Resenhas IME-USP 2003, Vol. 6, No. 1, 85 - 97.

STATISTICAL MEASURE OF QUADRATIC VECTOR FIELDS

Joan C. Artés and Jaume Llibre

Abstract: In [2] the authors classified the 44 topological phase portraits of all the structurally stable quadratic vector fields on the Poincaré sphere $\mathbb{S}^2$ modulo limit cycles. In this topological study, no information is given about the regions in the space of all coefficients where such phase portraits take place. In this paper we use a statistical method to provide estimations of the relative frequency for such regions. We also give estimations of the relative frequencies for the regions of phase portraits having nodes, foci and limit cycles.

1. Introduction

A vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $X = (P, Q)$ where $P = \sum a_{ij}x^i y^j$ and $Q = \sum b_{ij}x^i y^j$, $0 \le i + j \le n$, is called a *planar polynomial vector field of degree* $\le n$. If $\sum_{i+j=n}(|a_{ij}| + |b_{ij}|) \ne 0$, then we say that X has degree n. In particular, the polynomial vector fields of degree 2 are called *quadratic vector fields*. The $M = (n+1)(n+2)$ real numbers a_{ij}, b_{ij} are called the *coefficients of* X. The space of these vector fields, endowed with the structure of affine $\mathbb{R}^M$–space in which X is identified with the M–tuple $(a_{00}, a_{10}, \dots, a_{0n}, b_{00}, b_{10}, \dots, b_{0n})$ of its coefficients, is denoted by $\mathcal{P}_n(\mathbb{S}^2)$.

The *Poincaré compactification* of $X \in \mathcal{P}_n(\mathbb{S}^2)$, is defined to be the unique analytic vector field $p(X)$ tangent to the sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, whose restriction to the northern hemisphere $\mathbb{S}^2_+ = \{(x, y, z) \in \mathbb{S}^2 : z > 0\}$ is given by $z^{n-1}(f_+)_*(X)$, where f_+ is the central projection from $\mathbb{R}^2$ to $\mathbb{S}^2_+$, defined by $f_+(x, y) = (x, y, 1)/(x^2 + y^2 + 1)^{1/2}$. See Section 2 of [2] for more details. The closed northern hemisphere $\{(x, y, z) \in \mathbb{S}^2 : z \ge 0\}$ is also called the *Poincaré disc*.

Let $\mathbb{S}^1 = \{(x, y, z) \in \mathbb{S}^2 : z = 0\}$ be the equator of the Poincaré sphere. Then, the vector field $X \in \mathcal{P}_n(\mathbb{S}^2)$ is said to be *topologically structurally stable* if there is a neighborhood N of X under the given topology and a continuous map $h : N \to \mathrm{Hom}(\mathbb{S}^2, \mathbb{S}^1)$ (homeomorphisms of $\mathbb{S}^2$ which preserve $\mathbb{S}^1$) such that $h_X = \mathrm{Id}$ and h_Y maps orbits of $p(X)$ onto orbits of $p(Y)$, for every $Y \in N$. Again see Section 2 of [2] for more details. Define by $\sum$ the set of quadratic vector fields $X \in \mathcal{P}_2(\mathbb{S}^2)$ which are topologically structurally stable.

We denote by $\varphi : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{S}^2$ the *flow generated by the vector field* $p(X)$. We call *phase portrait of the vector field* $p(X) : \mathbb{S}^2 \to \mathbb{S}^2$ the decomposition of $\mathbb{S}^2$ as union of all the orbits of $p(X)$. We consider all the orbits, different from a singular point, oriented in the sense of the integral curves of the vector field $p(X)$, i.e. if the orbit is $\varphi_x(t)$, it is oriented in the sense of the t increasing. We denote the positive sense of the orbits drawing arrows in the pictures.

