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NILPOTENCE OF SUBNORMAL SUBGROUPS IN $\mathcal{U}(\mathbb{Z}G)$ [1]

M.M. Parmenter

To my teacher and friend Sudarshan K. Sehgal on his 65^{th} birthday.

Abstract: Let V be a subgroup of $\mathcal{U}(\mathbb{Z}G)$ such that $G \subset V$ and V is subnormal in a finite index subgroup of $\mathcal{U}(\mathbb{Z}G)$. In this paper, we present necessary and sufficient conditions for V to be either nilpotent on FC.

Key words: Group ring, nilpotent.

I'd like to start by recalling the following theorem, first proved for finite groups by Polcino Milies [9] and then extended to the general case by Sehgal and Zassenhaus [13].

Theorem 1

$\mathcal{U}(\mathbb{Z}G)$ is nilpotent if and only if G is nilpotent and the torsion subgroup T of G satisfies one of the following:

(i) T is central in G.

(ii) T is an Abelian 2-group and for $x \in G, t \in T, x^{-1}tx = t^{\pm 1}$.

(iii) $T = E \times K_8$ where $E^2 = 1$ and K_8 is the quaternion group of order 8. Moreover E is central in G and conjugation by $x \in G$ induces on K_8 one of the 4 inner automorphisms.

Proofs of this theorem can also be found in [10] and [11]. I like this result very much - probably because, unusually in unit groups of integral group rings, it actually gives a complete characterization of when a particular group property can hold.

For finite G, the fact that such properties rarely hold in $\mathcal{U}(\mathbb{Z}G)$ was explained by Hartley and Pickel in the following.

Theorem 2 [4]

Let G be a nonabelian finite group. Then $\mathcal{U}(\mathbb{Z}G)$ contains a noncommutative free group if and only if G is not a Hamiltonian 2-group.

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It is perhaps sometimes forgotten that Theorem 2 does not give a complete explanation of why $\mathcal{U}(\mathbb{Z}G)$ may fail to be nilpotent when G is not periodic. Families of nonperiodic nilpotent G such that $\mathcal{U}(\mathbb{Z}G)$ is solvable but not nilpotent can be obtained from Theorem 6.4.8 of [11] - for example, $G = < t, g | t^8 = 1, gtg^{-1} = t^3 >$. In such cases, an investigation of the proof of Theorem 1 is needed to understand why nilpotence fails.

Our goal in this brief note is to prove the following.

Theorem 3

Let G be an arbitrary group and let V be a subgroup of $\mathcal{U}(\mathbb{Z}G)$ such that $G \subseteq V$ and V is subnormal in a finite index subgroup of $\mathcal{U}(\mathbb{Z}G)$. The following are equivalent.

1. V is nilpotent

2. $\mathcal{U}(\mathbb{Z}G)$ is nilpotent

The corresponding theorem about when V contains a nonabelian free group was proved by Gonçalves, Ritter and Sehgal for G finite in [3], and a generalization to arbitrary G was given by Marciniak and Sehgal in [8]. As before, however, a complete proof of Theorem 3 still requires additional argument. We note that a partial description of when V is nilpotent is given in Corollary 2.2 of [8].

Proof of Theorem 3

(2) $\Rightarrow$ (1) is obvious. So assume V is nilpotent. Then G is nilpotent, and we will show directly that one of the three conditions listed in Theorem 1 must be satisfied.

Since V does not contain a nonabelian free subgroup, Theorem 0.4 of [8] tells us that all finite subgroups of G are normal in G. So $T(G)$ is Abelian or Hamiltonian. Assume first that $T(G)$ is Hamiltonian and contains an element x of odd order, and let H be the finite subgroup $K_8 \times < x >$ of $T(G)$. The main theorem of [3] then says that $V \cap \mathcal{U}(\mathbb{Z}H)$ contains a nonabelian free group, which is not the case. We conclude that $T(G)$ is a Hamiltonian 2-group, and the conditions of (iii) are met.

So assume from now on that $T(G)$ is Abelian. We'll show next that if $t \in T(G)$ and $x \in G$, then $xtx^{-1} = t$ or t^{-1}. Assume to the contrary that $xtx^{-1} = t^i$ where $1 < i < |t| - 1$. Let $k = \phi(|t|)$, and recall that $i^k \equiv 1 (\mathrm{mod}\ |t|)$.

Consider the Bass cyclic unit (see [12] for details)

$$u = (1 + t + \cdots + t^{i-1})^k + \frac{1 - i^k}{|t|}\hat{t}.$$

Note that u is nontrivial since $1 < i < |t| - 1$. Also $u^x = (1 + t^i + \cdots + t^{i(i-1)})^k + \frac{1-i^k}{|t|}\hat{t}$. In general, for any $r \geq 1$,

$$u^{x^r} = (1 + t^{i^r} + \cdots + t^{i^r(i-1)})^k + \frac{1 - i^k}{|t|}\hat{t}.$$

Hence $u^{x^k} = u$ and we get

$$uu^x u^{x^2} \cdots u^{x^{k-1}} = (1 + t + t^2 + \cdots + t^{i^{k-1}(i-1)+i^{k-2}(i-1)+\cdots+(i-1)})^k + \alpha\hat{t}$$

for some $\alpha \in \mathbb{Z}$.

$$\begin{aligned} \text{So} \quad uu^x u^{x^2} \cdots u^{x^{k-1}} &= (1 + t + t^2 + \cdots + t^{i^k-1})^k + \alpha\hat{t} \\ &= (1 + \beta\hat{t})^k + \alpha\hat{t} \text{ for some } \beta \in \mathbb{Z} \\ &= 1 \text{ using augmentation.} \end{aligned}$$

Next note that since V is nilpotent and $G \subseteq V, G$ is a subnormal subgroup of V. It follows that there exists a series of subgroups

$$G = V_r \lhd V_{r-1} \lhd \cdots \lhd V_1 < \mathcal{U}(\mathbb{Z}G)$$

where V_1 is of finite index in $\mathcal{U}(\mathbb{Z}G)$.

