Resenhas IME-USP 2002, Vol. 5, No. 4, 363 - 370.

Lie nilpotence in group algebras [1]

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To our friend Sudarshan K. Sehgal, on the occasion of his 65^{th} birthday.

Abstract: Let $*$ be the natural involution on a group algebra FG induced by setting $g \to g^{-1}$, for all $g \in G$. Here we survey on the results recently obtained on the Lie nilpotence of the symmetric and skew elements of FG under $*$.

Key words: Involution, group algebra, symmetric and skew elements.

Let FG be the group algebra of a group G over a field F. We shall give here a brief account of the results about the Lie nilpotence of FG and of certain significant subsets of FG.

Let $F\{X\}$ be the free associative algebra on the set $X = \{x_1, x_2, \ldots\}$ over F and let R be an F-algebra. If $f = f(x_1, \ldots, x_n)$ is a non-trivial polynomial in $F\{X\}$ and S is a subset of R, we say that f is a polynomial identity for S (or that S satisfies f) if $f(s_1, \ldots, s_n) = 0$ for all $s_1, \ldots, s_n \in S$. We shall be concerned with a special type of polynomial identity, the nth Lie commutator leading to the notion of Lie nilpotence. We shall also consider the case when R is a group algebra and S is the set of symmetric or skew elements of R under the canonical involution.

In general, group algebras satisfying a polynomial identity were classified by Isaacs and Passman [11] in characteristic zero and by Passman [17] in positive characteristic. Recall that a group G is p-abelian if the derived group G' is a finite p-group. We say that G is 0-abelian if it is abelian. With this terminology in mind, these results can be stated as follows.

Theorem 1 ([18, Teorems 5.3.8 and 5.3.9]) *Let F be a field such that char $F = p \geq 0$. Then FG satisfies a polynomial identity if and only if G has a p-abelian subgroup of finite index.*

Among polynomial identities one of the first studied is the Lie commutator of length n. Let $[\ ,\ ]$ be the usual Lie bracket, $[x_1, x_2] = x_1x_2 - x_2x_1$, and for $n \geq 2$ define inductively $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$. For subsets A, B of R, let us write $[A, B]$ for the additive subgroup of R generated by all commutators of the type $[a, b]$, $a \in A, b \in B$. We say that the subset S of the algebra R is Lie nilpotent if there exists $n \geq 2$ such that $[x_1, \ldots, x_n]$ is a polynomial identity for S. Let us say that a set S is Lie nilpotent of index $n \geq 2$ if n is the least positive integer such that $[x_1, \ldots, x_n]$ is an identity for S.

[1] Research partially supported by MURST (Italy) and FAPESP and CNPq (Brazil)

In general, what can one prove if a ring R is Lie nilpotent? Since $n \times n$ matrices over a field are Lie nilpotent if and only if $n = 1$, structure theory and standard arguments show that if R is Lie nilpotent, then the commutator ideal $[R, R]R$ of R must be locally nilpotent. Moreover Jennings in [12] proved that for any such ring, $[R, R, R]R$ is a nilpotent ideal. We remark that in this situation, one cannot hope to prove that, in general, the commutator ideal itself is nilpotent. As an example one can consider E, the Grassmann algebra on an infinite dimensional vector space over a field F of characteristic different from 2. Recall that E is generated by a countable set of elements $\{e_1, e_2, \ldots\}$ subject to the condition $e_i e_j = -e_j e_i$ for all $i, j \geq 1$. It is not hard to see that the algebra E is Lie nilpotent since it satisfies the polynomial identity $[x_1, x_2, x_3]$ (see for instance [5]). Still the commutator ideal of E is locally nilpotent but not nilpotent.

A characterization of group algebras which are Lie nilpotent was obtained by Passi, Passman and Sehgal as follows.

Theorem 2 ([16]) *If F is a field and char $F = p \geq 0$, then FG is Lie nilpotent if and only if G is a nilpotent p-abelian group.*

As a consequence of this theorem one can easily prove that, for the group algebras FG which are Lie nilpotent, the commutator ideal is nilpotent. In fact, in this case the previous theorem shows that G' is a finite p-group. Hence the augmentation ideal $\Delta(G, G')$ is nilpotent. But $\Delta(G, G')$ is actually the commutator ideal of FG.

