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Discrete group transforms on $SU(2) \times SU(2)$ and $SU(3)$

S. Grimm, F. Lemire, and J. Patera[1]

1 Introduction

From the point of view of this paper, the cosine transform [1, 21], which underlies important recent applications such as JPEG, is just one of the discrete transforms [7], which are based on certain discretizations of the simple Lie group $SU(2)$ in one dimension, on $SU(2) \times SU(2)$ in two dimensions, etc.. The traditional presentation of cosine transform techniques has no need of group theory.

In this paper we develop the group theoretical foundation for transforms on the compact semisimple Lie groups $SU(2) \times SU(2)$ and $SU(3)$, following the general reference [17]. Our goal is to show that the method is very versatile as to the discretizations with various densities of grid points, that it is computationally very efficient and easy-to-use.

Motivation for studying discrete Fourier-like methods stems from the fact that numerical computation of various transforms is being widely implemented in practical science and industry. The need arises in general from two directions: The first one is the processing of the large amounts of digital data which are being collected today. The second one is the need for exceptionally fast versions of harmonic analysis in many fields of science and technology. Generally known and extensively used are the now traditional methods of the fast Fourier analysis and the wavelet analysis using various types of wavelets [14]. Even if these methods are continuous in principle, they require discretization for really fast implementation.

In this article we describe the basics of the 1- and 2- dimensional versions of a very different and little known general multidimensional discrete transform [17] based on compact semisimple Lie groups. It is particularly suitable in situations where Fourier decompositions of similar type need to be performed with maximal speed many times over. The key to the efficiency of this approach is that a large body of auxiliary information (so called decomposition matrix for the problem) can be computed in advance once and used in all subsequent calculations.

In the past the method has been used for problems of rather challenging but theoretical interest. Thus the Fourier series in eight variables were computed, involving representations of the largest exceptional simple Lie group E_8 [9, 12, 11, 16] for example to determine the decomposition of E_8 plethysms ([11]). This problem could not be solved by other means without a major new computational effort and yet using the present method it is no different than any other Fourier series problem.

Most recently new interest in the method arose in connection with the 'good' behaviour of its continuous extension [5, 3, 4, 2] , opening new possibilities for

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applications in image processing and data compression. In turn this motivated further interest in the properties of the (continuous) expansion function which the method uses [19, 18].

Although this multidimensional method is most advantageous in the high dimensional context because its effectiveness grows with the order of the Weyl group of the underlying semisimple Lie group, we describe it in a ready-to-use form in the two lowest dimensional cases $\mathbf{SU}(2)$ and $\mathbf{SU}(3)$, i.e. the two least advantageous ones for the method, because of their practical importance and their relative transparency.

In [17] a discrete computational algorithm is described for decomposing certain class functions of an arbitrary semisimple compact Lie group G into sums of irreducible characters, or equivalently into sums of orbit functions. The idea of the method is to approximate the Lie group by a finite set of its discrete elements, suitably chosen for the problem at hand, so that the integration over the group is replaced by a discrete summation. The Fourier series must be finite, however, it can be arbitrarily long. An orbit function is a sum of exponential functions (two or one of them for $\mathbf{SU}(2)$; six, three or one for $\mathbf{SU}(3)$) related by the requirement or invariance under the Weyl group of the Lie group. The method is based on the use of elements of finite order in G and has applications ranging from computations in representation theory to signal processing.

From another perspective one may view this method as the Fourier expansion of functions defined on the fundamental region of the corresponding affine Weyl group. This region is known to be a simplex of the appropriate dimension. For $\mathbf{SU}(2)$ this is just a finite segment of the real line, for $\mathbf{SU}(3)$, it is the equilateral triangle, for other 2-dimensional cases these are triangles with angles $\{\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}\}$.

A simple minded version of the method for $\mathbf{SU}(2)$ can be stated as follows. Suppose there is a Fourier series with N terms equal to the function $f(\theta)$, $0 \leq \theta \leq \pi$ and that the task is to determine the coefficients of the series.

In this paper we develop this method for the Lie groups $G = \mathbf{SU}(2)$, $\mathbf{SU}(2) \times \mathbf{SU}(2)$, and $\mathbf{SU}(3)$. Our objective is to effectively determine the decompositions of class functions into linear combinations of orbit functions provided the class functions are known, à priori, to have nonzero coefficients in some fixed finite range. For most computational applications the functions may be assumed to satisfy these conditions. The main feature of the method is the construction of a decomposition matrix D, computed once and for all for a given range of functions. The coefficients for a function in this range are then determined by a simple matrix multiplication of D by a vector calculated by evaluating the function at certain points in the fundmental region.