It is known (see for instance [4]) that the separatrices of a compactified polynomial vector field $p(X)$ are the singular points, the local separatrices of the finite and infinite hyperbolic sectors, and the limit cycles. Every connected component of the complement of the set of separatrices of $p(X)$ in $\mathbb{S}^2$ is called *a canonical region*. Then, in order to determine the phase portrait of $p(X)$ in $\mathbb{S}^2$, it is sufficient to draw all the separatrices of $p(X)$ plus one orbit for every canonical region, see for details [6].

In [2] the authors characterize the phase portraits of the topologically structurally stable quadratic vector fields without limit cycles. Moreover, they prove for topologically structurally stable quadratic vector fields having limit cycles, that identifying the region bounded by the outermost limit cycle surrounding a focus with a single point, then it is obtained the phase portrait of a topologically structurally stable quadratic vector field without limit cycles. More precisely two of the main results of [2] can be stated as follows.

THEOREM 1. *The following statements hold.*
(a) If $X \in \sum$ has no limit cycles, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation to one of the 44 *phase portraits of Figure 1.*
(b) Each phase portrait of Figure 1 is realizable by some $X \in \sum$ without limit cycles.

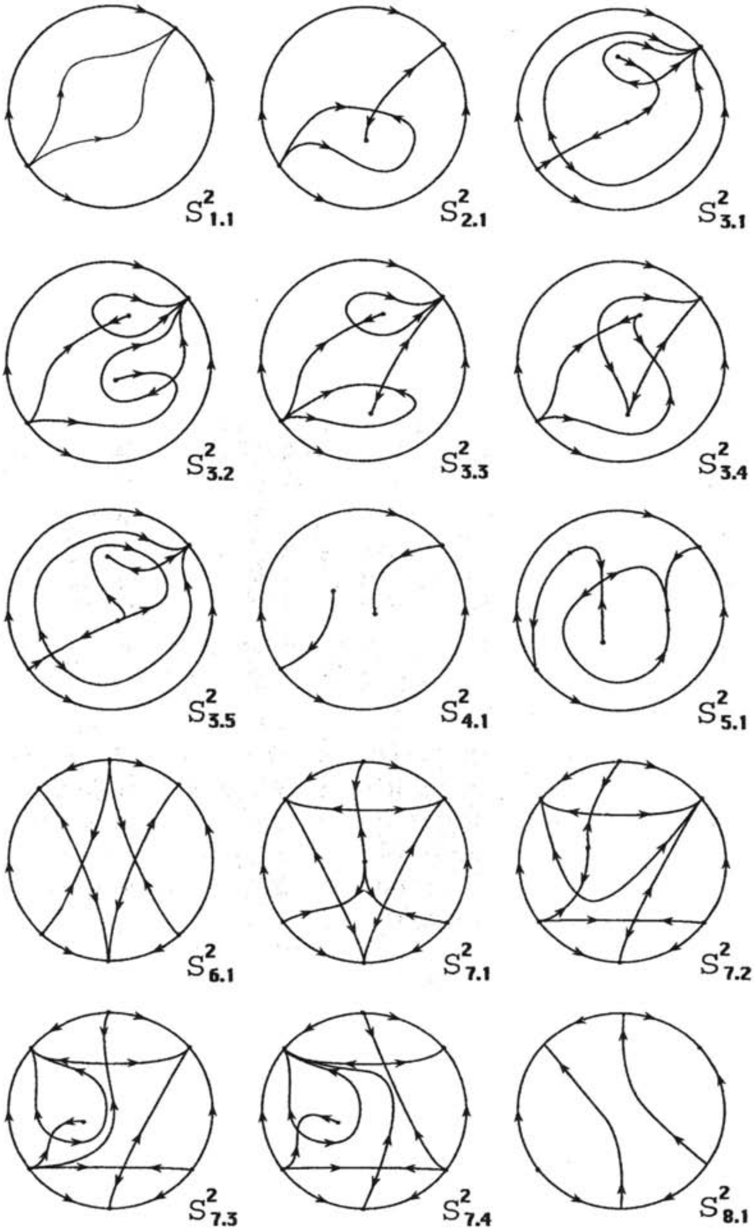

Figure 1. Phase portraits of the topologically structurally stable quadratic vector fields without limit cycles.