Recall that an involution $*$ on an F-algebra R is an antiautomorphism of R of order 2. For simplicity we shall assume throughout this paper that whenever $*$ is an involution on an F-algebra R, $*$ is the identity map on F and *char* $F \neq 2$. Let $R^+ = \{r \in R \mid r = r^*\}$ and $R^- = \{r \in R \mid r = -r^*\}$ be the sets of symmetric and skew elements of R respectively. Notice that R^+ is a Jordan subalgebra of R under the operation $\{a, b\} = \frac{1}{2}(ab + ba)$ and R^- is a Lie subalgebra of R under the bracket operation $[a, b] = ab - ba$. A natural question to ask in this setting is the following: if R^+ or R^- satisfies a polynomial identity does R itself satisfy a polynomial identity? The answer is yes, as it was shown by Amitsur in two subsequent papers [1], [2].

Let $F\{X, *\} = F\{x_1, x_1^*.x_2, x_2^*, \ldots\}$ be the free associative algebra with involution on the set X over F. Hence the elements of $F\{X, *\}$ are polynomials in the x_i's and the x_i^*'s. The algebra $F\{X, *\}$ is uniquely determined by the following universal property: for any algebra with involution R, any map $X \to R$ can be uniquely extended to a homomorphism of algebras with involution $F\{X, *\} \to R$. We say that a non-trivial polynomial $f(x_1, x_1^*, \ldots, x_n, x_n^*) \in F\{X, *\}$ is a $*$-polynomial identity for a subset S of the F-algebra R, if $f(s_1, s_1^*, \ldots, s_n, s_n^*) = 0$ for all $s_1, \ldots, s_n \in S$. With this terminology in mind, Amitsur's result can be stated as follows.

Theorem 3 ([2]) *If R satisfies a $*$-polynomial identity, then R satisfies a (ordinary) polynomial identity.*

We remark that Amitsur's theorem proves the existence of an ordinary polynomial identity for the algebra R but, in general, does not give any information on its degree. The reason for this failure is the following: the theorem was proved first for semiprime rings where, through structure theory, the degree of an identity for R is well related to that of the given $*$-identity; then the result was pushed to arbitrary rings by means of the so-called Amitsur's trick. In such procedure any information on the degree of the $*$-identity satisfied by R is lost. This problem has been recently solved in [3]. In fact, by using combinatorial methods pertaining to the asymptotic behaviour of a numerical sequence attached to the algebra R, it was shown that one can relate the degree of a $*$-polynomial identity satisfied by R to the degree of a polynomial identity for R by mean of an explicit function.

Another question in this setting is the following: if R satisfies some special kind of $*$-polynomial identity, what kind of ordinary identity can one get in Amitsur's theorem? Recalling that R^- is a Lie subalgebra of R under the bracket operation, it is natural to ask if, in particular, the Lie nilpotence of R^- implies the Lie nilpotence (or some other special type of identity) of R. The best known result in this direction is due to Zalesskii and Smirnov.

Theorem 4 ([22]) *Suppose that R is generated as a ring by R^- and 1. If R^- is Lie nilpotent then R is Lie nilpotent.*

As we shall see later, if R is not generated by its skew elements together with 1, then the conclusion of the above theorem cannot be true. Let us notice that for semiprime algebras standard arguments show that if R^- (respectively R^+) is Lie nilpotent, then R^- (respectively R^+) must be commutative. Such result cannot be improved as it is readily seen by considering the algebra $M_2(F)$ of 2×2 matrices over F. In fact, if $M_2(F)$ is endowed with the transpose involution, then $M_2(F)^-$ is commutative, whereas if it is endowed with the symplectic involution, then $M_2(F)^+$ is central and, so, commutative.

A group algebra FG has a natural involution induced linearly on FG by setting $g^* = g^{-1}$, for all $g \in G$. Throughout, when referring to an involution $*$ on FG, we shall always understand that $*$ is the one described above. With this in mind, how does the above question fit in the setting of group algebras? In this case the situation is quite clear. It turns out that the presence of 2-elements in the group G makes quite a difference for the final result. We start by showing, as in the Corollary in [7], that in case the center of G has infinitely many elements not of order 2, then the Lie nilpotence of the symmetric or skew elements forces the Lie nilpotence of the group algebra.