2 Class Functions and Orbit Functions on SU(2)

The Lie group $\mathbf{SU}(2)$ can be realized as the group of all 2×2 complex unitary matrices. A complex valued class function f on $\mathbf{SU}(2)$ is any map $f : \mathbf{SU}(2) \longrightarrow \mathbb{C}$ which is invariant under conjugation – i.e. $f(C^{-1}AC) = f(A)$ for all $A, C \in \mathbf{SU}(2)$. Let T denote the abelian subgroup of $\mathbf{SU}(2)$ consisting of all diagonal elements in the defining representation,

$$T = \left\{ \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \mid 0 \le \theta \le 1 \right\}$$

Since every unitary matrix can be diagonalized by a unitary transformation, every element of $\mathbf{SU}(2)$ is conjugate to an element of T. Consequently every class function f on $\mathbf{SU}(2)$ is completely determined by its restriction to the subgroup T. Moreover, we know that every element of $\mathbf{SU}(2)$ is conjugate to exactly one element in the set

$$F = \left\{ \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \mid 0 \le \theta \le 1/2 \right\} \subset T$$

The set of special functions on $\mathbf{SU}(2)$, called trace functions or characters, plays an important role. Consider for example $x(\theta) \in F$. Then $\operatorname{tr} x(\theta) = 2\cos(2\pi\theta)$ is a class function of $\mathbf{SU}(2)$ since $\operatorname{tr}\{C^{-1}x(\theta)C\} = \operatorname{tr}\{x(\theta)\}\}$ for all $C \in \mathbf{SU}(2)$.

For every finite dimensional representation (ρ, V) of $\mathbf{SU}(2)$ the character function denoted by $\chi_\rho : \mathbf{SU}(2) \longrightarrow \mathbb{C}$ is defined by $\chi_\rho(x) = \operatorname{tr}\rho(x)$, where $x \in \mathbf{SU}(2)$. Clearly each character function is a class function. Let $R = R(\mathbf{SU}(2))$ denote the complex algebra generated by the character functions of $\mathbf{SU}(2)$ where addition and multiplication are derived from the formation of direct sums and tensor products of representations. That is if (ρ_1, V_1) and (ρ_2, V_2) are finite dimensional representations of $\mathbf{SU}(2)$ then $\chi_{\rho_1} + \chi_{\rho_2} = \chi_{\rho_1 \oplus \rho_2}$ and $\chi_{\rho_1}\chi_{\rho_2} = \chi_{\rho_1 \otimes \rho_2}$. It is well known that R has a linear basis consisting of the characters of all finite dimensional irreducible representations. Thus every class function on $\mathbf{SU}(2)$ can be written as a linear combination of irreducible characters.

Another basis for the space of class functions on $\mathbf{SU}(2)$, more suitable for our purposes than the irreducible characters, consists of the orbit functions. With each irreducible representation of a semisimple Lie group, in particular $\mathbf{SU}(2)$, one associates a set of weights, linear integral combinations of the basis of fundamental weights. (In physics the weights are the projections of angular momenta, taking integer or half-odd values. In mathematics one prefers to work with twice the angular momenta in order to avoid the non-integer values.) The set of weights of a representation is a union of orbits of weights under the action of the Weyl group W. In the $\mathbf{SU}(2)$ case, W has only two elements, $W = \langle 1, -1 \rangle$. Suppose ω denotes the single fundamental weight of $\mathbf{SU}(2)$, then an irreducible finite dimensional representation of $\mathbf{SU}(2)$ is specified by a non-negative integer l. The corresponding integer or half-odd values. In mathematics one prefers to work with twice the angular momenta in order to avoid the non-integer values.) The set of weights of a representation is a union of orbits of weights under the action of the Weyl group W. In the $\mathbf{SU}(2)$ case, W has only two elements, $W = \langle 1, -1\rangle$. Suppose ω denotes the single fundamental weight of $\mathbf{SU}(2)$, then an irreducible finite dimensional representation of $\mathbf{SU}(2)$ is specified by a non-negative integer l. The corresponding set $\Omega(l)$ of all weights appearing in this representation contains $l+1$ weights:

$$\Omega(l) := \{\lambda = m\omega \mid m \in \{-l, -l+2, \ldots, l\}\}$$

The W-orbit of an $\mathbf{SU}(2)$-weight $\lambda = m\omega$ is the set

$$W\lambda = \begin{cases} \{\lambda, -\lambda\}, & \text{for } m \neq 0 \\ \{0\}, & \text{for } m = 0 \end{cases}$$

For each weight $\lambda = m\omega$, with m a non negative integer, we associate a function $\Phi_m : [0, \frac{1}{2}] \longrightarrow \mathbb{C}$ called an orbit function defined by

$$\Phi_m(\theta) = \sum_{\mu \in W\lambda} e^{2\pi i \mu(x(\theta))} = \begin{cases} e^{2\pi i m\theta} + e^{-2\pi i m\theta} = 2\cos(2\pi m\theta) & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$$

It is well known that the set of all orbit functions on $\mathbf{SU}(2)$ forms a linear basis for the algebra R of the class functions, i.e. every class function f of $\mathbf{SU}(2)$ can be decomposed into the Fourier series,

$$f(\theta) = \sum_{l=0}^{\infty} a_l \Phi_l(\theta)$$

Our goal is to describe an efficient computational technique for determining these coefficients a_i in the case where it is known à priori that the function f is a linear combination of finitely many orbit functions.