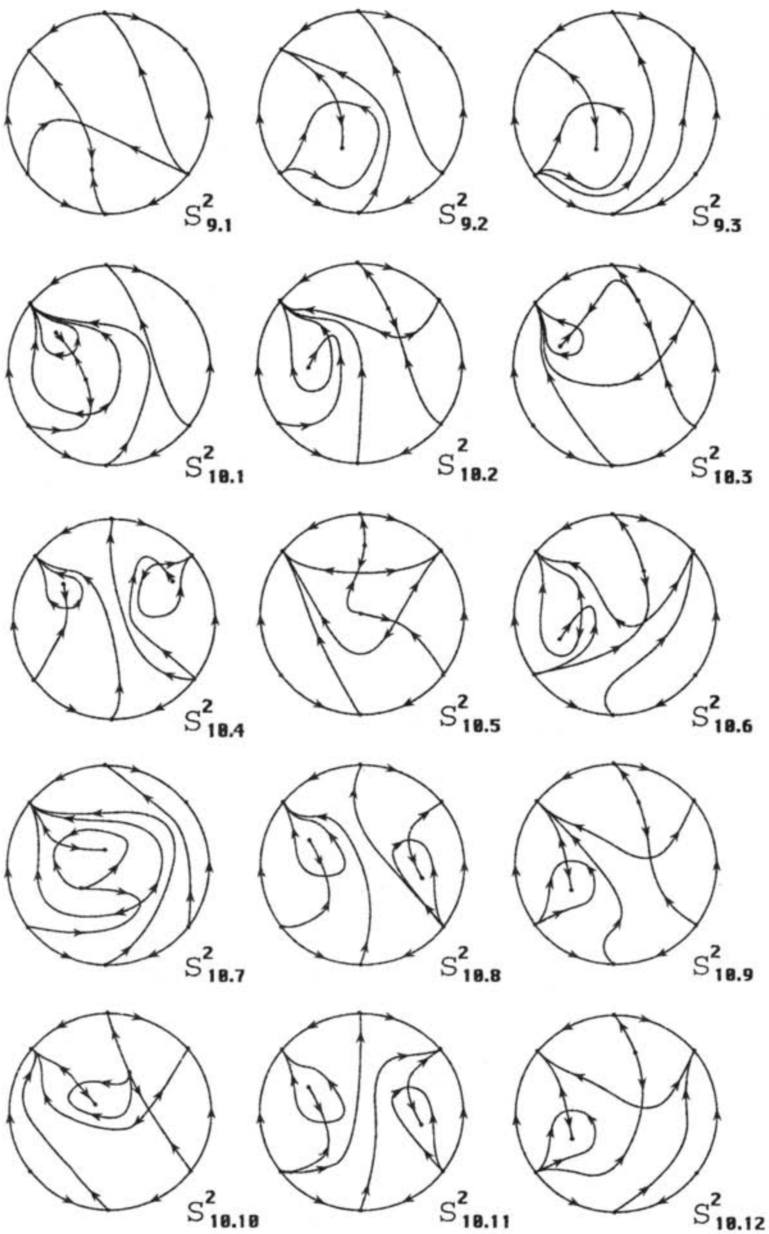

Figure 1. Continuation.

