

## Group rings of finite simple groups

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*S. K. Sehgal to his 65th birthday.*

**Abstract:** Let  $G$  be a finite non-abelian simple group. In the first part we consider the question whether  $\mathbb{C}G$  determines  $G$  up to isomorphism. This question is closely related to a recent conjecture of B. Huppert that  $G$  is determined up to a direct abelian factor by its set of ordinary character degrees. We sketch a proof that a finite simple group is determined by all its group algebras over a field. This proof involves also arguments of modular group algebras of  $G$ . The second part deals with conjugacy questions in the unit group of  $\mathbb{Z}G$ . A survey on the known results of conjectures of Zassenhaus and variations of these conjectures is given with respect to finite simple groups.

**Key words:** Group rings, finite simple groups, character degrees, Zassenhaus conjectures, automorphisms, blocks.

## 1 Introduction

The question up to which extent a finite group  $G$  is determined by its group ring  $RG$  -  $R$  an arbitrary commutative ring - is a topic with a long history. The case when  $R = \mathbb{Z}$  has been studied first by G.Higman in his thesis [29]. Further work on such questions have been stimulated by R.Brauer in his well known lectures on modern representation theory [9].

M. Hertweck has shown that the isomorphism problem for integral group rings, i.e. the question whether  $\mathbb{Z}G \cong \mathbb{Z}H$  implies that  $G$  and  $H$  are isomorphic, has in general a negative answer [21].

For many classes of interesting groups however this problem has a positive answer and this is not only the case when the coefficient ring of the group ring is  $\mathbb{Z}$ .

The object of the first part of this survey concerns some recent results - motivated by a conjecture of B. Huppert - for the case when  $G$  is finite simple and  $R$  is a field.

In the second part we consider torsion units and properties of torsion subgroups in integral group rings of finite simple groups. If  $R$  is an integral domain of characteristic zero such that  $|G|$  is not invertible then it is known for a finite simple group  $G$  that  $RG \cong RH$  if and only if  $G$  and  $H$  are isomorphic. Even more, if  $RG \cong RH$ , then  $G$  and  $H$  have the same chief series [42]. If  $G$  is abelian simple then already G. Higman showed that the torsion units of  $\mathbb{Z}G$  are just the elements of  $G$  and the group of normalized - i.e. augmentation preserving -

automorphisms of  $\mathbb{Z}G$  coincides with  $\text{Aut}G$ .

Thus the question is what about the torsion units and torsion subgroups of the unit group of  $\mathbb{Z}G$  can be said when  $G$  is finite non-abelian simple.

In this survey we do not consider infinite simple groups. For recent work and aspects on group rings of locally finite simple groups see [57] and [78].

## 2 Ordinary and modular group rings

**Problem 1 [Conjecture of B. Huppert 2000, [32]]** Denote for a finite group by  $\text{cd}G$  its set of ordinary character degrees. Assume that  $G$  is finite simple. If  $\text{cd}G = \text{cd}H$  as set then  $H \cong [G, G] \times A$  where  $A$  is a finite abelian group.

B. Huppert gave evidence for his conjecture by proving it among others for the following simple groups:

$A_1(q)$ ,  ${}^2B_2(q)$ ,  $q$  arbitrary  $A_n$  for  $n \leq 11$ , a finite number of small simple groups of Lie type, 15 of the 26 sporadic simple groups including all Mathieu groups and all Janko groups [32], [33].

We remark that we formulated Huppert's conjecture in such a way that it contains a positive solution for simple abelian groups.

The next problem was posed as a question for a general finite group.

**Problem 2 [9, Problem 2\*]** If two groups  $G_1$  and  $G_2$  have isomorphic group algebras over every ground field  $\Omega$  are  $G_1$  and  $G_2$  isomorphic ?

E. Dade gave a counterexample to this problem [14]. The counterexamples are metabelian. For finite abelian groups it is easy to see that the answer is positive. It is certainly natural first to study the question whether Problem 2 has a positive solution in the case when  $G$  is a finite non-abelian simple group and then to go on Huppert's conjecture.

Huppert's conjecture indicates also that much more might be true. If one assumes that  $\text{cd}G$  coincides with  $\text{cd}H$  and the multiplicities of the character degrees agree, then this is equivalent to an isomorphism of the complex group algebras  $\mathbb{C}G$  and  $\mathbb{C}H$ . This leads naturally to the following question.

**Problem 3** Let  $K$  be a field and let  $G$  be a finite simple group. Is it true that  $KG \cong KH$  implies that  $G$  is simple ?