3 Elements of finite Order and Decomposition matrices

For each positive integer N let T_N denote the set of all elements of T having adjoint order N (the order in the adjoint representation) divides N. It is clear (see [10] or [15] §4) that there are exactly $N+1$ conjugacy classes of such elements namely

$$\left\{ x_{N,k} = \begin{pmatrix} e^{\frac{2\pi i k}{2N}} & 0 \\ 0 & e^{-\frac{2\pi i k}{2N}} \end{pmatrix} \mid k = 0, 1, \ldots, N \right\}$$

The order $C_{N,k}$ of the conjugacy class of the element $x_{N,k}$ is given by

$$C_{N,k} = \begin{cases} 1 \text{ if } k = 0, N \\ 2 \text{ otherwise.} \end{cases}$$

We now define a sesquilinear form on R by setting for all $f, g \in R$

$$\langle f, g \rangle_N = \sum_{x \in T_N} f(x)\overline{g(x)}$$

By taking advantage of the conjugacy structure in T_N we can reduce this form to

$$\langle f, g \rangle_N = \sum_{k=0}^{N} C_{N,k} f(x_{N,k})\overline{g(x_{N,k})}$$

It is easily verified that the set of orbit functions $\mathcal{O}_N = \{\Phi_k \mid k = 0, \cdots, N\}$ are orthogonal with respect to the form $\langle \cdot, \cdot \rangle_N$. In particular, for $0 \leq a, b \leq N$ we have

$$\frac{\langle \Phi_a, \Phi_b \rangle_N}{\langle \Phi_a, \Phi_a \rangle_N} = \delta_{ab}.$$

More precisely for $0 \leq a \leq N$ and $b \geq 0$

$$\langle \Phi_a, \Phi_b \rangle_N = K_{N,(a,b)} 2N \tag{1}$$

where

$$K_{N,(a,b)} = \begin{cases} 0 & \text{if } a \neq \pm b \pmod{2N} \\ 1 & \text{if } a = b = 0 \\ 4 & \text{if } a = N, b = nN \text{ for } n > 0 \\ 2 & \text{otherwise.} \end{cases}$$

Our proposed computational method for decomposing class functions as linear combinations of orbit functions hinges on this orthogonality.

More precisely, if $f \in R$ with $f = \sum_{k=0}^{N} a_k \Phi_k$ then we can compute the coefficients a_k from the fact that

$$\langle f, \Phi_k \rangle_N = \begin{cases} 2Na_0 & \text{if } k = 0 \\ 8Na_N & \text{if } k = N \\ 4Na_k & \text{otherwise.} \end{cases}$$

This calculation can be formalized into a matrix multiplication as follows. Let $D_N = (D_N^{i,j})$ denote the square $N+1 \times N+1$ matrix whose $(i,j)^{th}$ component $D_N^{i,j}$ is given by

$$D_N^{i,j} := \frac{K_i C_j \overline{\Phi_i(x_{N,j})}}{8N}, \quad \text{where } K_i = \begin{cases} 4 & \text{if } i = 0 \\ 1 & \text{if } i = N \\ 2 & \text{otherwise.} \end{cases}$$

If we define $v(N, f) = (f(x_0), \cdots, f(x_N))^T$. Then

$$D_N \; v(N, f) = \begin{pmatrix} a_0 \\ \vdots \\ a_N \end{pmatrix}.$$

More generally, if $P > N$ and $f \in R$ with $f = \sum_{k=0}^{P} a_k \Phi_k$ then again forming the vector $v(N,f) = (f(x_{N,0}), \cdots, f(x_N))^T$ we have that the i^{th} component of $D_N\, v(N,f)$ is equal to the sum of all coefficients of f with index equal to $\pm i$ (mod $2N$).

In order to illustrate the concepts introduced in this section we consider a simple example where $N = 5$. For convenience we set $\omega = e^{\frac{2\pi i}{10}}$, a primitive tenth root of unity. The values of the first 10 orbit functions of $\mathbf{SU}(2)$ on the elements $x_i = x_{5,i} \in T_5$ are given
in the following table:

	x_0	x_1	x_2	x_3	x_4	x_5
Φ_0	1	1	1	1	1	1
Φ_1	2	$\omega+\omega^9$	$\omega^2+\omega^8$	$\omega^3+\omega^7$	$\omega^4+\omega^6$	$2\omega^5$
Φ_2	2	$\omega^2+\omega^8$	$\omega^4+\omega^6$	$\omega^4+\omega^6$	$\omega^2+\omega^8$	2
Φ_3	2	$\omega^3+\omega^7$	$\omega^4+\omega^6$	$\omega+\omega^9$	$\omega^2+\omega^8$	$2\omega^5$
Φ_4	2	$\omega^4+\omega^6$	$\omega^2+\omega^8$	$\omega^2+\omega^8$	$\omega^4+\omega^6$	2
Φ_5	2	$2\omega^5$	2	$2\omega^5$	2	$2\omega^5$
Φ_6	2	$\omega^4+\omega^6$	$\omega^2+\omega^8$	$\omega^2+\omega^8$	$\omega^4+\omega^6$	2
Φ_7	2	$\omega^3+\omega^7$	$\omega^4+\omega^6$	$\omega+\omega^9$	$\omega^2+\omega^8$	$2\omega^5$
Φ_8	2	$\omega^2+\omega^8$	$\omega^4+\omega^6$	$\omega^4+\omega^6$	$\omega^2+\omega^8$	2
Φ_9	2	$\omega+\omega^9$	$\omega^2+\omega^8$	$\omega^3+\omega^7$	$\omega^4+\omega^6$	$2\omega^5$
Φ_{10}	2	2	2	2	2	2

