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Remarks on Analytic Hypoellipticity [1]

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Abstract: We will compare the following ideas: analytic hypoellipticity on open subsets of Euclidean space; global analytic hypoellipticity; analytic hypoellipticity in the sense of germs. We present a new operator which posseses Treves curves, yet is analytic hypoelliptic in the sense of germs. That is, the analog of the Treves conjecture, in the sense of germs, is false.

Key words: Analytic hypoellipticity, sum of squares.

1 Introduction

The problem of analyticity of solutions of partial differential equations is one of the oldest in analysis. Indeed, Hilbert [19] wrote the following, concerning his 19^{th} problem:

"*One of the most remarkable facts in the elements of the theory of analytic functions appears to me to be this: That there exist partial differential equations whose integrals are all of necessity analytic functions of the independent variables, that is, in short, equations susceptible of none but analytic solutions. The best known partial differential equations of this kind are the potential equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

and certain linear differential equations investigated by Picard;"

In today's language we say that the Laplace operator is analytic hypoelliptic. This is a property it shares with all elliptic linear partial differential equations with analytic coefficients. When one leaves the class of elliptic equations, the question of analytic regularity becomes quite complicated.

We begin by presenting a short survey comparing various notions of analytic hypoellipticity. This is the focus of Sections 2 through 5. In Section 6 we present a new operator $\mathcal{H}$ on $\mathbb{R}^3$. The operator $\mathcal{H}$ has an infinite number of Treves curves over the origin, yet is analytic hypoelliptic at the origin in the sense of germs. We state our results concerning $\mathcal{H}$ in Section 7. We give a short outline of the proofs in Section 8. Complete details will appear elsewhere.

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2 Sums of Squares and Analytic Hypoellipticity

Let $\mathcal{M}$ be a real analytic manifold and let $X_1, \dots X_r$ be real analytic vector fields on $\mathcal{M}$. We study an operator P of the form "sum of squares". That is P has the form

$$P = {X_1}^2 + \ldots + {X_r}^2.$$

Definition 1 *We say that P is analytic hypoelliptic (in the strong sense) on $\mathcal{M}$ if for every open $O \subset \mathcal{M}$ we have the following: Pu analytic on O implies that u is analytic on O. Here u is a distribution on O.*

We always assume that the X_j satisfy a "finite type" condition. That is, at each point of $\mathcal{M}$, the Lie algebra generated by the X_j (under the commutation bracket) has dimension equal to $\dim\mathcal{M}$.

Under these conditions a classical result of Hörmander [21] guarantees the hypoellipticity of P. However analytic hypoellipticity will not hold unless further assumptions are made.

If we assume that Σ, the characteristic set of P, is a symplectic manifold and that the principal symbol of P vanishes precisely to second order on Σ, then P is analytic hypoelliptic. This follows from a result of Treves [27] and Tartakoff [26].

Further work in this direction around this time was done by Métivier and Sjöstrand. See for example [23], [24].

When Σ is not symplectic, analytic hypoellipticity may fail. We have the following example due to Baouendi - Goulaouic [1]. Consider the operator on $\mathbb{R}^3$ given by

$$B = {\partial_x}^2 + {\partial_y}^2 + x^2{\partial_t}^2. \tag{1}$$

If we use natural coordinates $(x, y, t; \xi, \eta, \tau)$ on the cotangent bundle $T^*(\mathbb{R}^3)$, we see that the characteristic set Σ for B is defined by the equations

$$\xi = \eta = x = 0.$$

Σ is not symplectic and B is not analytic hypoelliptic on any open set that intersects $x = 0$.

Note that

$$\Sigma = \{f = g = h = 0\}$$

with df, dg, dh independent and the following condition on the Poisson brackets

$$\{f, g\} = 1, \{f, h\} = 0, \{g, h\} = 0.$$

Σ is foliated by the integral curves of the Hamilton field of h (bicharacteristics of h). We will say that such a Σ is of *Baouendi - Goulaouic type.*

These bicharacteristics are examples of what are now called Treves curves. Indeed, let $(0, 1) \subset \mathbb{R}$ denote the open unit interval. We have the following :

Definition 2 *Let $\Sigma \subset T^*\mathcal{M}$ be an analytic submanifold and let $\gamma : (0,1) \to \Sigma$ be a non-constant analytic curve. We call γ a Treves curve for Σ if*

$$\frac{d\gamma}{dt}(t) \in (T_{\gamma(t)}\Sigma)^{\perp} \tag{2}$$

for all $t \in (0,1)$. Note that $T_{\gamma(t)}\Sigma$ is the tangent space to Σ at the point $\gamma(t)$ and $(T_{\gamma(t)}\Sigma)^{\perp}$ is the orthogonal space with respect to the symplectic form.