More generally is of course the question, what invariants of a general finite group  $G$  are determined by  $KG$ . The case when  $K$  is the field of complex numbers has been studied in [34] with respect to soluble groups. Assume that  $\mathbb{C}G \cong \mathbb{C}H$ . Then it is shown in [34] that  $G$  is nilpotent if, and only if,  $H$  is nilpotent. Moreover let  $K$  be a normal Hall - subgroup of  $G$  then  $H$  has a normal Hall subgroup  $L$  such that  $\mathbb{C}K \cong \mathbb{C}L$  and  $\mathbb{C}G/K' \cong \mathbb{C}H/L'$ .

Note that results may depend on the field. The field of complex numbers determines only the order of a finite abelian group whileas  $\mathbb{Q}$  determines finite abelian groups up to isomorphism.

Since there are only few simple groups with the same order, it should follow that if Problem 3 has an affirmative answer for some field that  $G$  and  $H$  are

isomorphic.

The object of the remainder of this section is a sketch of the proof of the following result.

**Theorem 2.1** [30] *Let  $G$  be a finite simple group then Problem 2 has an affirmative answer.*

Note, if  $G$  is simple abelian, then Problems 1,2, and 3 have a positive solution.

Let  $K$  be a field and let  $G$  be a finite non-abelian simple group. Starting with an isomorphism  $KG \cong KH$  we see that  $H$  has to be perfect and that  $|H| = |G|$ . Assume that  $H$  is not simple. Then  $H$  has a non-abelian simple image  $Q$  of order dividing  $|H|$ . In particular  $|Q|$  has to divide  $|G|$ .

Divide the finite non-abelian simple groups into three families, the alternating groups  $A_n$ , the simple groups of Lie type and the 26 sporadic simple groups.

Case 1.  $G$  is a simple group of Lie Type.

**Proposition 2.2** [40] *Let  $G$  be a finite simple group of Lie type and let  $p$  be its describing characteristic. Let  $F$  be a field of characteristic  $p$ . Then  $FG \cong FH$  implies  $H$  is simple.*

Clearly it suffices to show the proposition in the case when  $F$  is algebraically closed. The main ingredient for the proof of the proposition is that  $FG$  has precisely two  $p$ -blocks, one of defect zero and  $F$ -dimension  $|P|^2$ , where  $P$  denotes a Sylow  $p$ -subgroup of  $G$  [31, section 5]. Using Clifford theory for blocks [1, Theorem 4.4] it is shown that then  $H$  has to be simple as well.

There are only few non-isomorphic simple groups with the same order namely the alternating group  $A_8$  and the linear group  $A_2(4)$  and the infinite series of pairs  $B_n(q), C_n(q)$  with  $n \geq 3$  and  $q$  odd.

A look at the degrees of the ordinary characters and of the Brauer characters shows that Problem 3 has a positive solution for  $A_8$  and  $A_2(4)$  with respect to each field  $K$ . Moreover Huppert established his conjecture for these two groups [32].

It is known that the minimal degree of a faithful ordinary character of  $B_n(q)$  differs from that one of  $C_n(q)$ , [46], [74]. In case of  $C_n(q)$  it is  $\frac{q^n-1}{2}$  or  $\frac{q^n+1}{2}$  while as the minimal degree for  $B_n(q)$  is greater equal than  $\frac{q^{2n}-1}{q^2-1}$ , cf. [74, Theorem 5.2, Theorem 6.1].

This settles case 1.

Case 2.  $G$  is an alternating group of degree  $n$ .

Case 2a.  $Q = A_m$ .

If  $G = A_5$  then the smallest ordinary character degree is 3. If  $n \geq 6$ , then by a result of W. Burnside  $n-1$  is the smallest ordinary character degree [10, Appendix C, p.468].

This shows immediately that if  $CA_n$  maps onto  $CA_m$  then  $n = m$ .

Case 2b.  $Q$  is a sporadic group.

The minimal degrees of ordinary characters of the symmetric group have been classified by R. Rasala [61]. Using his results we get the following for the alternating groups.

**Proposition 2.3** *Let*

$$d_0 = 1, d_1 = n - 1, d_2 = \frac{1}{2}n(n - 3), d_3 = \frac{1}{2}(n - 1)(n - 2),$$

$$d_4 = \frac{1}{6}n(n - 1), d_5 = \frac{1}{6}(n - 1)(n - 2)(n - 3), d_6 = \frac{1}{3}n(n - 2)(n - 4),$$

$$d_7 = \frac{1}{24}n(n - 1)(n - 2)(n - 7), d_8 = \frac{1}{24}(n - 1)(n - 2)(n - 3)(n - 4).$$

- a) *Suppose that  $9 \leq n \leq 14$ . Then the first four minimal ordinary character degrees for  $A_n$  are given by  $d_0, d_1, d_2, d_3$ .*
- b) *If  $15 \leq n \leq 21$ , then the seven smallest ordinary character degrees of  $A_n$  are  $d_0, \dots, d_6$ .*
- c) *If  $n \geq 22$ , then the nine smallest ordinary character degrees of  $A_n$  are  $d_0, \dots, d_8$ .*

*In all cases the multiplicity of the degrees  $d_i$  is 1.*

We show how this proposition is used to show that  $Q$  cannot be the Babymonster B.