It is easily verified that the orbit functions $\mathcal{O}_5 = \{\Phi_0, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5\}$ are orthogonal with respect to the form $\langle\cdot,\cdot\rangle_5$. In particular

$$\begin{aligned}\langle\Phi_1,\Phi_1\rangle_5 = 2\cdot\overline{2} + 2(\omega+\omega^9)^2 + 2(\omega^2+\omega^8)^2 \\ + 2(\omega^3+\omega^7)^2 + 2(\omega^4+\omega^6)^2 + (2\omega^5)^2 = 20 = 4\times 5\end{aligned}$$

and

$$\begin{aligned}\langle\Phi_1,\Phi_2\rangle_5 = 2\cdot\overline{2} + 2(\omega+\omega^9)\overline{(\omega^2+\omega^8)} + 2(\omega^2+\omega^8)\overline{(\omega^4+\omega^6)} \\ + 2(\omega^3+\omega^7)\overline{(\omega^4+\omega^6)} + (2\omega^5)\overline{(2)} = 0.\end{aligned}$$

The decomposition matrix D_5 is given by

$$\frac{1}{40}\begin{pmatrix} 4 & 8 & 8 & 8 & 8 & 4 & \\ 4 & 4(\omega+\omega^9) & 4(\omega^2+\omega^8) & 4(\omega^3+\omega^7) & 4(\omega^4+\omega^6) & 4\omega^5 & \\ 4 & 4(\omega^2+\omega^8) & 4(\omega^4+\omega^6) & 4(\omega^4+\omega^6) & 4(\omega^2+\omega^8) & 4 & \\ 4 & 4(\omega^3+\omega^7) & 4(\omega^4+\omega^6) & 4(\omega+\omega^9) & 4(\omega^2+\omega^8) & 4\omega^5 & \\ 4 & 4(\omega^4+\omega^6) & 4(\omega^2+\omega^8) & 4(\omega^2+\omega^8) & 4(\omega^4+\omega^6) & 4 & \\ 2 & 2(2\omega^5) & 2(2) & 2(2\omega^5) & 2(2) & 2(\omega^5) & 2 \end{pmatrix}$$

Now assume that f is a class function on $\mathbf{SU}(2)$ which is known to be a linear combination of the orbit functions from the set $\mathcal{O}_5$. Assume further that we are given that

$$f(x_0) = 7 \quad f(x_1) = 1 - \omega - \omega^9 \quad f(x_2) = \omega^4 + \omega^6$$
$$f(x_3) = 1 - \omega^3 - \omega^7 \quad f(x_4) = \omega^2 + \omega^8 \text{ and } \quad f(x_5) = 3$$

Then we determine the coefficients of the orbit functions in the decomposition of f by applying D_5 :

$$\frac{1}{40}\begin{pmatrix} 4 & 8 & 8 & 8 & 8 & 4 & \\ 4 & 4(\omega+\omega^9) & 4(\omega^2+\omega^8) & 4(\omega^3+\omega^7) & 4(\omega^4+\omega^6) & 4\omega^5 & \\ 4 & 4(\omega^2+\omega^8) & 4(\omega^4+\omega^6) & 4(\omega^4+\omega^6) & 4(\omega^2+\omega^8) & 4 & \\ 4 & 4(\omega^3+\omega^7) & 4(\omega^4+\omega^6) & 4(\omega+\omega^9) & 4(\omega^2+\omega^8) & 4\omega^5 & \\ 4 & 4(\omega^4+\omega^6) & 4(\omega^2+\omega^8) & 4(\omega^2+\omega^8) & 4(\omega^4+\omega^6) & 4 & \\ 2 & 2(2\omega^5) & 2(2) & 2(2\omega^5) & 2(2) & 2(\omega^5) & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 1-\omega-\omega^9 \\ \omega^4+\omega^6 \\ 1-\omega^3-\omega^7 \\ o^2+\omega^8 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

We conclude that $f = \Phi_0 + \Phi_2 + \Phi_3 + \Phi_4$.

4 Decomposition of Class functions on $\mathbf{SU}(2) \times \mathbf{SU}(2)$

The decomposition of 1-dimensional class functions on $\mathbf{SU}(2)$ can easily be extended to 2-dimensional class functions in a trivial manner by using the compact Lie group $\mathbf{SU}(2) \times \mathbf{SU}(2)$. The group $G = \mathbf{SU}(2) \times \mathbf{SU}(2)$ denotes the set of all ordered pairs (A, B), with $A, B \in \mathbf{SU}(2)$. Multiplication in $\mathbf{SU}(2) \times \mathbf{SU}(2)$ is defined componentwise – i.e. if $(A, B), (C, D) \in \mathbf{SU}(2) \times \mathbf{SU}(2)$ then $(A, B)(C, D) = (AC, BD)$. Every element in G is conjugate to an element in the subgroup $T_{\mathbf{SU}(2)\times\mathbf{SU}(2)} := T \times T$ and from our previous results we have that every element in G is conjugate to exactly one element in the set

$$\left\{ \left(\begin{pmatrix} e^{2\pi i\phi} & 0 \\ 0 & e^{-2\pi i\phi} \end{pmatrix}, \begin{pmatrix} e^{2\pi i\psi} & 0 \\ 0 & e^{-2\pi i\psi} \end{pmatrix} \right) \mid 0 \le \phi, \psi < \frac{1}{2} \right\}.$$