Now $u^s \in V_1$ for some $s \in \mathbb{Z}$. Since $\mathbb{Z} < t >$ is commutative, we have

$$u^s(u^s)^x \cdots (u^s)^{x^{k-1}} = 1$$

which can be rewritten as

$$x^{u^s} x^{u^{2s}} \cdots x^{u^{(k-1)s}} (x^{-(k-1)})^{u^{ks}} u^{ks} = 1.$$

Since $x \in G \subseteq V_2$, we know $x^{u^s}, x^{u^{2s}}, \cdots x^{u^{(k-1)s}}$ and $(x^{-(k-1)})^{u^{ks}}$ are all in V_2. Hence $u^{ks} \in V_2$.

But this argument can be repeated - at the next step beginning with $u^{ks}(u^{ks})^x \cdots (u^{ks})^{x^{k-1}} = 1$. Continuing we get $u^e \in V_r = G$ for some e. Since t is of finite order, this means $u^f = 1$ for some f, contradicting the fact that u is of infinite order.

So $xtx^{-1} = t^{\pm 1}$, as desired. The rest of the proof follows in exactly the same way as that of Theorem 1.

□

When $V = \mathcal{U}(\mathbb{Z}G)$, the Bass cyclic unit argument just presented gives a proof of one direction of Theorem 1 which differs from the original (this alternate proof can also be found in [10]).

In a similar spirit to Theorem 3, we have the following.

Theorem 4

Let G be an arbitrary group and let V be a subgroup of $\mathcal{U}(\mathbb{Z}G)$ such that $G \subseteq V$ and V is subnormal in a finite index subgroup of $\mathcal{U}(\mathbb{Z}G)$. Then the following are equivalent.

(1) V is FC

(2) $\mathcal{U}(\mathbb{Z}G)$ is FC

Proof

The proof combines the Bass cyclic unit argument given in Theorem 3 with a few details from the proof of Theorem 6.5.3 in [11]. Note that a slight generalization of Lemma 6.5.1 in [11] is needed to complete the argument.

□

Remark

For periodic groups one measure of how far from nilpotent $\mathcal{U}(\mathbb{Z}G)$ usually is can be found in [5] where it is shown that $Z_3(\mathcal{U}(\mathbb{Z}G)) = Z_2(\mathcal{U}(\mathbb{Z}G))$ for all periodic G (for finite G, this had been done earlier in [2]). Moreover, $Z_2(\mathcal{U}(\mathbb{Z}G))$ has been completely determined in such cases (in terms of $Z(\mathcal{U}(\mathbb{Z}G))$) and is contained in $G \cdot Z(\mathcal{U}(\mathbb{Z}G))$ ([6], also [1] when G is finite). Some progress in finding appropriate analogues for these results when G is not periodic can be found in [7] – interestingly, the Bass cyclic unit argument seen in Theorems 3 and 4 was also helpful in proving Lemma 2.6 of this latter paper.

REFERENCES

1. Satya R. Arora, A.W. Hales and I.B.S. Passi, Jordan decomposition and hypercentral units in integral group rings, Commun. in Alg. 21 (1993), 25-35.

2. Satya R. Arora and I.B.S. Passi, Central height of the unit group of an integral group ring, Comm. in Alg. 21(1993), 3673-3683.

3. Jairo Gonçalves, Jürgen Ritter and Sudarshan Sehgal, Subnormal subgroups in $\mathcal{U}_1\mathbb{Z}G$, Proc. Amer. Math. Soc. 103 (1988), 375-382.

4. B. Hartley, and F. Pickel, Free subgroups in the unit groups of integral group rings, Canad. J. Math. 32 (1980), 1342-1352.

5. Yuanlin Li, The hypercentre and n-centre of the unit group of an integral group ring, Canad. J. Math. 50 (1998), 401-411.

6. Yuanlin Li and M.M. Parmenter, Hypercentral units in integral group rings, Proc. Amer. Math. Soc. 129 (2001), 2235-2238.

7. Yuanlin Li and M.M. Parmenter, Some results on hypercentral units in integral group rings, Comm. in Alg. (To appear).

8. Zbigniew S. Marciniak and Sudarshan K. Sehgal, Subnormal subgroups of group ring units, Proc. Amer. Math. Soc. 126 (1998), 343-348.

9. C. Polcino Milies, Integral group rings with nilpotent unit groups, Canad. J. Math. 28 (1976), 954-960.

10. Cesar Polcino Milies and Sudarshan K. Sehgal, An Introduction to Group Rings, Kluwer, 2002.

11. S.K. Sehgal, Topics in Group Rings, Marcel Dekker, 1978.

12. S.K. Sehgal, Units in Integral Group Rings, Longman, Essex, 1993.

13. S.K. Sehgal and H.J. Zassenhaus, Integral group rings with nilpotent unit groups, Commun. in Alg. 5 (1977), 101-111.

M.M. Parmenter
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, Newfoundland, Canada
A1C 5S7