The order of  $B$  is

$$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47.$$

The smallest degrees of non-trivial complex irreducible representations of  $B$  are  $\delta_1(B) = 4371$  and  $\delta_2(B) = 96255$ . Therefore the initial range for  $n$  is  $n \leq 4372$ . If  $n = 4372$  we get  $d_2(n) > 96255$ . Thus  $CA_{4372}$  cannot map onto  $CB$ .

If  $n \leq 4371$  then the degree 4371 of  $B$  must appear as one of the larger degrees of  $A_n$ . Looking again at  $d_2(n) = \frac{1}{2}n(n - 3)$  we see that  $d_2(n) > \delta_2(B)$  provided  $n \geq 96$ .

If  $n < 95$  we have  $d_2(n) < \delta_1(B)$  which again means that  $\delta_1(B)$  appears as a larger degree of  $A_n$ . Now  $d_3(n) > \delta_1(B)$  for  $n > 31$ . On the other hand 47 divides  $|B|$ . Thus  $n \geq 47$ . Consequently  $B$  cannot occur as  $Q$ .

With similar calculations it is shown that all other sporadic simple groups also cannot occur as  $Q$ .

Case 2c.  $Q$  is a simple group of Lie type.

As simple group of Lie type  $Q$  has a non-trivial ordinary character degree of prime power order  $p^n$  namely the degree of the Steinberg character. By [2], if  $n \geq 10$ , the  $A_n$  has a non-trivial character degree of prime power order if and only if  $n = p^f + 1$  and  $p^f$  is the only one of this kind. The degree  $d = p^f$  of the Steinberg character is the order of a Sylow  $p$ -subgroup of  $Q$ . Moreover looking at the order of a finite simple group  $Q$  of Lie type one sees that

$$|Q| \leq d^3.$$

This shows that all character degrees of  $Q$  are smaller than  $d^3/12$ . On the other hand looking at Proposition 2.3 the degrees of the characters of the alternating groups grow rapidly. More precisely there are at most 4 character degrees less than  $d^3/12$  if  $n \geq 15$ . These have multiplicity 1 and so  $Q$  has at most 4 different character degrees provided  $n \geq 15$ . But there is no non-abelian simple group with less than 5 conjugacy classes. A direct inspection of the alternating groups of degree less than 15 yields the result that  $Q$  can only exist in the cases  $n = 5, 6, 8, 9$  and in these cases  $Q$  is isomorphic to  $A_n$ .

Case 3.  $G$  is a sporadic group.

Case 3a.  $Q$  is sporadic. Looking at the Atlas [12] one sees that  $\text{cd}Q$  is not a subset of  $\text{cd}G$ .

Case 3b.  $Q$  is alternating. Because  $n - 1$  is the smallest non-trivial degree for  $A_n$  the smallest degree for  $G$  bounds  $n$ . This and the fact that each prime dividing  $|Q|$  divides the order of  $G$  shows that  $Q$  cannot be an alternating group.

Case 3c.  $Q$  is simple of Lie type. Only the groups  $M_{11}, M_{12}, M_{24}, \text{Co}_2, \text{Co}_3$  have a non-trivial ordinary character degree of prime power order. The prime powers are 11, 23 and 16. As  $Q$  is a simple group of Lie type, the Steinberg character of  $Q$  gives a character degree of prime power order. Thus there are few possibilities to be checked and a direct inspection shows finally that this case can also not occur.

In [47] all non-trivial ordinary character degrees of prime power degree of finite simple groups of Lie type are classified. There are not many cases where such degrees are not given by the Steinberg character. It appears to be possible to use this together with [46], [53] and [74] to avoid the use of Proposition 2.2. This should lead to the result that the complex group algebra of a finite simple group  $G$  determines  $G$  up to isomorphism. I doubt that at present sufficiently much is known about the degrees of ordinary characters in order to prove Huppert's conjecture for each finite simple group.