Lemma 1 *Let G be a group whose center Z satisfies $\mid Z^2 \mid= \infty$. If FG^+ or FG^- is Lie nilpotent of index n then FG is Lie nilpotent of index n .*

Proof. We give the proof for FG^-. Since FG^- is Lie nilpotent of index n, $[x_1 - x_1^*, \ldots, x_n - x_n^*]$ is a $*$-polynomial identity for FG^-. Now, for any $z \in Z$,

$$[zx_1 - z^*x_1^*, x_2 - x_2^*, \ldots, x_n - x_n^*] \tag{1}$$

and

$$z^*[x_1 - x_1^*, x_2 - x_2^*, \ldots, x_n - x_n^*] \tag{2}$$

both vanish when evaluated in FG. Since any Lie commutator is multilinear and *char* $F \neq 2$, putting together (1) and (2) we get that

$$(z - z^*)[x_1, x_2 - x_2^*, \ldots, x_n - x_n^*]$$

also vanishes in FG. It is clear that a repeated application of this argument leads to the conclusion that for all $z_1, \ldots, z_n \in Z$,

$$(z_1 - z_1^*) \cdots (z_n - z_n^*)[x_1, \ldots, x_n] \tag{3}$$

vanishes in FG. Recalling that for all $i = 1, \ldots, n$, $z_i^* = z_i^{-1}$, then (3) says that $(z_1^2 - 1) \cdots (z_n^2 - 1)[x_1, \ldots, x_n]$ vanishes in FG for all $z_1, \ldots, z_n \in Z$. It is easy to see that in a group algebra this leads to the conclusion that $[x_1, \ldots, x_n]$ is a polynomial identity for FG (see for instance [7, Lemma 1]).

In the absence of 2-elements Giambruno and Sehgal proved that the Lie nilpotence of FG^+ or FG^- forces the Lie nilpotence of FG.

Theorem 5 ([7]) *Suppose that G has no 2-elements. If FG^+ or FG^- is Lie nilpotent then FG is Lie nilpotent.*

Later G. Lee settled the case when FG^+ has 2-elements. Let K_8 denote the quaternion group of order 8. The final result is the following.

Theorem 6 ([13]) *If $K_8 \not\subseteq G$ then FG^+ is Lie nilpotent if and only if FG is Lie nilpotent. In case $K_8 \subseteq G$, then FG^+ is Lie nilpotent if and only if $G = K_8 \times E \times P$ where $E^2 = 1$ and P is either 1 or a finite p-group in case char $F = p > 2$.*

The case when FG^- is Lie nilpotent and the group G has 2-elements seems more complicated. We start with an example in order to show that one can have quite wild situations in this case.

Lemma 2 *Suppose that G contains an abelian subgroup A of index 2. If either $A^2 = 1$ or $(G \setminus A)^2 = 1$, then FG^- is Lie nilpotent of index 2.*

Proof. Since FG^- is spanned by all elements of the type $g - g^{-1}$, $g \in G$, it is enough to show that any two such elements commute. Let $g, h \in G$ and suppose first that $A^2 = 1$. If either g or h are in A, we have that $g^2 = 1$ or $h^2 = 1$ and thus either $g - g^{-1} = 0$ or $h - h^{-1} = 0$ and there is nothing to prove. So, we may assume that g and h are both in xA where $x \in G \setminus A$. Write $g = xa$ and $h = xb$ with $a, b \in A$ and compute:

$$[xa - (xa)^{-1}, xb - (xb)^{-1}] = xaxb + a^{-1}x^{-1}b^{-1}x^{-1} - xbxa - b^{-1}x^{-1}a^{-1}x^{-1}.$$

Since $A \lhd G$ we can write $ax = xa'$ and $bx = xb'$ with $a', b' \in A$. Hence

$$[xa - (xa)^{-1}, xb - (xb)^{-1}] = x^2a'b + a^{-1}b'^{-1}x^{-2} - x^2b'a - b^{-1}a'^{-1}x^{-2}.$$

As $x^2 \in A$ and $A^2 = 1$ we have that $x^4 = 1$ so $x^2 = x^{-2}$ and it follows that $[g - g^{-1}, h - h^{-1}] = 0$.

Assume now that $(xA)^2 = 1$. As above, we need only to consider the case when $g, h \in A$ but, since A is abelian, the result follows immediately.

In case the group algebra FG is semiprime, it was proved by the authors that actually the example given in the above lemma is exhaustive of all possibilities. Recall that such group algebras were characterized by Passman (see [18, Theorem 4.2.13] or [20, Theorem 6.6.4]: FG is semiprime if and only if either char $F = 0$ or char $F = p > 0$ and the FC-subgroup of G is a p'-group.

Theorem 7 ([9]) *Suppose that FG is semiprime. Then FG^- is Lie nilpotent if and only if one of the following occurs.*

(i) G is abelian.

(ii) $A = \langle g \in G \mid o(g) \neq 2\rangle$ is a normal abelian subgroup of G and $(G \setminus A)^2 = 1$.