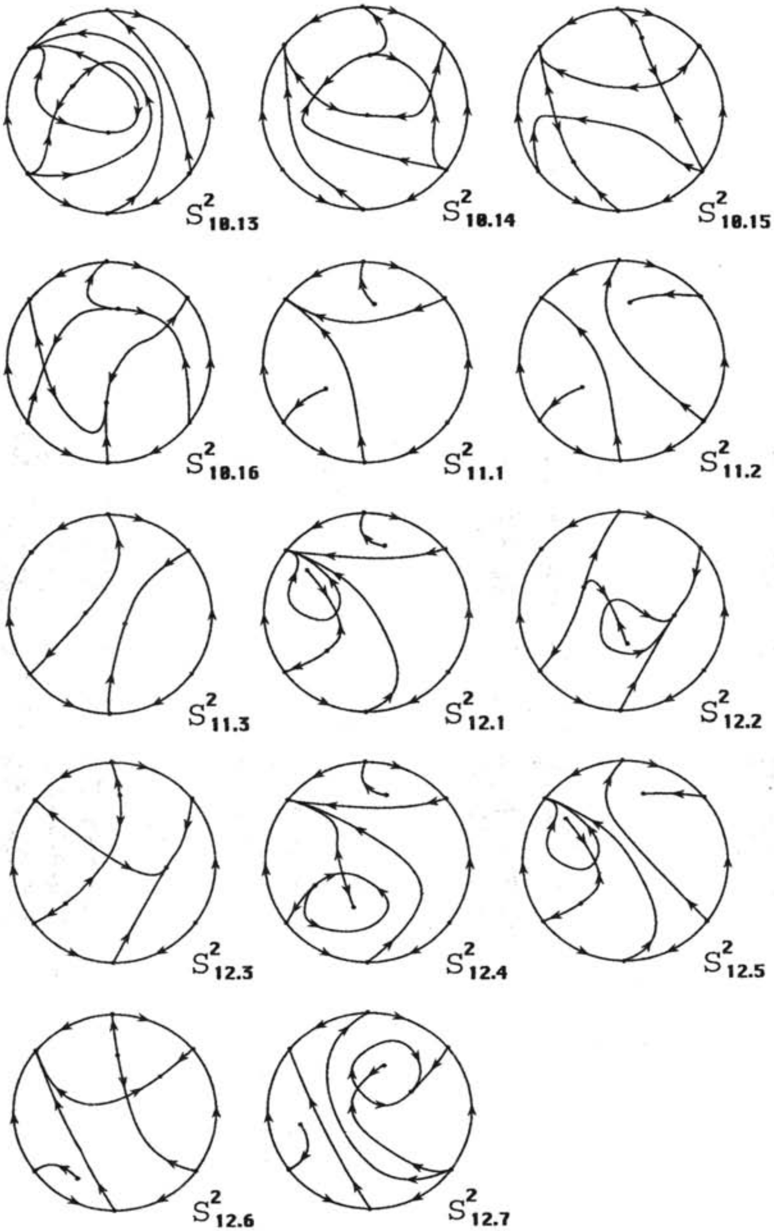

Figure 1. Continuation.

One of the main reasons for introducing the results modulo orientation is to reduce the number of different phase portraits to be drawn. Many of these phase portraits also can be considered with the opposite orientation, while the others are self symmetrical with respect to the orientation.

THEOREM 2. *If $X \in \sum$ has limit cycles and we identify the region bounded by the outermost limit cycle surrounding a focus with a single point, then we obtain the phase portrait of some $Y \in \sum$ without limit cycles.*

In [2] the authors proved that the 44 different phase portraits for structurally stable quadratic vector fields on the Poincaré sphere $\mathbb{S}^2$ modulo limit cycles can be grouped according to Table 1. In this table every capital letter corresponds to a pair of infinite singular points while lower letters correspond to finite singular points in the Poincaré compactification of the polynomial vector field; for more details see [2]. Letter S stands always for saddles while P stands for antisaddles.

Family	Singularities	# of classes
1	P	1
2	$s\,p\,P$	1
3	$p_1\,p_2\,s_1\,s_2\,P$	5
4	$p_1\,p_2\,S$	1
5	$p_1\,p_2\,p_3\,s\,S$	1
6	$s_1\,s_2\,P_1\,P_2\,P_3$	1
7	$s_1\,s_2\,s_3\,p\,P_1\,P_2\,P_3$	4
8	$S\,P_1\,P_2$	1
9	$p\,s\,S\,P_1\,P_2$	3
10	$p_1\,p_2\,s_1\,s_2\,S\,P_1\,P_2$	16
11	$p_1\,p_2\,S_1\,S_2\,P$	3
12	$p_1\,p_2\,p_3\,s\,S_1\,S_2\,P$	7

Table 1. Number of structurally stable phase portraits for a given configuration of the singularities in $\mathbb{S}^2$.