### 3 Integral group rings

Notations.  $\varepsilon : RG \rightarrow R, \sum_{x \in G} r_x \cdot g \mapsto \sum_{x \in G} r_x$  denotes the augmentation map and its kernel, the augmentation ideal, is denoted by  $I_R(G)$ . The group of units of  $RG$  is  $U(RG)$  and its subgroup consisting of the units with augmentation 1 is denoted by  $V(RG)$ .

H. Zassenhaus stated with respect to the structure of  $U(RG)$  of integral group rings  $RG$  three well known conjectures, cf. [69].

**Problem 4 [The Zassenhaus conjecture (Z2)]** Let  $H$  be a group basis of  $ZG$ . Then  $H$  is conjugate by a unit of  $QG$  to  $G$ .

It has been shown by K. W. Roggenkamp and L. L. Scott that this conjecture is in general not true [45], [63], [73]. For finite simple groups however the answer is open.

The Zassenhaus conjecture (Z2) may be also phrased in terms of automorphisms. An  $R$ -algebra automorphism  $\sigma$  of  $RG$  is called normalized if it preserves the augmentation, i.e.

$$\sigma(\varepsilon(g)) = \varepsilon(\sigma(g)).$$

In the case when the isomorphism problem for  $ZG$  has a positive solution (Z2) is equivalent to the following.

**(AUT)** Let  $X$  be a group basis of  $ZG$ .<sup>1</sup> Any normalized automorphism  $\sigma$  of  $ZG$  may be written as the composition of one induced from a group automorphism of  $X$  followed by a central automorphism.

Note that an automorphism is called central if it fixes the centre elementwise. By the Skolem - Noether theorem these are precisely the automorphisms of  $ZG$  given by conjugation with a unit  $u$  of  $QG$  which stabilizes  $ZG$ .

If  $G$  belongs to a class of finite groups closed under direct products, then (Z2) holds for each group for this class provided (AUT) is valid for each group of the class [38, Lemma 5.3 a)].

The following so - called  $F^*$  - theorem has been discovered by K. W. Roggenkamp and L. L. Scott [65].

**$F^*$  - theorem.** Denote by  $R$  the  $p$ -adic integers (or more generally the integral closure of  $p$ -adic integers in a finite extension field of  $\mathbb{Q}_p$ ). Let  $G$  be a finite group, with a normal  $p$ -subgroup  $N$  containing its centralizer. Then for any augmented ring automorphism  $\sigma$  of  $RG$  which stabilizes  $I_R(N) \cdot RG$ , the groups  $G$  and  $\sigma(G)$  are conjugate in the units of  $RG$ .

Unfortunately the proof of Roggenkamp and Scott was not quite complete. In the mean time a complete proof exists. The strong results of A. Weiss on  $p$ -permutation modules are a basis for it [76]. The proof can be collected from [23], [24] and [26].

The assumption on  $G$  in the  $F^*$  - theorem is equivalent to the condition that the generalized Fitting subgroup  $F^*(G)$  is a  $p$ -group. This explains the name of the theorem. Note also that the groups satisfying the assumption of the  $F^*$  - theorem form a class closed under direct products. Thus we get the following conclusion.

**Corollary.** Assume that the generalized Fitting subgroup  $F^*(G)$  is a  $p$ -group then the Zassenhaus conjecture (Z2) is valid. In particular, if  $G$  is a finite soluble group with  $O_{p'}(G) = 1$ , then the (Z2) - conjecture is valid.

<sup>1</sup>A group basis of  $ZG$  is a subgroup of  $V(ZG)$  of the same order as  $G$ . It automatically spans  $ZG$ .

This leads naturally to the following open question.

**Problem 5.** Let  $G$  be a finite group. Assume that  $O_{p'}(G) = 1$  for some prime  $p$ . Is the Zassenhaus conjecture (Z2) valid for  $G$ ?

A positive answer to Problem 5 would lead to an obstruction theory for the (Z2) - conjecture for a general finite group as described for soluble groups in [44]. Clearly finite simple groups are a first test object whether Problem 5 has an affirmative answer. In contrast to the case of soluble groups there is no similar approach known to show this conjecture for simple or almost simple groups. Also many module - theoretic techniques of the modular representation theory work only for  $p$  - constrained groups. Thus it is not obvious how to extend the proof of the  $F^*$  - theorem to general finite groups.

In the case of simple or almost simple groups character - theoretical arguments and the theory of blocks with cyclic defect are up to now almost the only methods which lead to results. If the structure of the group ring is at least locally explicitly known - but this is only the case for very few groups - then the automorphisms of the group ring can be precisely computed. For some groups this seems the only possibility to decide whether conjecture (Z2) is valid or not. With respect to the complex reflection group  $3.A_6$  ( the non-split central extension of the alternating group  $A_6$  with a cyclic group of order 3 ) this was the only way to show finally that (Z2) holds for this group, see [28]. For the other exceptional complex reflection groups (Z2) has been proved in [20].

