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Relative category $\mathcal{O}$, blocks, and representation type

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1 Introduction

Let A be a finite dimensional algebra over a field k. We can place A into one of three classes, according to the indecomposable modules the algebra admits. The algebra has *finite representation type* if it has only finitely many indecomposable modules, up to isomorphism. (As a very special case, A is *semisimple* if its only indecomposable modules are simple.) Otherwise it has *infinite representation type*. Algebras of infinite representation type are either *tame* or *wild*. Tame algebras are the ones where there is some reasonable chance of classifying all the indecomposable modules.

Classifying algebras by their representation type is a first step towards understanding the underlying module category. This has already been done for the classical Schur algebras [Erd, DN, DEMN], quantum Schur algebras [EN], and the algebras corresponding to the blocks of category $\mathcal{O}$ [FNP, BKM], among others.

This article presents a summary of work carried out jointly with Daniel K. Nakano and appearing in [BN].

2 Basic Algebras and Quivers

Let P be the direct sum of the projective indecomposable modules for A, (so P is a progenerator for A). Set $\Lambda = \text{End}_A(P)^{\text{op}}$, the *basic algebra* for A. The Morita Theorem says that A is Morita equivalent to Λ. In particular, they have the same representation type, so it suffices to study the representation type of basic algebras.

A *quiver* is simply a directed graph (with loops and multiple edges allowed). A *Dynkin quiver* is a quiver obtained from a Dynkin diagram by assigning a direction to each edge. An *extended Dynkin quiver* is defined similarly.

Let $Q(\Lambda)$ be the Ext^1-quiver for Λ; that is, the directed graph with one vertex i for each simple module L_i of Λ, and with n arrows from i to j where $n = \dim \text{Ext}^1_\Lambda(L_i, L_j)$.

2.1 Separating a quiver

Given a quiver Q, form a new quiver Q' having two vertices i', i'' for each vertex i of Q, and an arrow from i' to j'' for each arrow from i to j in Q. Now decompose Q' as a union of connected components. This process is called *separating the quiver* Q. An example is illustrated in Figure 1, in which a quiver is separated into two A_4 (Dynkin) quivers.

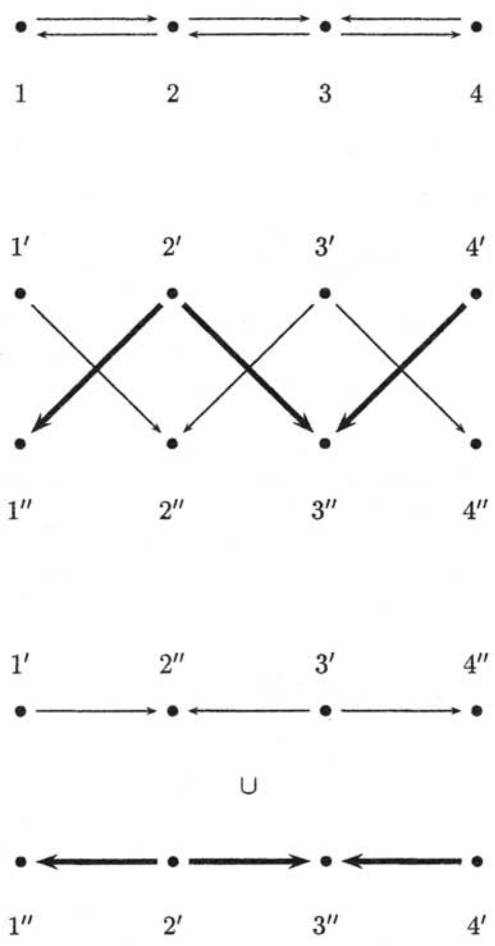


Figure 1: Separating a quiver

Let J be the Jacobson radical of Λ. We say Λ is *two-nilpotent* if $J^2 = 0$.

Theorem (Gabriel [Gab], Dlab-Ringel [DR]). *Let Λ be a basic algebra.*

1. *If Λ is two-nilpotent, then:*

   (a) Λ has finite representation type $\iff$ $Q(\Lambda)$ can be separated into a finite union of Dynkin quivers.

(b) Λ *has tame representation type* $\iff$ $Q(\Lambda)$ *can be separated into a finite union of Dynkin and extended Dynkin quivers (including at least one extended Dynkin quiver).*

2. In general, $(\Rightarrow)$ *holds in (a) and (b).*

3 Relative Category $\mathcal{O}$

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $k = \mathbb{C}$. Let Φ and Δ denote the set of roots and simple roots, respectively. Given a subset $S \subset \Delta$, we associate in the usual way a standard parabolic subalgebra with Levi decomposition $\mathfrak{p}_S = \mathfrak{m}_S + \mathfrak{u}_S$. (When S is fixed we usually drop the subscript S.)