Clearly this set can be identified geometrically with a square and therefore the complex valued class functions on $\mathbf{SU}(2) \times \mathbf{SU}(2)$ are in 1–1 correspondence with the functions on the square

$$f : \square \longrightarrow \mathbb{C}.$$

The dominant weights λ of G are given by the set of all functions $\lambda_{(a,b)} = a\omega_1 + b\omega_2$ (in short (a, b)), where $a, b \in \mathbb{Z}^{\ge 0}$ and ω_1 (resp. ω_2) are the fundamental weights in first (resp. second) copy of $\mathbf{SU}(2)$. Further the Weyl Group of G is simply the Cartesian product of the Weyl group of $\mathbf{SU}(2)$ with itself. Then we associate with each dominant weight $\lambda_{(a,b)}$ an orbit function $\Phi_{(a,b)}$ defined by

$$\Phi_{(a,b)} = \sum_{(c,d)\in W\times W(a,b)} e^{2\pi i(c\omega_1 + d\omega_2)} = \Phi_a \Phi_b$$

where Φ_a and Φ_b are orbit functions (2) on the respective $\mathbf{SU}(2)$ components of G. These orbit functions of G are known to be a basis for the space of all class functions on G. As before the goal is to decompose a given class function on G into a linear combination of orbit functions.

5 Decompositon of Class functions on $\mathbf{SU}(2) \times \mathbf{SU}(2)$

For any pair of positive integers N and M, $T_N \times T_M$ is an abelian subgroup of $T_{\mathbf{SU}(2)\times\mathbf{SU}(2)}$. The conjugacy classes of elements in $T_N \times T_M$ are in one-one correspondence with the set

$$\left\{(x_{N,k}, x_{M,l}) = \left(\begin{pmatrix} e^{\frac{2\pi ik}{2N}} & 0 \\ 0 & e^{-\frac{2\pi ik}{2N}}\end{pmatrix}, \begin{pmatrix} e^{\frac{2\pi il}{2M}} & 0 \\ 0 & e^{-\frac{2\pi il}{2M}}\end{pmatrix}\right)\right\}.$$

where $0 \leq k \leq N, 0 \leq l \leq M$. Further, for $(x_{N,k}, x_{M,l})$ we have

$$\begin{aligned}\Phi_{a,b}((x_{N,k}, x_{M,l}) &= \Phi_a(x_{N,k})\Phi_b(x_{M,l}) \\ &= \left(e^{2\pi i\frac{ka}{N}} + e^{-2\pi i\frac{ka}{N}}\right)\left(e^{2\pi i\frac{lb}{M}} + e^{-2\pi i\frac{lb}{M}}\right), \quad (a, b \neq 0)\end{aligned}$$

As in the case $\mathbf{SU}(2)$, we can now define a sesquilinear form on the space of all class functions on G by setting

$$\begin{aligned}\langle f, g\rangle_{N,M} &= \sum_{(x,y)\in T_N\times T_M} f(x,y)\overline{g(x,y)} \\ &= \sum_{k=1}^{N-1}\sum_{l=1}^{M-1} C_{N,k}C_{M,l}f(x_{N,k}, x_{M,l})\overline{g(x_{N,k}, x_{M,l})}.\end{aligned}$$

In particular if we apply this form to orbit functions we observe that

$$\begin{aligned}\langle\Phi_{(a,b)}, \Phi_{(c,d)}\rangle_{N,M} &= \sum_{(x,y)\in T_N\times T_M} \Phi_{(a,b)}(x,y)\overline{\Phi_{(c,d)}(x,y)} \\ &= \sum_{x\in T_N}\sum_{y\in T_M} \Phi_a(x)\Phi_b(y)\overline{\Phi_c(x)\Phi_d(y)} \\ &= \sum_{x\in T_N} \Phi_a(x)\overline{\Phi_c(x)} \sum_{y\in T_N} \Phi_b(y)\overline{\Phi_d(y)} \\ &= \langle\Phi_a, \Phi_c\rangle_N \langle\Phi_b, \Phi_d\rangle_M\end{aligned}$$

Therefore, choosing integers a, b, c, d such that $0 \leq a < N$, $0 \leq b < M$, and $c, d \geq 0$, we obtain with (1)

$$\langle \Phi_{(a,b)}, \Phi_{(c,d)} \rangle_{N,M} = K_{N,(a,c)} K_{M,(b,d)} 4NM$$

where, $K_{N,(a,c)}$ and $K_{M,(b,d)}$ are defined in (1). It follows then that the orbit functions in the set

$$\mathcal{O}_{N,M} = \{\Phi_{(a,b)} \mid 0 \leq a < N, 0 \leq b < M\}$$

are orthogonal with respect to the form $\langle \cdot, \cdot \rangle_{N,M}$:

$$\langle \Phi_{(a,b)}, \Phi_{(c,d)} \rangle_{N,M} = 0 \quad \text{for } \Phi_{(a,b)} \neq \Phi_{(c,d)} \in \mathcal{O}_{N,M}.$$

Therefore, in analogy with the case of orbit functions on the group $\mathbf{SU}(2)$, if f is an orbit function on G which is known a priori to be a linear combination of the orbit functions in the set $\mathcal{O}_{N,M}$ then we can use the form $\langle \cdot, \cdot \rangle_{N,M}$ to determine the coefficients of this decomposition. In fact this calculation can be formalized into a matrix multiplication by a decomposition matrix $D_{N,M}$ which is equal to the tensor product of the decomposition matrices D_N and D_M used to decompose orbit functions on $\mathbf{SU}(2)$.