May be the condition  $O_{p'}(G) = 1$  in Problem 5 should be replaced in such a way that the statement is only a statement for the principal block  $B_0$ , see also the remarks about this in [71].

**Problem 6 [The principal block variation (B-Z2) $_{0,p}$ ].** Let  $H$  be a group basis of  $\mathbb{Z}G$ . Let  $p$  be a rational prime dividing  $|G|$  and let  $B_1, \dots, B_k$  be the blocks of the Wedderburn decomposition of  $\mathbb{C}G$  associated to the irreducible  $\mathbb{C}$  - characters of  $G$  which belong to the principal  $p$ -block. Let  $\pi$  be the projection of  $\mathbb{C}G$  onto  $B_0 := \bigoplus_{i=1}^k B_i$ . Then  $\pi(G)$  and  $\pi(H)$  are conjugate within  $B_0$ .

There is also no counterexample known to the following block variation of conjecture (Z2).

**Problem 7 [The block variation (B-Z2) $_C$ ].** Let  $H$  be a group basis of  $\mathbb{Z}G$ . Let  $B$  be a block of the Wedderburn decomposition of  $\mathbb{C}G$ . Let  $\pi$  be the projection of  $\mathbb{C}G$  onto  $B$ . Then  $\pi(G)$  and  $\pi(H)$  are conjugate within  $B$ .

We remark that the block variations stated as above also make sense  $p$ -adically and modularly. It might be possible that projections of group bases of  $\mathbb{Z}G$  are conjugate within the principal block of  $KG$  where  $K$  denotes an algebraically closed field of characteristic  $p$ .

Other variations of (Z2) which have been considered in the last years deal with Sylow-like theorems.

**Problem 8.**

a) [The Zassenhaus conjecture (Z1)] Let  $x$  be a unit of  $V(\mathbb{Z}G)$  of finite

- order. Then  $x$  is conjugate within  $\mathbb{Q}G$  to an element of  $G$ .
- b) Let  $X$  and  $Y$  be group bases of  $\mathbb{Z}G$ . Let  $S \in \text{Syl}_p(X)$  and  $T \in \text{Syl}_p(Y)$ . Are then  $S$  and  $T$  conjugate within  $\mathbb{Q}G$  ?
- c) Let  $U$  be a  $p$ -subgroup of  $V(\mathbb{Z}G)$ . Is  $U$  conjugate within  $\mathbb{Q}G$  to a subgroup of  $G$ ?

Clearly Problems 8b and 8c describe a Sylow-like variation for the unit group of integral group rings. For Problem 8a this is as well the case when the unit has prime power order. Problem 8b is an important special case of 8c. It contains also a statement about normalized automorphisms in the case when  $X$  and  $Y$  are isomorphic.

It is a consequence of the  $F^*$ -theorem that Problem 8b is true for finite soluble groups [43]. It follows from the results in [77] that Problem 8c holds provided  $G$  is nilpotent. This was extended to nilpotent-by-nilpotent groups and soluble groups with abelian Sylow subgroups [15]. In [16] it is shown that Problem 8c is valid for Frobenius groups which do not have the symmetric group  $S_5$  as a homomorphic image.

To the conjecture (Z1) no counterexample is known. In the area of soluble groups it is among other shown for  $G$  nilpotent [77] and certain metabelian groups [48], [60], [59], [55].

Problems 6, 7 and 8 are closely related to the following about defect groups.

**Problem 9 [The defect group problem [72, p.257]], see also [71].** Let  $G$  and  $H$  be finite groups. Denote by  $\mathbb{Z}_p$  the  $p$ -adic integers and let  $B$  be a block of group rings  $\mathbb{Z}_pG$  and  $\mathbb{Z}_pH$ . Let  $D_G, D_H$  resp. be defect groups of  $G$  and  $H$  with respect to  $B$ . Identify  $D_G$  and  $D_H$  with their projections on  $B$ . Is it then true that after suitable normalization  $D_G$  and  $D_H$  are conjugate by a unit in  $B$ .

Note that the group rings  $\mathbb{Z}_pG$  and  $\mathbb{Z}_pH$  have just this common block  $B$ . So they may be non-isomorphic. But of course the defect group problem may be specialized to the situation where  $H$  is a group basis of  $\mathbb{Z}G$ . If then the defect group problem holds for the principal block  $B_0$ , then Problem 8b has a positive answer. Note that the suitable normalization in case of the principal block is just given from the augmentation. If  $G$  has cyclic Sylow  $p$ -subgroups the defect group problem has a positive answer [72].