Let $\mathcal{O}_S$ be the full subcategory of $\mathfrak{g}$-modules V satisfying:

1. V is finitely-generated over $U(\mathfrak{g})$;
2. V is a direct sum of finite dimensional irreducible $\mathfrak{m}_S$-modules;
3. V is locally $\mathfrak{u}_S$-finite,

called *relative* (or *parabolic*) category $\mathcal{O}$. (When $S = \varnothing$, $\mathcal{O}_S$ is the classical Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$.)

Given a weight λ which is dominant integral on the roots in S, form the finite dimensional $\mathfrak{p}$-module $F(\lambda)$ of highest weight λ. Define the *generalized Verma module* (GVM)

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F(\lambda).$$

These are the "standard objects" in $\mathcal{O}_S$. $V(\lambda)$ has a unique simple quotient, $L(\lambda)$, and all simple modules in $\mathcal{O}_S$ are obtained in this way. Denote by $P(\lambda)$ the indecomposable projective cover of $L(\lambda)$.

The category $\mathcal{O}_S$ decomposes into *blocks* $\mathcal{O}_S^\mu$, consisting of modules having generalized infinitesimal character associated to the weight μ (which we may and do assume to be antidominant, by the Harish-Chandra homomorphism). Each block has only finitely many simple modules, and their projective covers have finite length. We can therefore associate to each block a finite dimensional basic algebra $\Lambda = \mathrm{End}_{\mathcal{O}_S}(P)^{\mathrm{op}}$ (where P is a progenerator—the direct sum of all the indecomposable projectives in the block), whose module category is Morita equivalent to $\mathcal{O}_S^\mu$. The central question becomes, What is the representation type of Λ? (We will refer to this as the representation type of the block $\mathcal{O}_S^\mu$.)

Assume henceforth that μ is integral (and antidominant). Set

$$J = \{\, \alpha \in \Delta \mid (\mu + \rho, \alpha) = 0 \,\},$$

and let $\Phi_S,\ \Phi_J \subset \Phi$ be the root subsystems of Φ generated by S, J. We say that μ is *regular* if $J = \varnothing$, otherwise *singular*. By the Translation Principle, $\mathcal{O}_S^\mu \simeq \mathcal{O}_S^{\mu'}$ if $J = J'$, so we may focus on J instead of μ, and write O_S^μ as $\mathcal{O}(\Phi, \Phi_S, \Phi_J)$.

4 Representation type of blocks of $\mathcal{O}_S$

4.1 Ordinary $\mathcal{O}$

The representation type of the blocks of category $\mathcal{O}$ (where $S = \varnothing$) was worked out in 2001, independently by Futorny-Nakano-Pollack [FNP] and Brüstle-König-Mazorchuk [FNP]. The results are summarised in the following table.

Φ	Φ_J	Rep. Type
Φ	Φ	Semisimple
A_1	$\varnothing$	Finite
A_2	A_1	
A_3	A_2	Tame
B_2	A_1	
All others		Wild

4.2 Regular blocks of $\mathcal{O}_S$

This case (where $J = \varnothing$) also has a complete, straightforward answer [BN}, as summarised below. Notice that there are no tame blocks in this setting.

Φ	Φ_S	Rep. Type
Φ	Φ	Semisimple
A_n	A_{n-1}	Finite
B_n	B_{n-1}	
C_n	C_{n-1}	
G_2	A_1	
All others		Wild

4.3 Mixed case

Assume that $S \neq \varnothing, J \neq \varnothing$. A complete answer was obtained in [BN] for the representation type of these blocks when $S \cap J = \varnothing$; we call this the *mixed case.* We found several infinite familes of each type (semisimple, finite, tame). The answers are the same for types B and C, so we list them together as BC. Observe that the blocks in the mixed case are all wild unless $S \cup J = \Delta$.

Φ	Φ_S	Φ_J	Conditions	Rep. Type
A_n	A_{n-r}	A_r	$1 \leq r \leq n$	Semisimple
BC_n	A_1	BC_{n-1}		
BC_n	BC_{n-1}	A_1		
G_2	A_1	A_1		
A_n	$A_1 \times A_r$	A_{n-r-1}	$1 \leq r \leq n-2$	Finite
A_n	A_{n-r-1}	$A_1 \times A_r$	$r = 1, 2$	
BC_n	BC_{n-2}	A_2		
BC_n	A_r	BC_{n-r}	$r = 2, 3$	
BC_3	A_2	A_1		
BC_4	A_3	A_1		
D_n	A_r	D_{n-r}	$r = 1, 2$	
D_n	D_{n-1}	A_1		
E_6	D_5	A_1		
A_n	A_{n-4}	$A_1 \times A_3$		Tame
BC_n	BC_{n-3}	A_3		
D_n	D_{n-2}	A_2		
D_5	A_1	A_4		
All others				Wild

5 Some techniques

In this section we describe a few of the techniques used to prove the results tabulated in Sections 4.2 and 4.3.