(iii) G contains an elementary abelian 2-group of index 2.

We now turn to the non semiprime case. Clearly in this case *char* $F = p > 2$. We split the conclusion into two different cases according as G is a torsion group or it contains elements of infinite order.

Theorem 8 ([9]) *Suppose that char $F = p > 2$ and G is a torsion group. If FG^- is Lie nilpotent, then*

(i) the p-elements form a nilpotent p-abelian subgroup P of G (hence FP is Lie nilpotent);

(ii) $\bar{G} = G/P$ is either abelian or it contains a normal abelian subgroup $\bar{A}$ such that $(\bar{G} \setminus \bar{A})^2 = 1$ or it contains an elementary abelian 2-group of index 2.

Theorem 9 ([9]) *Suppose that char $F = p > 2$ and G is not a torsion group. Then FG^- is Lie nilpotent if and only if there exists a normal subgroup H of G such that H is nilpotent, p-abelian and $(G \setminus H)^2 = 1$.*

We finish this short survey with an application of Theorem 7 to the study of the unitary group of a group algebra.

For an algebra R let $\mathcal{U}(R)$ denote the group of units of R. If R is endowed with an involution, then one defines in a natural way the group of unitaries of R as

$$Un(R) = \{u \in \mathcal{U}(R) \mid uu^* = 1\}.$$

There is a close relation between the group of unitaries of an algebra and its skew elements. For instance one can easily prove the following result.

Theorem 10 ([9]) *Let R be a finite dimensional semisimple algebra with involution over an algebraically closed field F. Then $Un(R)$ satisfies a group identity if and only if R^- is commutative.*

Recall that an element $u \in Un(R)$ is called a Cayley unitary if there exists a quasi-regular element $a \in R^-$ such that $u = (1+a)(1-a)^{-1}$. The above relation between group identities on $Un(R)$ and the commutativity of R^- should not be surprising if we look at the general linear group. In fact, it was shown in [4] that the Cayley unitaries, fill in the orthogonal group or a subgroup of the symplectic group of index 2.

For group algebras, it was proved in [6] and [8] that, for a torsion group G over an infinite field F, a group identity in $\mathcal{U}(FG)$ forces a polynomial identity on FG. This result led to a complete classification of torsion groups G such that the group of units of FG satisfies a group identity ([19], [14], [15]).

In the case of the unitary group $Un(FG)$, the classification of torsion groups for which $Un(FG)$ satisfies a group identity is still open in general. About partial results, in [10] Gonçalves and Passman classified all finite groups G such that $Un(FG)$ does not contain a free group of rank 2, provided the field F is non-absolute; i.e., it is not an algebraic extension of a finite field.

Inspired by the connection between skew elements and unitary units and the classification given in Theorem 7 above, in [9] the authors proved a result in case $Un(FG)$ satisfies a group identity subject to a small restriction and $char\ F = 0$.

Since every free group of finite rank can be embedded in a free group of rank 2, every group which satisfies a group identity also satisfies a group identity in two variables. Moreover, by a suitable change of variables (by applying suitable endomorphisms of the free group) one can always assume that w is of the form:

$$w(x,y) = yx^{\epsilon_1}y^{-1}x^{\epsilon_2}yx^{\epsilon_3}y^{-1}\cdots y^{\eta}x^{\epsilon_t}, \tag{4}$$

where $\epsilon_1, \ldots, \epsilon_t \in \{\pm 1, \pm 2\}$ and $\eta \in \{1, -1\}$.

We say that a word $w(x,y)$ is 2-*free* if it does not become trivial when evaluated on elements of order 2. For instance, if in (4) we take $\epsilon_1, \ldots, \epsilon_t \in \{\pm 1\}$, then w becomes 2-free. Notice that groups which are n-Engel satisfy a group identity which is 2-free.

As a consequence of Theorem 7, we get the following.

Theorem 11 ([9])*Let F be a field of characteristic 0, G any group and T the set of torsion elements of G. Suppose that $Un(FG)$ satisfies a group identity which is 2-free. Then T is a subgroup and one of the following conditions holds*

(i) T is an abelian group.

(ii) $A = \langle g \in T \mid o(g) \neq 2 \rangle$ is a normal abelian subgroup of G.

(iii) T contains an elementary abelian 2-subgroup B such that $[T : B] = 2$.

Conversely, if $G = T$ is a torsion group and G satisfies one of conditions (i), (ii), (iii), then $Un(FG)$ satisfies a group identity.

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