Conjecture (Z2) holds for finite Coxeter groups [6]. Denote by  $\text{AutCT}(G)$  the character table automorphisms of  $G$  and let  $W$  be a finite irreducible Coxeter group. Then the sequence

$$1 \longrightarrow \text{Inn}W \longrightarrow \text{Aut}W \longrightarrow \text{AutCT}(W) \longrightarrow 1$$

is exact [6, Theorem 1.1]. This is a very special property and not typical for other groups (even for complex reflection groups it is in general not true [20]).

The counterexamples to (Z2) give the impression that character table automorphisms coming from Galois automorphisms of the character field of a block of



the Wedderburn decomposition of  $\mathbb{C}G$  give an obstruction for the validity of (Z2). This makes it reasonable to study the following rational variation.

**Problem 10 [The  $\mathbb{Q}$ -variation (Z2) $_{\mathbb{Q}}$ ].** Let  $X$  and  $Y$  be group bases of  $\mathbb{Z}G$ . Then there is an automorphism  $\sigma$  of  $\mathbb{Q}G$  such that  $\sigma(X) = Y$  and  $\sigma$  fixes each component of the Wedderburn decomposition of  $\mathbb{Q}G$ .

We first give a survey about known results with respect to the variations and simple groups.

**Theorem 3.1** [8, 2.2 and 2.3]

- a) *Problem 8b is valid for all sporadic simple groups and their automorphism groups.*
- b) *The variation (Z2) $_{\mathbb{Q}}$  is valid for all sporadic simple groups and their automorphism groups.*

With respect to Problems 8a and 8c very little is known for simple and related groups. (Z1) is true with respect to the alternating group  $A_5$  and the symmetric group  $S_5$  [54], [55]. It follows from [16] that Problem 8c has a positive answer for  $A_5, S_5$  and  $A_1(5)$ . In [75] it is shown that for  $G = A_1(p^f)$  with  $f = 1$  or  $f = 2$  Problem 8a holds for torsion units of order  $p$ .

**Theorem 3.2** *Let  $G$  be an alternating group.*

- a) [37, Corollary 2.8.2] *Problem 8b holds for  $G$ .*
- b) [38, Satz 5.9] *(Z2) $_{\mathbb{Q}}$  holds for  $G$ .*

Theorem 3.2 b) was first proved in [38, pp.91 – 95]. The proof is similar to that one given by G. Peterson for the Zassenhaus conjecture for symmetric groups in [58], see also Theorem 3.5 below. Note that the proof given in [38] is different from the sketchy arguments given in [39, p.98]. Though these arguments are not complete another less technical proof could be written up with it. Later A. Giambruno gave his own variant for a proof of Theorem 3.2 b) [18].

It remains to establish a Sylow - like theorem and (Z2) $_{\mathbb{Q}}$  for simple groups of Lie type. Because (Z2) implies that Problem 8b has a positive answer as well as (Z2) $_{\mathbb{Q}}$ , both variations hold for the simple groups of Lie type listed in Theorem 3.6 below.

Next we turn to results on the Zassenhaus conjecture (Z2) itself and the block variations.

**Theorem 3.3** [8, Theorem 2.5] [7] *The conjecture (Z2) is valid for a sporadic simple group if in Table 1 the corresponding column is marked with a +-sign. Analogously the variations (B-Z2) $_{\mathbb{C}}$  and (B-Z2) $_{0,p}$  for all  $p$  hold if there is a + sign in the table. The last column lists all primes  $p$  for which the principal block conjecture is unknown.*

**Theorem 3.4** [8, 2.1] *Let  $X$  be a finite sporadic simple group with nontrivial outer automorphism group. Then (Z2) is valid for  $\text{Aut}(X) = X.2$ .*

Table 1: Results on (Z2) and the variations of (Z2)