5.1 Rank reduction

Theorem. *Assume $S \cap J = \varnothing$. Let $\Delta' \subset \Delta$ and $\Phi' = \Phi_{\Delta'}$. Then if $\mathcal{O}(\Phi', \Phi_S \cap \Phi', \Phi_J \cap \Phi')$ is not semisimple (resp. not finite, not tame) then neither is $\mathcal{O}(\Phi, \Phi_S, \Phi_J)$.*

The theorem is proved via a combination of two techniques: the induction–restriction process of [FNP], and a generalization of an equivalence of categories of Enright-Shelton [ES] to the singular setting.

Corollary. *If $|\Delta - (S \cup J)| \geq 2$ then $\mathcal{O}(\Phi, \Phi_S, \Phi_J)$ is wild.*

Proof. Take $\Delta' = \Delta - (S \cup J)$ and use the ordinary category $\mathcal{O}$ result. □

5.2 Wild poset configurations

Let W (resp. W_S, W_J) be the Weyl group of Φ (resp. Φ_S, Φ_J). The isomorphism classes of simple modules in a block $\mathcal{O}(\Phi, \Phi_S, \Phi_J)$ are parametrized by a subset

${}^S W^J$ of W; specifically, ${}^S W^J$ is the set of minimal length coset representatives for $W_S \backslash W / W_J$. This set inherits from W a partial ordering by the Bruhat order.

Definition. A *diamond* in ${}^S W^J$ is a subposet of the form  (where the edges represent length one Bruhat order relations).

Proposition. *If* ${}^S W^J$ *contains a diamond then* $\mathcal{O}(\Phi, \Phi_S, \Phi_J)$ *is wild.*

To prove this, one looks at the Ext^1-quiver. If there is an extension between one of the simple modules parametrized by the diamond and some fifth irreducible, then the quiver does not split into a union of extended Dynkin diagrams; hence the block is wild by the Gabriel-Dlab-Ringel theorem. If there is no such extension, then the block contains a subcategory Morita equivalent to $\mathcal{O}(A_1 \times A_1, \varnothing, \varnothing)$, which is wild by the classical $\mathcal{O}$ result. Now use the rank reduction theorem.

The diamond condition is an easy condition to check, because the poset ${}^S W^J$ is straightforward to compute. In particular, it can be used to check that many low-rank "base cases" are wild. This can then be combined with rank reduction to prove wildness for many infinite families. We also found a similar poset configuration which can be used to prove wildness in certain cases which do not contain diamonds.

5.3 Detailed structure of generalized Verma modules

In a few cases we needed to use the full force of the Kazhdan-Lusztig theory to compute the radical filtrations of the GVMs in a block. Via reciprocity, we could then deduce the structure of the indecomposable projectives. Finally, we used results of Gabriel and others to determine the representation type of the block.

Example. If there are n simples in the block, and the GVMs have radical filtrations

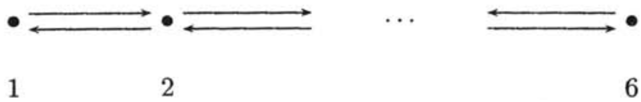
$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ & 1 & & n-1 \\ & & & \vdots \\ & & & 1 \end{array}$$

then the representation type is semisimiple if $n = 1$, finite if $n = 2$ or 3, tame if $n = 4$, and wild otherwise. But if the GVMs have radical filtrations

$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ & 1 & & n-1 \end{array}$$

then the representation type is finite (independent of n).

The block $\mathcal{O}(G_2, \varnothing, A_1)$ is of the first type, with $n = 6$, so it is wild. However, the block $\mathcal{O}(G_2, A_1, \varnothing)$ is of the second type, with $n = 6$, so it has finite representation types. Both examples have same Ext^1-quiver:



which separates into two A_6-quivers. So the block $\mathcal{O}(G_2, \varnothing, A_1)$ illustrates the failure of the converse of the Gabriel-Dlab-Ringel Theorem.

This is an instance of the theory of Koszul duality, due to Beilinson-Ginsburg-Soergel and Backelin [BGS, Bac]. If w_0 is the longest element of W, the blocks $\mathcal{O}(\Phi, \Phi_S, \Phi_J)$, $\mathcal{O}(\Phi, \Phi_J, \Phi_{-w_0(S)})$, and $\mathcal{O}(\Phi, \Phi_{-w_0(J)}, \Phi_S)$ all have naturally isomorphic Ext^1-quivers, but they often have different representation type.

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