In [27] Treves conjectured that when the characteristic set Σ is a manifold and contains such curves, the associated operator is not analytic hypoelliptic. Later [28], Treves extended his conjecture. Both conjectures are still open at this time. Recent progress has been made by Chanillo [4].

Next we have the example of Métivier [22]:

$$M = {\partial_x}^2 + (x^2 + y^2){\partial_y}^2. \tag{3}$$

M is not analytic hypoelliptic on any open set containing the origin. (M is elliptic away from the origin.) Note that the characteristic set Σ is given by

$$\Sigma = \{\xi = x = y = 0\},$$

and hence is of Baouendi–Goulaouic type.

3 Global Analytic Hypoellipticity

Definition 3 *We say that P is globally analytic hypoelliptic on $\mathcal{M}$ if for every distribution u on $\mathcal{M}$ we have the following: Pu analytic on $\mathcal{M}$ implies that u is analytic on $\mathcal{M}$.*

We have the surprising fact that the Baouendi - Goulaouic operator is globally analytic hypoelliptic on the 3 dimensional torus $\mathbb{T}^3$. This result is due to Cordaro – Himonas [12].

Other interesting global results have been obtained by Himonas – Petronilho [20] and by Bergamasco – Cordaro – Malagutti [3].

Later Tartakoff [25] proved that the Baouendi–Goulaouic operator

$$B = {\partial_x}^2 + {\partial_y}^2 + x^2{\partial_t}^2$$

is globally analytic hypoelliptic on

$$\mathbb{S}^1 \times V.$$

Here $\mathbb{S}^1$ is the circle in y space and V is any open set in (x,t) space. Note that here the Treves curves are compact.

4 Several Complex Variables

We turn to related results from the theory of functions of several complex variables. We begin with the results of Chen [6] and [5].

Theorem 1 *Let $n \geq 2$. Let $\mathcal{D} \subset \mathbb{C}^n$ be a bounded complete Reinhardt domain with real analytic pseudoconvex boundary. Then*

(1) The $\bar{\partial}$- Neumann problem is analytic hypoelliptic.

(2) The Szegö projector preserves analyticity.

Similar results in this direction were obtained by Derridj and Tartakoff [13].

Note that Christ [9] has shown that there exist domains in $\mathbb{C}^2$ as above, such that the associated "sum of squares" is not analytic hypoelliptic (in the strong sense). Here Treves curves exist and are compact.

Christ [11] has also shown the existence of bounded pseudoconvex domains with analytic boundary such that the Szegö projector does not preserve analyticity.

The analytic singularities of the Bergman and Szegö kernels are determined by the Treves curves for tube domains in $\mathbb{C}^2$, Francsics – Hanges [14], [15]. It is not clear how these off–diagonal singularities affect the mapping properties of these kernels.

5 Analytic Hypoellipticity in the Sense of Germs

Definition 4 *We say that P is analytic hypoelliptic at $x \in \mathcal{M}$ if we have the following: Pu analytic near x implies that u is analytic near x. Here u is a distribution defined near x.*

Consider the tube domain

$$\mathcal{D} = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_2 > (\Im z_1)^m\}.$$

Here $m \geq 3$ is an odd integer. If we parametrize the boundary using $x_1 = \Re z_1, y_1 = \Im z_1, x_2 = \Re z_2$, we see that the natural CR structure is generated by the vector field

$$L = \frac{\partial}{\partial x_1} + m{y_1}^{m-1}\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_1}. \tag{4}$$

L is analytic hypoelliptic at any point where $y_1 = 0$. On the other hand, L is not analytic hypoelliptic at any point where $y_1 \neq 0$. Much more general results of this nature were obtained by Baouendi and Treves [2]. Also observe that the operator $\bar{L}L$ is analytic hypoelliptic at any point where $y_1 = 0$.