A quadratic vector field is defined by 12 parameters, that is, it can be identified with a point in $\mathbb{R}^{12}$, or even better, by means of a rescaling of the variable t, with a point of the compact sphere $\mathbb{S}^{11}$. The structurally stable vector fields occupy an open and dense set inside the space of parameters (i.e. the space of all coefficients of the quadratic polynomial vector field) see [8]. In the complementary region of measure zero live all bifurcations of the quadratic vector fields. Some of these bifurcations will be of algebraic type, but others will be analytical. Concretely, the bifurcations which distinguish among the 12 cases described in Table 1 are algebraic. All them deal with collision of singular points and these formulas are algebraic. But most of the bifurcations that distinguish among the different phase portraits inside a same case are not algebraic. They deal with saddle to saddle connections, and except in few cases where these connections are algebraic, most of the times are not. So, it is clear that it will not be possible to achieve an estimate of the size by algebraic or analytic means of the different regions in which the parameter space is divided.

Now, by using the computer program named P4 (Planar Polynomial Phase Portraits) we can draw the global phase portrait of our quadratic vector fields in the Poincaré disc (see [1]). In fact, P4 is prepared for drawing the phase portrait of any planar polynomial vector field, whether it is structurally stable or not. Really P4 locates and draws the separatrices of a polynomial vector field. As it is well known such vector fields have finitely many separatrices which are: the separatrices associated to a hyperbolic sector of a finite or infinite singular point, finite and infinite singular points, and limit cycles.

The program can be used to determine the phase portrait of any given vector field in a family of Table 1, its phase portrait. The program P4 is able to detect limit cycles numerically. It may not detect small limit cycles, or distinguish between two limit cycles which are very close, but it may assure the existence of at least one limit cycle, and sometimes even 2 or 3 cycles surrounding the same focus.

In short, using these two tools (the classification of all structurally stable quadratic vector fields on the sphere $\mathbb{S}^2$ and the program P4), and a statistical method we can provide: first, a relative frequency of the 44 regions of the parameter space associated to the different structurally stable quadratic phase portraits, see Table 3; second, a relative frequency of nodes or foci, see Tables 4, 5 and 6; and third, a relative frequency of the existence of limit cycles in quadratic vector fields which can be summarized saying that 3.23% of quadratic vector fields have at least one limit cycle.

In Section 2 we expose the working method used to search and compute the sample, and in Section 3 we summarize the statistical results and point out to several remarkable facts suggested by the data presented in this work.

2. Working method

We generate in a random way 20 million quadratic vector fields. We want to obtain a uniform distribution on the parameter space, that is $\mathbb{R}^{12}$. Since any phase portrait of a polynomial system remains invariant if we multiply all its coefficients by a non zero constant, and the origin of coordinates is not a structurally stable system and has null measure, we will take all them in $\mathbb{S}^{11}$. In order to generate random uniformly distributed points in $\mathbb{S}^{11}$ we cannot take 11 angles and polar coordinates since this method concentrates points in some "poles". The best way consists in generating 12 Gaussian random variables with mean 0 and standard deviation 1 and then normalize this tuple (see [5]).

In concrete, we start generating random numbers in the interval $(0, 1)$ with 16 decimal digits using a constant density function which is equal to 1 on this interval and 0 in the complement. The method used for generating such random numbers is a prime modulus m multiplicative linear congruential generator method, see [7] for more details. Later on we take them in couples and generate two Gaussian distributions using the polar form of the Box–Muller transformation (see [3]). Finally we normalize the 12–tuple to $\mathbb{S}^{11}$.

