H. O. Pflugfelder and J. D. H. Smith (Eds.), Quasigroups and loops: theory and applications. (Sigma Series in Pure Mathematics, 8. Heldermann Verlag, Berlin, 1990).

[4] B. Hartley, *Topics in the theory of nilpotent groups* (Group theory, Academic Press, London, 1984, pp. 61–120.)

[5] K. Kunen, Moufang Quasigroups, *J. Algebra* **183** (1996), no. 1, 231–234.

[6] E.N. Kuz'min, On a relation between Malcev algebras and analytic Moufang loops, *Algebra Logika* **10**, no. 1 (1971), 3–22. English transl.: *Algebra Logic* **10** (1972), 1–14.

[7] A.I. Malcev, Analytic loops, *Mat. Sb. (N. S.)* 36 (78), no.3 (1955), 569-575.

[8] J.W. Milnor and J.C. Moore, On the structure of Hopf algebras, *Ann. of Math.* (2) **81** (1965), 211–264.

[9] P.J. Morandi, J.M. Pérez–Izquierdo and S. Pumplün, On The Tensor Product of Composition Algebras, *J. Algebra* **243** (2001), 41–68.

[10] J. Mostovoy, On the notion of the lower central series for loops, *preprint*.

[11] J. Mostovoy, J.M. Pérez–Izquierdo, Dimension filtration on loops, *preprint*.

[12] J.M. Pérez–Izquierdo, An Envelope for Bol Algebras, *to appear in J. Algebra*.

[13] J.M. Pérez–Izquierdo, Algebras, hyperalgebras, nonassociative bialgebras and loops, http://mathematik.uibk.ac.at/mathematik/jordan/.

[14] J.M. Pérez–Izquierdo and I.P. Shestakov, An Envelope for Malcev Algebras, *J. Algebra* **272** (2004), 379–393.

[15] H.O. Pflugfelder, *Quasigroups and loops: introduction* (Sigma Series in Pure Mathematics, 7. Heldermann Verlag, Berlin, 1990).

[16] D.G. Quillen, On the associated graded ring of a group ring, *J. Algebra* **10** (1968), 411–418.

[17] L.V. Sabinin, *Smooth Quasigroups and Loops* (Mathematics and Its Applications, 492, Kluwer Academic Publishers, 1999).

[18] L.V. Sabinin and P.O. Mikheev, Analytic Bol loops, *Webs and quasigroups* (Kalinin. Gos. Univ., Kalinin, 1982) 102–109.

[19] L.V. Sabinin and P.O. Mikheev, On local analytic loops and their corresponding hyperalgebras, *Proceedings of 9th conference of young researchers* (VINITI Press, Moscow, 1986) 34–54.

[20] L.V. Sabinin and P.O. Mikheev, Infinitesimal theory of local analytic loops, *Soviet Math. Dokl.* **36** (1988), no. 3, 545-548.

[21] V. Shcherbacov and V. Izbash, On quasigroups with Moufang identity, *Buletinul A. S. a R. M. Matematica* **27** (1998), no. 2, 109–116.

[22] I.P. Shestakov and U.U. Umirbaev, Free Akivis algebras, primitive elements, and hyperalgebras, *J. Algebra* **250** (2002), no. 2, 533–548.

[23] E.M. Sweedler, *Hopf algebras* (Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969).

José M. Pérez–Izquierdo
Dpto. Matemáticas y Computación
Universidad de La Rioja
26004, Logroño
Spain
e-mail: jm.perez@dmc.unirioja.es