On the other hand, the sum of squares

$$T = (\frac{\partial}{\partial x_1} + m{y_1}^{m-1}\frac{\partial}{\partial x_2})^2 + (\frac{\partial}{\partial y_1})^2$$

is not analytic hypoelliptic at any point where $y_1 = 0$. This is due to Grigis - Sjöstrand [16] when $m = 3$ and to Hanges - Himonas [18] for general odd m. A proof valid for all $m \geq 3$ was later given by Christ [8].

Observe that T and $\bar{L}L$ differ by a first order operator. It would be interesting to study the influence of lower order terms on regularity questions.

Also note that the characteristic set here has codimension 2. However the set of characteristic points that are "weakly pseudoconvex" is a manifold of Baouendi–Goulaouic type.

6 A New Example

Let (x_1, x_2, t) be coordinates on $\mathbb{R}^3$ and let

$$\mathcal{H} = \frac{\partial^2}{\partial t^2} + t^2\,\triangle_x + \frac{\partial^2}{\partial \theta^2},$$

where

$$\frac{\partial}{\partial \theta} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}$$

and

$$\triangle_x = (\frac{\partial}{\partial x_1})^2 + (\frac{\partial}{\partial x_2})^2.$$

Note that $\mathcal{H}$ is hypoelliptic with loss of one derivative, see [21]. The characteristic set is given by

$$\Sigma = \{\tau = t = x_1\xi_2 - x_2\xi_1 = 0\}.$$

Observe that $\xi \neq 0$ on Σ, hence Σ is of Baouendi–Goulaouic type. As a consequence Σ is foliated by Treves curves.

7 Treves Curves for $\mathcal{H}$ and Statements of Results

The Treves curves (bicharacteristics) are contained in Σ and are integral curves of the non-degenerate vector field

$$x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + \xi_1\frac{\partial}{\partial \xi_2} - \xi_2\frac{\partial}{\partial \xi_1},$$

which is the Hamilton field of the function $x_1\xi_2 - x_2\xi_1$.

Characteristic points have the form

$$(x,t,\xi,\tau) = (x^0, 0, \xi^0, 0)$$

where

$$x_1^0\xi_2^0 - x_2^0\xi_1^0 = 0.$$

The Treves curve through such a point is given by the equations $t=0$, $\tau=0$ and

$$x_1(s) = x_1^0\cos(s) - x_2^0\sin(s)$$
$$x_2(s) = x_2^0\cos(s) + x_1^0\sin(s)$$
$$\xi_1(s) = \xi_1^0\cos(s) - \xi_2^0\sin(s)$$
$$\xi_2(s) = \xi_2^0\cos(s) + \xi_1^0\sin(s)$$

where $0 \le s \le 2\pi$.

Note that these curves are always compact. When $x^0 \neq 0$, each curve is the intersection of Σ with the torus

$$|x|^2 = |x^0|^2, \quad |\xi|^2 = |\xi^0|^2.$$

When $x^0 = 0$, each curve is a circle in ξ space, centered at 0, with radius equal to $|\xi^0|$.

In particular, the origin is the projection of an infinite number of compact Treves curves.

Given $p, q > 0$ we define

$$\Omega_{p,q} = \{(x,t) \in \mathbb{R}^3 : |x|^2 < p^2, t^2 < q^2\}.$$

We have the following results concerning the operator $\mathcal{H}$.

Theorem 2 *Let $\Omega \subset \mathbb{R}^3$ be open. If Ω intersects the hyperplane $\{t=0\}$, then $\mathcal{H}$ is not analytic hypoelliptic (in the strong sense) on Ω .*

Theorem 3 *The operator $\mathcal{H}$ is globally analytic hypoelliptic on $\Omega_{p,q}$ for every $p, q > 0$.*

Theorem 4 *The operator $\mathcal{H}$ is analytic hypoelliptic at the origin.*

Note that Theorem 2 is consistent with the Treves conjectures. Also observe that Theorem 4 is an immediate consequence of Theorem 3.

Even though the origin is the projection of an infinite number of Treves curves, it is impossible to find a solution that is singular there. That is, the analog of Treves conjecture in the sense of germs, is false.