Under these conditions we are using 192 digits for determining a quadratic vector field of the sample which are later on numerically affected by square roots and logarithms, and consequently the probability to get a non structurally stable quadratic vector field is almost zero.

We have checked that this sample is significative in the sense that we have also worked previously with a sample of 2 millions and the relative frequencies of the different phase portraits were essentially the same with both samples. More specifically, the differences between the relative frequencies obtained with these two samples were of order 0.01.

First, by studying the singularities of the given quadratic vector fields we can decide to which of the 12 families of Table 1 they belong. Some of these 12 families are easy to study, because there is only one possible phase portrait for them. These are Families 1, 2, 4, 5, 6 and 8. However, Families 2, 4 and 5, have finite antisaddles and from the computations of P4 we can know if these points are nodes or foci. Moreover, in this last case, there can be one or more limit cycles around the focus which can be numerically detectable by the program. This is something that we will like to provide a relative frequency too.

We recall that if a quadratic vector field has a limit cycle, then in its interior there is only a unique singular point which is a focus, see [9].

The large number of phase portraits of the sample will allow us to have an accurate relative frequency of the main regions associated to the 12 families of Table 1. For every one of the Families 2, 3, 4, 5, 7, 9, 10, 11 and 12, we prepare a number of vector fields and draw their phase portraits. Firstly, we decide to which subfamily they belong (if needed); secondly, we determine the number of nodes and foci; and finally, we look for the limit cycles (if they exist). With these relative frequencies, for such families, and their weighted proportion with respect to the other families, we determine all the estimations. We have taken a number of vector fields of each family according to the number of different subfamilies expected. Concretely, we have taken one thousand vector fields multiplied by the number of expected phase portraits in each family. That is, one thousand for Family 2, five thousands for Family 3, and so on, including sixteen thousands for Family 10. We remark that these numbers of thousands taken in each family were the first of each family which appear from the 20 millions of the sample. We note that if instead of taking these numbers of thousands we take four times such numbers, the relative frequencies do not present any significative modification.

We have not needed to draw all $41,000$ phase portraits one by one. We have prepared a reduced version of program P4 so that the computer looks for singular points (finite and infinite), detects the saddles and their separatrices, and integrates the separatrices during a large period of time. In most cases the separatrices reach an antisaddle (finite or infinite). We detect how many separatrices end at these antisaddles, and by comparing with the phase portraits of the 44 structurally stable quadratic vector fields, the computer determines the phase portraits. When the separatrix is too slow to reach an antisaddle, or it turns around a limit cycle, then the program marks that case as unknown and consequently, we need to study more carefully only these cases. In total, we have needed to draw about $2,000$ phase portraits.

However, even $16,000$ (respectively $7,000$) phase portraits of Family 10 (respectively 12) do not seem to be enough to catch all possible phase portraits in these families. More specifically, among the selected samples there are neither vector fields having phase portraits 10.1, 10.4, 10.7, 10.8, 10.11 and 10.13, nor 12.5 and 12.7, using the notation of Figure 1. Thus, their relative frequency inside their families (and even more, their relative frequency in the whole parameter space) must be very small. Moreover, among the $7,000$ vector fields of Family 12, there are only 4 and 9 vector fields having phase portraits 12.1 and 12.4, respectively. So, the statistical relative frequency given here for such subfamilies can be subject to a big error. We remark that taking four times the sample for Families 10 and 12 have not detected the existence of the missing phase portraits.

3. Statistical Results

The 20 million random phase portraits split among the 12 families according to Table 2.

Family	**Vector Fields**	**Percentage**
1	$5,084,479$	25.42%
2	$5,084,875$	25.42%
3	$1,479,151$	7.40%
4	$1.549,910$	7.75%
5	$367,813$	1.84%
6	$1,835,737$	9.18%
7	$482,608$	2.41%
8	$993,161$	4.97%
9	$1,760,477$	8.80%
10	$961,671$	4.81%
11	$285,820$	1.43%
12	$114,298$	0.57%

Table 2. Percentage of realization of each of the 12 main families.