| Group            | (Z2) | (B-Z2) <sub>C</sub> | (B-Z2) <sub>0,p</sub><br>valid $\forall p$ | (B-Z2) <sub>0,p</sub><br>? |
|------------------|------|---------------------|--|----------------------------|
| M <sub>11</sub>  | +    | +                   | +  | —                          |
| M <sub>12</sub>  | +    | +                   | +  | —                          |
| M <sub>22</sub>  | +    | +                   | +  | —                          |
| M <sub>23</sub>  | +    | +                   | +  | —                          |
| M <sub>24</sub>  | +    | +                   | +  | —                          |
| J <sub>2</sub>   | +    | +                   | +  | —                          |
| Suz              | +    | +                   | +  | —                          |
| H'S              | +    | +                   | +  | —                          |
| McL              | open | open                | open                                       | $p = 2, 5, 11$             |
| Co <sub>3</sub>  | +    | +                   | +  | —                          |
| Co <sub>2</sub>  | +    | +                   | +  | —                          |
| Co <sub>1</sub>  | +    | +                   | +  | —                          |
| He               | open | +                   | open                                       | $p = 2, 5$                 |
| Fi <sub>22</sub> | open | +                   | open                                       | $p = 2, 3$                 |
| Fi <sub>23</sub> | open | open                | open                                       | $p = 2, 3, 5, 23$          |
| Fi <sub>24</sub> | open | +                   | open                                       | $p = 2, 3, 23$             |
| H'N              | +    | +                   | +  | —                          |
| Th               | +    | +                   | +  | —                          |
| B                | +    | +                   | +  | —                          |
| M                | open | open                | open                                       | $p = 2, 3, 13, 17$         |
| J <sub>1</sub>   | +    | +                   | +  | —                          |
| O'N              | +    | +                   | +  | —                          |
| J <sub>3</sub>   | +    | +                   | +  | —                          |
| Ly               | open | open                | open                                       | $p = 3, 5, 67$             |
| Ru               | +    | +                   | +  | —                          |
| J <sub>4</sub>   | open | open                | open                                       | $p = 2, 11, 43$            |

We note that if in the open cases the principal block conjecture (B-Z2)<sub>0,p</sub> held for each prime  $p$  then our results would show that already the conjecture (Z2) is valid in these open cases.

With respect to alternating groups  $A_n$  one knows only small cases where (Z2) is valid. This is the case for  $2 \leq n \leq 10$  and  $n = 12$  [3]. For central extensions of alternating groups the problem seems to be even harder. The only general result in this area of simple groups is the following of G. Peterson.

**Theorem 3.5** [58] (Z2) is valid for symmetric groups.

**Theorem 3.6** [3], [4], [5]. The conjecture (Z2) is valid for all finite simple groups of Lie type of rank 1 and of rank 2 which are not of type  ${}^2A_3$  or  ${}^2A_4$ .

In the last part of this section we consider the structure of the normalizer of a finite subgroup of  $G$  within  $V(\mathbb{Z}G)$ .

**Problem 11 .**

- a) [**The normalizer problem [70, Problem 43]**]. Consider  $G$  as subgroup of  $U = U(\mathbb{Z}G)$ . Is it true that the normalizer  $N_U(G)$  coincides with  $Z(U) \cdot G$ , where  $Z(U)$  denotes the centre of  $U$ ?
- b) Let  $K$  be a subgroup of  $G$ . Is  $N_G(K) = C_U(K) \cdot N_G(K)$ ?

Problem 11a played in the last years a prominent role in the solution of the isomorphism problem of integral group rings.

M. Mazur showed that a counterexample for a finite group to Problem 11a would lead to a counterexample for the isomorphism problem for integral group rings of infinite groups [49], [50]. K. W. Roggenkamp and A. Zimmermann constructed a counterexample to the semilocal analogue of Problem 11a [66], [67]. More precisely they constructed a finite group  $G$  such that  $N_U(G) > Z(U) \cdot G$ , where  $U$  denotes the group of units of  $\mathbb{Z}_{\pi(G)}G$  and  $\mathbb{Z}_{\pi(G)}$  is the intersection of all the local rings  $\mathbb{Z}_p$ ,  $p$  a prime dividing  $|G|$ . M. Hertweck finally constructed a global counterexample to Problem 11a, i.e. one over  $\mathbb{Z}$  [21]. With M. Mazur's construction this gave as well a counterexample to the isomorphism problem for infinite groups. But M. Hertweck used his counterexample to Problem 11a also in a crucial way for his construction of a counterexample to the isomorphism problem for finite groups.

The normalizer problem has been considered in many articles within the last years, see [35], [36], [22], [27], [41], [51], [52]. We restrict ourselves here to results involving simple groups.

If  $x$  is a normalizing element of  $K$ , then conjugation by  $x$  induces an automorphism of  $K$ . If  $K = G$  then this automorphism is class preserving. By W. Feit and G. Seitz class preserving automorphisms of finite simple groups are inner [17]. Thus Problem 11a has a positive answer with respect to finite simple groups. The normalizer problem has been considered also for infinite groups. It follows from [36] or [25] that for groups with no finite normal subgroup the normalizer problem is valid, so in particular for infinite simple groups.