REMARK. It is clear that we can extend this technique to decompose n-dimensional class functions on the compact Lie group consisting of the Cartesian product of n copies of $\mathbf{SU}(2)$. In this case the class functions are identified with functions on an n-dimensional cube – i.e. the fundamental region of the group $\mathbf{SU}(2) \times \cdots \times \mathbf{SU}(2)$.

6 Class Functions and Orbit Functions on $\mathbf{SU}(3)$

There exist 2 dimensional class functions defined on geometric regions corresponding to the fundamental regions for each of the rank 2 compact Lie groups. In this section we consider the class functions on the group $\mathbf{SU}(3)$ which consists of the complex unitary 3×3 matrices. Again, since every unitary matrix can be diagonalized by a unitary transformation, it suffices to restrict the domain of class functions on $\mathbf{SU}(3)$ to the diagonal matrices:

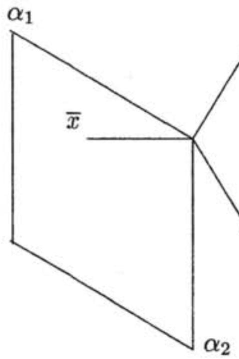
$$T := \left\{ \Theta(x, y, z) := \begin{pmatrix} e^{2\pi i x} & 0 & 0 \\ 0 & e^{2\pi i y} & 0 \\ 0 & 0 & e^{2\pi i z} \end{pmatrix} \mid x + y + z = 0 \right\}$$

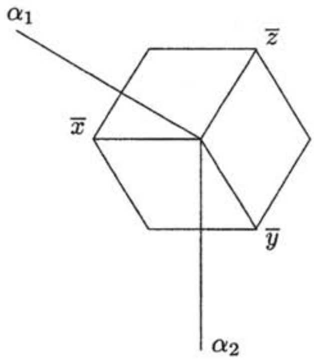
(Note that (x, y, z) and $(x + a, y + b, z + c)$, where a, b, c are integers with $a + b + c = 0$, represent the same element in T.) Let $\alpha_1 = \overline{(1, -1, 0)} = \overline{x} - \overline{y}$ and

$\alpha_2 = \overline{(0,1,-1)} = \overline{y} - \overline{z}$ be the projections of $(1,-1,0)$ and $(0,1,-1)$ to the plane defined by $x+y+z=0$. Then

$$T \cong \{a\alpha_1 + b\alpha_2 \mid 0 \le a, b < 1\} \\ \cong \{a\alpha_1 + b\alpha_2 \mid -1 < (2a-b), (2b-a), (a+b) \le 1\}. \quad (2)$$

Both are fundamental domains with respect to translations by the lattice generated by α_1 and α_2. The latter being a regular hexagon centered at the origin. This is shown in the following pictures, which are the projections onto the plane defined by the equation $x+y+z=0$:

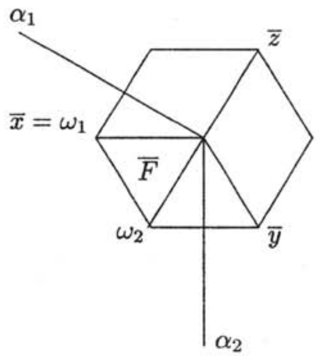




Two elements in T are conjugate, iff they differ by a permutation of their diagonal entries. Therefore the conjugacy classes in $\mathbf{SU}(3)$ are in 1–1 correspondence with the points of

$$F = \{(x,y,z) \in T \mid x \ge y \ge z\},$$

and F can be visualized in the projection to the plane $x+y+z=0$:



Denote the verices of F as in the above picture by $\omega_1 = \frac{1}{3}(2\alpha_1 + \alpha_2) = \overline{x}$ and $\omega_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2) = \overline{x} + \overline{y}$. The ω_1 and ω_2 are the *fundamental weights* of $\mathbf{SU}(3)$. Note that with the usual scalar product we have $(\omega_i, \alpha_j) = \delta_{ij}$. Then

$$F = \{a\omega_1 + b\omega_2 \mid a, b \ge 0,\ a+b \le 1\}$$

and a class function $f : \mathbf{SU}(3) \to \mathbb{C}$ can be identified with a function on the (equilateral) triangle $f : F \to \mathbb{C}$.

The Weyl group W of $\mathbf{SU}(3)$ can be seen as the group generated by the reflections in the lines orthogonal to α_1 and α_2, i.e. the lines along ω_1 and ω_2. Then for any weight $\lambda = a\omega_1 + b\omega_2$ the W orbit of λ is given by

$$\begin{aligned} W\lambda = \{ & a\omega_1 + b\omega_2,\ -a\omega_1 + (a+b)\omega_2,\ (a+b)\omega_1 - b\omega_2, \\ & b\omega_1 - (a+b)\omega_2,\ -(a+b)\omega_1 + a\omega_2,\ -b\omega_1 - a\omega_2\}. \end{aligned}$$