The 44 structurally stable quadratic vector fields in $\mathbb{S}^2$, modulo limit cycles, appear with the frequencies given in Table 3.

Family	%	Family	%	Family	%
1.1	25.42%	**2.1**	25.42%	**3.1**	1.411%
3.2	0.23%	**3.3**	0.84%	**3.4**	4.07%
3.5	0.85%	**4.1**	7.75%	**5.1**	1.84%
6.1	9.18%	**7.1**	1.52%	**7.2**	0.60%
7.3	0.13%	**7.4**	0.16%	**8.1**	4.97%
9.1	8.39%	**9.2**	0.13%	**9.3**	0.28%
10.1	0.00%	**10.2**	0.04%	**10.3**	0.04%
10.4	0.00%	**10.5**	0.08%	**10.6**	0.01623%
10.7	0.00%	**10.8**	0.00%	**10.9**	0.03%
10.10	0.04%	**10.11**	0.00%	**10.12**	0.04%
10.13	0.00%	**10.14**	0.23%	**10.15**	0.38%
10.16	3.91%	**11.1**	0.37%	**11.2**	0.17%
11.3	0.89%	**12.1**	0.00073%	**12.2**	0.03%
12.3	0.45%	**12.4**	0.00147%	**12.5**	0.00%
12.6	0.09%	**12.7**	0.00%		

Table 3. Percentage of realization of each of the 44 phase portraits modulo limit cycles.

Families 2, 7 and 9 have one antisaddle. Thus, it can be a node or a focus. If it is a focus, it can also have one or more limit cycles. The percentages are given in Table 4. We only mention the limit cycle configurations as (n) (where n is the number of numerically detected limit cycles) that have appeared. Other configurations may be possible but too small to be appreciated in this sample.

Family	Nodes	Foci	(1)	(2)
2	25.10%	74.90%	2.10%	0.00%
7	40.20%	59.80%	0.85%	0.00%
9	73.93%	26.07%	0.67%	0.00%

Table 4. Percentage of nodes, foci and limit cycles in the families with one antisaddle.

Families 3, 4, 10 and 11 have two antisaddles. Thus, they can be nodes or foci, or both. Around the foci it can also have limit cycles. The percentages are given in Table 5. We only mention the limit cycles configurations as (n), (n, m) (where n and m are the number of numerically detected limit cycles around one or two foci) that have appeared. Other configurations may be possible but too small to be appreciated in this sample.

Family	**N-N**	**N-F**	**F-F**	**(1)**	**(2)**	**(1,1)**
3	4.58%	24.06%	71.36%	4.2%	0.02%	0.06%
4	18.80%	42.10%	39.10%	18.80%	0.10%	2.50%
10	50.65%	45.19%	4.16%	0.53%	0.00%	0.00%
11	37.53%	39.27%	23.20%	2.70%	0.00%	0.00%

Table 5. Percentage of nodes, foci and limit cycles in the families with two antisaddles.

The Families 5 and 12 have three antisaddles. Thus, they can be either three nodes, two nodes and one focus, or one node and two foci (three foci are not possible, see [9]). As before it can also have limit cycles surrounding some focus. The percentages are given in Table 6. Other configurations may be possible but too scarce to be detected in this sample.

Family	**N-N-N**	**N-N-F**	**N-F-F**	**(1)**	**(2)**
5	1.90%	26.40%	71.70%	7.10%	0.10%
12	34.49%	53.41%	12.10%	1.50%	0.00%

Table 6. Percentage of nodes, foci and limit cycles in the families with three antisaddles.

Adding all this statistical information, we can say that foci are more common than nodes; more precisely, from every $10,000$ detected antisaddles, $4,021$ of them are nodes and $5,979$ are foci.