8 Construction of the Inverse of $\mathcal{H}$

If $u \in \mathcal{S}$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^3$, we define the partial Fourier transform by

$$\hat{u}(\xi, t) = \int_{\mathbb{R}^2} e^{-i<x,\xi>} u(x,t)dx.$$

We will use polar coordinates (ρ, φ) in ξ space, that is

$$\xi_1 = \rho\cos(\varphi), \xi_2 = \rho\sin(\varphi).$$

If we have $u, f \in \mathcal{S}$ such that $\mathcal{H}u = f$, then this is equivalent to

$$\frac{\partial^2 \hat{u}}{\partial\varphi^2}(\rho,\varphi,t) + \frac{\partial^2 \hat{u}}{\partial t^2}(\rho,\varphi,t) - t^2\rho^2\hat{u}(\rho,\varphi,t) = \hat{f}(\rho,\varphi,t).$$

Next we expand $\hat{u}, \hat{f}$ in Fourier series

$$\hat{u}(\rho,\varphi,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(\rho,t)e^{ik\varphi},$$

$$\hat{f}(\rho,\varphi,t) = \sum_{k=-\infty}^{\infty} \hat{f}_k(\rho,t)e^{ik\varphi}.$$

It follows that

$$H_{\rho^2,k^2}\hat{u}_k(\rho,t) = \hat{f}_k(\rho,t),$$

where

$$H_{\rho^2,k^2} = \frac{\partial^2}{\partial t^2} - \rho^2 t^2 - k^2.$$

Note that H_{ρ^2,k^2} has an inverse on $\mathcal{S}(\mathbb{R})$ for all $\rho \neq 0$ and $k \in \mathbb{R}$. Indeed, an explicit integral formula exists for the inverse, see for example [27].

Hence we have

$$\hat{u}_k(\rho,t) = \int_{-\infty}^{\infty} K_{k,\rho}(t,t')\hat{f}_k(\rho,t')dt',$$

where $K_{k,\rho}(t,t')$ is the distribution kernel of the inverse of H_{ρ^2,k^2}. It follows that the solution u is given by

$$u(x,t) =$$

$$\int_0^\infty \int_0^{2\pi} e^{i(x_1\rho\cos(\varphi)+x_2\rho\sin(\varphi))} \sum_{k=-\infty}^{\infty} e^{ik\varphi} \int_{-\infty}^{\infty} K_{k,\rho}(t,t')\hat{f}_k(\rho,t')dt' \frac{\rho d\rho d\varphi}{(2\pi)^2}.$$

We may write $u = \mathcal{E}f$, where we define

$$(\mathcal{E}f)(x,t) = \int_{\mathbb{R}^3} E(x,t;x',t')f(x',t')dx'dt'$$

where the kernel E is determined from above. There are several ways to write E.

One way to write E is in terms of Bessel functions of integral order. Recall that

$$J_k(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi+it\sin(\varphi)}d\varphi$$

satisfies Bessel's equation

$$J_k''(t) + \frac{1}{t}J_k'(t) + (1 - k^2/t^2)J_k(t) = 0.$$

Hence it follows that

$$E(x,t;x',t') =$$

$$\int_0^\infty \sum_{k=-\infty}^{\infty} e^{ik(\theta-\theta')}\overline{J_k(r\rho)}J_k(r'\rho)K_{k,\rho}(t,t')\frac{\rho d\rho}{2\pi}.$$

Note that we have used the notation $x_1 = r\cos(\theta), x_2 = r\sin(\theta)$ with similar notation for x'.

The following formula is useful in obtaining estimates that are uniform in k and ρ :

$$K_{k,\rho}(t,t') = (1/\sqrt{\rho})\int_0^\infty e^{-sk^2/\rho}K_1(s;t\sqrt{\rho},t'\sqrt{\rho})ds.$$

Here K_1 is the inverse for the Hermite operator

$$H_{1,1} = \frac{\partial^2}{\partial t^2} - t^2.$$

It follows from our construction and classical estimates, see for example [27], that we have

Proposition 1 *$\mathcal{H}$ is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ with inverse $\mathcal{E}$.*

An important preliminary observation is the following

Proposition 2 *Let $(x,t;x',t') \in \mathbb{R}^3 \times \mathbb{R}^3$. Assume that either $t \neq t'$ or $|x| \neq |x'|$. Then E is analytic near $(x,t;x',t')$.*

The results of Section 7 follow from these Propositions. Complete details will appear elsewhere.

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