Recall the definition of the orbit function:

$$\Phi_\lambda = \sum_{\mu \in W\lambda} e^{2\pi i \lambda}$$

where

$$e^{2\pi i \lambda}(x\alpha_1 + y\alpha_2) = e^{2\pi i (ax+by)}.$$

It follows then that the orbit function $\Phi_{(a,b)}$ associated with the dominant integral weight $\lambda = a\omega_1 + b\omega_2$ is given by

$$\begin{aligned} \Phi_{(0,0)} &= 1 \\ \Phi_{(a,0)} &= e^{2\pi i a\omega_1} + e^{2\pi i(-a\omega_1 + a\omega_2)} + e^{-2\pi i a\omega_2} \\ \Phi_{(0,b)} &= e^{2\pi i b\omega_2} + e^{2\pi i(b\omega_1 - b\omega_2)} + e^{-2\pi i b\omega_1} \\ \Phi_{(a,b)} &= e^{2\pi i(a\omega_1 + b\omega_2)} + e^{2\pi i(-a\omega_1 + (a+b)\omega_2)} + e^{2\pi i((a+b)\omega_1 - b\omega_2)} + \\ & \quad e^{2\pi i(b\omega_1 - (a+b)\omega_2)} + e^{2\pi i(-(a+b)\omega_1 + a\omega_2)} + e^{2\pi i(-b\omega_1 - a\omega_2)} \end{aligned}$$

7 Decomposition of class functions on $\mathbf{SU}(3)$

Let T_N denote the subgroup of T containing all elements of adjoint order N. Clearly T_N contains $3N^2$ elements and the conjugacy classes of elements in T_N are in 1–1 correspondence with the elements in

$$T_N \cap F = \{\frac{a}{N}\omega_1 + \frac{b}{N}\omega_2 \in T \mid a, b \in \mathbb{Z}^{\geq 0}, a + b \leq N\}.$$

For computational purposes it is convenient to observe that the elements in $T_N \cap F$ can be written in terms of the $\{\alpha_1, \alpha_2\}$ basis. In fact these elements exactly correspond to the elements $c\alpha_1 + d\alpha_2$ where $c, d \in \frac{1}{3N}\mathbb{Z}^{\geq 0}$ with $c + d = 0 \pmod 3$, $2c \geq d$ and $2d \geq c$.

As before we use T_N to define a sesquilinear form on the set of all class functions on $\mathbf{SU}(3)$. In fact, for any class functions f, g on $\mathbf{SU}(3)$ we define

$$\langle f, g \rangle_N = \sum_{x \in T_N} f(x)\overline{g(x)}$$

The key observation is that the set of orbit functions $\{\Phi_{(a,b)} \mid a,b \geq 0, a+b \leq N\}$ is a maximal set of orthogonal functions with respect to this form. In fact, for $(a,b),(c,d) \in \{(x,y) \mid x,y \in \mathbb{Z}^{\geq 0}, x+y \leq N\}$

$$\frac{\langle \Phi_{(a,b)}, \Phi_{(c,d)} \rangle_N}{\langle \Phi_{(a,b)}, \Phi_{(a,b)} \rangle_N} = \delta_{(a,b),(c,d)}$$

(a,b). As in the previous section then we can take advantage of these orthogonality relations to compute the coefficients of a class function on the group $\mathbf{SU}(3)$. To illustrate we present a simple example. Assume that $N = 2$. There are exactly 12 elements of adjoint order 2 and these can be collected into 6 conjugacy classes

$$x_0 = 0 = (0,0), \qquad x_1 = \frac{1}{2}\omega_1 = \frac{1}{6}(2,1), \qquad x_2 = \frac{1}{2}\omega_2 = \frac{1}{6}(1,2),$$
$$x_3 = \omega_1 = \frac{1}{3}(2,1), \qquad x_4 = \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2 = \frac{1}{2}(1,1), \qquad x_5 = \omega_2 = \frac{1}{3}(1,2)$$

where we have listed the coordinates in the α-basis. The elements $\{x_0, x_3, x_5\}$ are central elements and hence their conjugacy classes have order one. The order of the conjugacy classes for the elements $\{x_1, x_2, x_4\}$ is three. The following table provides the values of the initial set of orbit functions on these elements:

	x_0	x_1	x_2	x_3	x_4	x_5
$\Phi_{(0,0)}$	1	1	1	1	1	1
$\Phi_{(1,0)}$	3	$\omega^2 + 2\omega^5$	$2\omega + \omega^4$	$3\omega^4$	-1	$3\omega^2$
$\Phi_{(0,1)}$	3	$2\omega + \omega^4$	$\omega^2 + 2\omega^5$	$3\omega^2$	-1	$3\omega^4$
$\Phi_{(2,0)}$	3	$3\omega^4$	$3\omega^2$	$3\omega^2$	3	$3\omega^4$
$\Phi_{(1,1)}$	6	-2	-2	6	-2	6
$\Phi_{(0,2)}$	3	$3\omega^2$	$3\omega^4$	$3\omega^4$	3	$3\omega^2$
$\Phi_{(3,0)}$	3	-1	-1	3	-1	3
$\Phi_{(2,1)}$	2(3)	$2(\omega^2 + 2\omega^5)$	$2(2\omega + \omega^4)$	$2(3\omega^4)$	$2(-1)$	$2(3\omega^2)$
$\Phi_{(1,2)}$	2(3)	$2(2\omega + \omega^4)$	$2(\omega^2 + 2\omega^5)$	$2(3\omega^2)$	$2(-1)$	$2(3\omega^4)$
$\Phi_{(0,3)}$	3	-1	-1	3	-1	3

where $\omega = e^{\frac{2\pi i}{6}}$. Note also that

$$\Phi_{(3,0)} = \Phi_{(0,3)} = \frac{1}{2}\Phi_{(1,1)}, \quad \Phi_{(2,1)} = 2\Phi_{(1,0)}, \quad \Phi_{(1,2)} = 2\Phi_{(0,1)}.$$