Limit cycles are not difficult to find. Thus, it is estimated that 3.23% of quadratic vector fields have at least one limit cycle. Even cases with 2 limit cycles are not rare. We have found 29 vector fields with configuration $(1,1)$ and 3 with configuration (2). All of them in Families 3, 4 and 5, with the remarkable fact that 2.5% of phase portraits of Family 4 have the configuration $(1,1)$. Families with 3 or 4 limit cycles are more rare and have not appeared in our sample.

Some other possibilities have not appeared. Thus, either they are very rare, or there is a geometrical impossibility for their realization. We will not mention here cases with more than one limit cycle since they may be too scarce to appear in this sample, but other absences seem more curious. For example:

(1) There are no vector fields with Phase Portrait 3.2 with 2 nodes.

(2) The most common phase portrait in Family 7 is 7.1. However, there are no

vector fields having Phase Portrait 7.1 with limit cycles, but there are limit cycles for some of the phase portraits of the Subfamilies 7.2, 7.3 and 7.4.

(3) Family 10 is strongly concentrated towards Phase Portrait 10.16 which occupies 81.3% of the family. Altogether with Phase Portraits 10.14 and 10.15 they occupy 94.1% of the family. However, no phase portrait of the $15,048$ studied from these three subfamilies has a limit cycle. All limit cycles in Family 10 seem to be concentrated on Subfamilies 10.2, 10.3, 10.5, 10.6 and 10.10. We cannot say that limit cycles appear only on these subfamilies since there are many subfamilies which do not appear in our sample, but it seems that either they are not possible in 10.14, 10.15 and 10.16, or that the probabilities are very small there.

Other remarkable facts are:

(1) The family where more often appears the limit cycle configuration $(1,1)$ is the 4.1. It is also the most common case for having at least one limit cycle.

(2) Family 9 is strongly concentrated towards the Phase Portrait 9.1 having $95,37\%$ of the region corresponding to this family. Limit cycles do not seem very common in this family. Even this we have found examples of limit cycles in all the three subfamilies.

(3) Family 12 apart from being the one with less presence in the parameter space (just a 0.57%) is also heavily concentrated towards the Phase Portrait 12.3 with 78.24%.

Acknowledgments: We thank to Jorge Sotomayor and to the referee their comments and suggestions that improved the presentation of our results.

The authors are partially supported by a MCYT grant BFM2002-04236-C02-02 and by a DGICIT grant 2001SGR 00173.

References

[1] J. C. Artes, F. Dumortier, C. Herssens and J. Llibre, *Planar Polynomial Phase Portraits*, Computer Program, 1997.

[2] J. C. Artes, R. Kooij, and J. Llibre, *Structurally stable quadratic vector fields*, vol. **134** (639), Memoirs Amer. Math. Soc., 1998.

[3] G.E.P. Box and M.E. Muller, *A note on the generation of random normal deviates*, Ann. Math. Stat **29** (1958), 610–611.

[4] Li Weigu, J. Llibre, M. Nicolau and Xiang Zhang, *On the differentiability of first integrals of two dimensional flows*, Proc. Amer. Math. Soc. **130** (2002), 2079–2088.

[5] G. Marsaglia, *Choosing a point from the surface of a sphere*, Ann. Math. Stat **43** (1972), 645–646.

[6] D. A. Newmann, *Classification of continuous flows on 2–manifolds*, Proc. Amer. Math. Soc. **48** (1975), 73–81.

[7] S. K. Park and K. W. Miller, *Random number generators: good ones are hard to find*, Comm. ACM **31** (1988), 1192–1201.

[8] J. Sotomayor, *Stable planar polynomial vector fields*, Rev. Mat. Iberoamericana **1** (1985), 15–23.

[9] Ye Yanqian, *Theory of limit cycles*, Transl. of Math., Amer. Math. Soc, Vol. 66, 1984.

Joan C. Artés and Jaume Llibre
Departament de Matemàtiques,
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain.
E-mails: artes@mat.uab.es, jllibre@ mat.uab.es.