Coleman's Lemma, see [11] or [35, 2.6 Theorem] says that Problem 11b has a positive solution provided  $K$  is a  $p$ -subgroup. Coleman's Lemma implies that the automorphism of  $G$  induced by a normalizing element of  $U(\mathbb{Z}G)$  restricted to a Sylow  $p$ -subgroup coincides with an inner automorphism of  $G$ . Thus a group automorphism of  $G$  whose restriction to an arbitrary Sylow subgroup  $S$  coincides with an inner automorphism is called a Coleman automorphism. A

group automorphism of a finite group which is restricted to a Sylow  $p$ -subgroup the identity is called  $p$ -central. Clearly for each prime  $p$  a Coleman automorphism may be modified by an inner automorphism  $\gamma_p$  such that the composition is  $p$ -central. In [27] it is shown that for each finite simple group  $G$  exists at least one prime  $p$  such that  $p$ -central automorphisms of  $G$  are inner [27]. In particular Coleman automorphisms of finite simple groups are inner. This also proves the normalizer problem for finite simple groups. Even a little bit more follows.

**Proposition 3.7** *Let  $K$  be a simple subgroup of the finite group  $G$ . Assume that  $K$  is subnormal in  $G$  then Problem 11b holds for  $K$ .*

**Proof.** Let  $x$  be an element of  $N_U(K)$ . Let  $p$  be a prime such that  $p$ -central automorphisms of  $K$  are inner and choose  $P \in \text{Syl}_p(K)$ . Clearly there is  $k \in K$  such that  $x \cdot k$  normalizes  $P$ . By Coleman's Lemma we may write  $x \cdot k = g \cdot c$  with  $g \in N_G(P)$  and  $c \in C_U(K)$ . Now  $K^g$  is subnormal because  $K$  is by assumption subnormal.  $K^g \cap K$  contains  $P$ . On the other hand the intersection of subnormal subgroups is subnormal again. Because  $K$  is simple it follows that  $K^g = K$ . Hence conjugation by  $g^{-1} \cdot x \cdot k$  is an automorphism of  $K$  and because  $g^{-1} \cdot x \cdot k = c$  it is a  $p$ -central automorphism. By choice of  $p$  it is conjugation with some  $l \in K$ . Thus conjugation by  $x$  on  $K$  coincides with conjugation by  $g \cdot l \cdot k^{-1}$ . This proves the proposition. q.e.d.

The result on Coleman automorphisms of finite simple groups is used to prove part a) of the following result.

**Theorem 3.8** [27] *Let  $G$  be a finite group.*

- a) *Assume that the center of  $F^*(G)$  is of odd order and that  $G/F^*(G)$  has no chief factor of order 2 then the normalizer problem has a positive answer. In particular this is the case when  $G$  has no composition factor of order 2.*
- b) *If  $G$  is soluble and  $G/O_2(G)$  has no chief factor of order 2, then the normalizer problem has a positive answer for  $G$ .*

The results used to show that Problem 11a is valid for finite simple groups both that of [17] and that of [27] depend on the classification of the finite simple groups. Using a result of J. Krempa it is shown in [35, 3.5 Theorem] that the normalizer problem has a positive solution provided each 2-central automorphism of order 2 is inner. If  $G$  is simple with abelian Sylow 2-subgroups, then the latter follows from Glauberman's  $Z^*$ -theorem [19, Corollary 5]. This gives a classification free solution for the normalizer problem for such finite simple groups and illustrates the importance of the  $Z^*$ -theorem.

Most of the applications of representation theory to the structure of finite groups require only ordinary character theory. Glauberman's celebrated  $Z^*$ -theorem is proved with the aid of modular representations. J. Alperin remarked in his lecture at the conference "Richard Brauer (1901 - 1977): Taking his ideas

into the 21st century ” - held in Stuttgart 25.3. - 31.3. 2001 - that no major application of integral representation theory to the theory of finite groups has been given so far. The analogue to the  $Z^*$  - theorem for odd primes  $p$ , the so-called  $Z_p^*$  - theorem has been linked with the existence of non-trivial central units of order  $p$  in the principal  $p$  - block of a possible counterexample [62]. Can integral representation theory contribute to a classification free proof of the  $Z_p^*$  - theorem ?

The knowledge about torsion units of integral group rings of finite soluble groups has been grown up considerably in the last twenty years [13], [21], [44], [64], [73], [70], [76], [77]. This survey makes the attempt to show that some progress has been made in the area close to finite simple groups as well. But in contrast to the soluble area there is no general obstruction theory to questions like the isomorphism problem. Many results - for example the  $F^*$  - theorem - wait for an analogue for insoluble groups in order to become a result for a general finite group.

The whole development becomes clear if one compares the results known nowadays to the content of the classical books on group rings by S. K. Sehgal [68] and D. S. Passman [56]. These two books certainly have had a big influence on our present knowledge about group rings.

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