By direct computation we find that

$$\begin{aligned}
\langle \Phi_{(0,0)}, \Phi_{(0,0)} \rangle_2 &= 12 \\
\langle \Phi_{(1,0)}, \Phi_{(1,0)} \rangle_2 &= 36 = 3 \times 12 \\
\langle \Phi_{(2,0)}, \Phi_{(2,0)} \rangle_2 &= 108 = 9 \times 12 \\
\langle \Phi_{(1,1)}, \Phi_{(1,1)} \rangle_2 &= 144 = 12 \times 12
\end{aligned}$$

8 Eigenfunctions for the Neumann problem on an equilateral triangle

The orbit functions of dominant integral weights on the compact Lie group $\mathbf{SU}(3)$ are related directly to the eigenfunctions for the Neumann problem on an equilateral triangle [20]. In fact the fundamental region of $\mathbf{SU}(3)$ with respect to conjugation is the equilateral triangle

$$F = \{a\alpha_1 + b\alpha_2 \mid a, b \geq 0; a + b \leq 1\}.$$

For each dominant integral weight $\mu = m\omega_1 + n\omega_2$ the associated orbit function Φ_μ can be viewed as a complex valued function with domain F. If we identify α_1 and α_2 with the vectors in the plane having rectangular coordinates $(3/2, -\sqrt{3}/2)$ and $(0, \sqrt{3})$ respectively then in terms of rectangualr coordinates we have that

$$F = \{(x, y) \mid 0 \leq y \leq x\sqrt{3}; y \leq \sqrt{3}(1 - x)\}.$$

The orbit function Φ_μ can then be expressed in terms of rectangualar coordinates as

$$\Phi_\mu(x, y) = \frac{1}{C_{mn}} \sum_{(k,l)} exp\left(\frac{2\pi i}{3}(kx + l\sqrt{3}y)\right)$$

where the sum is taken over the indices $(k, l) \in \{(2m + n, \pm n), (n - m, \pm(m + n)), -(2n + m), \pm m)\}$, and C_{mn} denotes the orbit size of the weight μ. By direct calculation one can easily verify that Φ_μ is an eigenfunction for the Neumann problem on the equilateral triangle F – i.e. $f = \Phi_\mu$ satisfies

$$\begin{aligned} \Delta f + \lambda f &= 0 \quad \text{on } D \\ \frac{\delta f}{\delta n} &= 0 \quad \text{on } \delta D \end{aligned}$$

with the eigenvalue $\lambda = \frac{16\pi^2}{9}(m^2 + n^2 + mn)$.

By comparing with [6, 20] we see that, in fact, the orbit functions associated with the dominant integral weights of the compact Lie group $\mathbf{SU}(3)$ exactly coincide with the solutions to the Neumann problem. It follows then that the problem of decomposing a class function on $\mathbf{SU}(3)$ in terms of orbit functions is equivalent to the decomposition of the class function in terms of eigenfunctions for the Neumann problem on an equilateral triangular region in the plane.

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F. Lemire
Dep. of Mathematics
University of Windsor
Windsor, Ontario
Canada
e-mail: lemire@uwindsor.ca

J. Patera
Univ. Montreal
Canada
e-mail: patera@crm.umontreal.ca

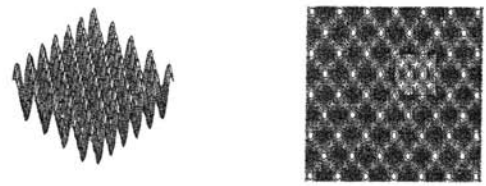

Figure 1: Dependence of the $SU(2)\times SU(2)$ orbit function $\Phi_{2,3}(x,y)$ on continuous variables within $-1.5 < x, y < 1.5$. Lighter color square indicates the fundamental region of the group.

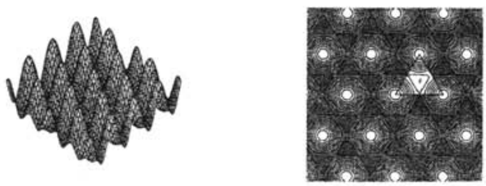

Figure 2: Dependence of the $SU(3)$ orbit function $\Phi_{1,1}(x,y)$ on continuous variables within $-1.5 < x, y < 1.5$. Lighter color triangle indicates the fundamental region of the group.

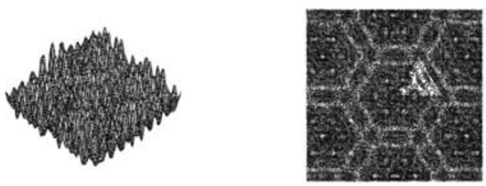

Figure 3: Real part of the orbit function $\Re\Phi_{2,3}(x,y)$ within $-1.5 < x, y < 1.5$. Lighter color triangle indicates the fundamental region of the group.

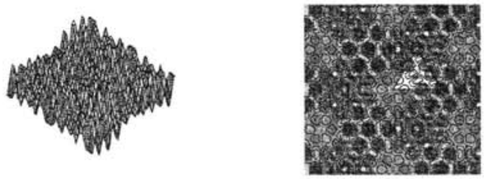

Figure 4: Imaginary part of the orbit function $\Im\Phi_{2,3}(x,y)$ within $-1.5 < x, y < 1.5$. Lighter color triangle indicates the fundamental region of the group.