Polynomial Identities in T-prime Algebras [1]

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Abstract: We survey results concerning the polynomial identities satisfied by "important" algebras. We discuss classical and new facts about the polynomial identities satisfied by the matrix algebra of order two, by the Grassmann (or exterior) algebra, and by its tensor square. We give details about two situations that are quite different. First when one considers algebras over a field of characteristic 0, using methods from the theory of representations of the symmetric group, one obtains quite complete descriptions. On the other hand, when the base field is of positive characteristic, the picture is still rather unclear. We discuss ordinary, weak and graded polynomial identities, and give some recent results concerning algebras over infinite fields of positive characteristic.

Key words: Polynomial identities, Relatively free algebras, Specht problem, Graded identities.

One of the most important classes of algebras in the PI theory is that of the T-prime algebras. Their importance was first revealed by the results of Kemer; they are the "building blocks" in the structure theory developed by Kemer for the T-ideals in the free associative algebra, see [22] for a detailed account. Let us recall some of the definitions and conventions used in the sequel.

Let K be a fixed infinite field, all algebras, vector spaces etc. are considered over this field K. Suppose that $X = \{x_1, x_2, \ldots\}$ is an infinite (countable) set, and let $K(X)$ be the free associative algebra freely generated over K by the set X. This means that $K(X)$ is a vector space with a basis consisting of 1 and all (noncommutative) monomials in the variables from X, and the multiplication is induced by the concatenation of monomials. If A is an algebra, the polynomial $f(x_1, \ldots, x_n) \in K(X)$ is a *polynomial identity* (abbreviated as PI), or an identity, for A if $f(a_1, \ldots, a_n) = 0$ in A for every choice of $a_i \in A$. The set of all identities $T(A)$ of A is an ideal of $K(X)$ which is called the T-ideal of A. An easy verification shows that an ideal I of $K(X)$ is T-ideal if and only if it is closed under all endomorphisms of the algebra $K(X)$. The class of all algebras that satisfy the identities of $T(A)$ is the variety of algebras var A generated by A.

The algebra A is called verbally prime (or T-prime) if its T-ideal $T(A)$ is T-prime i.e., prime in the class of all T-ideals of $K(X)$. In other words $T(A)$ is T-prime if for every T-ideals I and J, the inclusion $IJ \subseteq T(A)$ implies $I \subseteq T(A)$ or $J \subseteq T(A)$. The T-ideal $T(A)$ (and the algebra A, and the variety var A) is *semiprime* if $I^2 \subseteq T(A)$ implies $I \subseteq T(A)$ for every T-ideal I. This means that there do not exist nilpotent T-ideals that contain $T(A)$. The quotient algebra $K(X)/T(A)$ is the relatively free algebra in the variety var A. Denote by

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x_i (the same letters) the images of the $x_i \in X$ under the canonical homomor-
phism $K(X) \longrightarrow K(X)/T(A)$. It is easy to show that if $B \in \operatorname{var} A$ and $b_i \in B$
are arbitrary then there exists unique homomorphism $K(X)/T(A) \longrightarrow B$ such
that $x_i \mapsto b_i$ for every i. This justifies the name "relatively free algebra" for
$K(X)/T(A)$.

Now we recall the definition of some important algebras.

Let $M_n(K)$ be the full matrix algebra of order n over K, G the Grassmann (or
exterior) algebra of an infinite dimensional vector space V over K. One chooses
a basis $\{e_i \mid i \in \mathbb{N}\}$ of V such that $e_i e_j = \delta_{ij}$ in G (here $\delta_{ij} = 0$ if $i \neq j$ and
$\delta_{ii} = 1$ is the Kronecker symbol), and then 1 and the monomials $e_{i_1} e_{i_2} \ldots e_{i_k}$,
$i_1 < i_2 < \ldots < i_k$, $k \geq 1$, form a basis of G. In this notation $G = G_0 \oplus G_1$
where G_0 is the centre of G and is spanned by all monomials of even length, i.e.
the monomials such that k is even, and G_1 is spanned by the monomials of odd
length. Set $M_n(G)$ the algebra of the $n \times n$ matrices with entries from G

The algebra $M_{k,l}$ consists of the matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A \in M_k(G_0)$,
$D \in M_l(G_0)$, B and C being matrices $k \times l$ and $l \times k$ respectively, whose entries
belong to G_1.

One of the most important ingredients in the structure theory of T-ideals
developed by Kemer is the following theorem.

Theorem 1 *[22, Chapter 1.3, pp. 21–25] Let char $K = 0$. Then:*

*1. If V is a nontrivial variety, then $V = N_k W$ where N_k is the variety of all
nilpotent of class $\leq k$ algebras, W is the largest semiprime variety contained in
V, and the product of two varieties MN consists of all algebras A having an ideal
I that belongs to N, and whose quotient A/I lies in M.*

*2. The T-ideal I is semiprime if and only if $I = I_1 \cap \ldots \cap I_q$ where the T-ideals
I_j are T-prime.*

*3. The only T-prime ideals are the T-ideals of the algebras $M_n(K)$, $M_n(G)$,
$M_{k,l}$.*

Hence the description of the T-prime ideals is extremely important. In char-
acteristic 0, the theory of Kemer gives such a description in terms of the iden-
tities satisfied by some concrete and "nice" algebras. On the other hand, when
char $K = p > 0$, there is no such good description. The theory is relatively well
developed for finitely generated algebras only. See for example [24] for future ref-
erence. The above T-prime ideals remain T-prime in positive characteristic but
there appear new ones, the so-called irregular T-prime ideals. Their description
is still far from our understanding, and the picture is quite unclear, see [25]. Note
only that every associative algebra over a field of positive characteristic $p > 0$
satisfies some standard identity $s_n(x_1, x_2, \ldots, x_n)$ ([23]). Here the *standard poly-
nomial* s_n is defined as

$$s_n(x_1, x_2, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$$

for σ running over the symmetric group S_n acting on $\{1, 2, \ldots, n\}$, and $(-1)^\sigma$
standing for the sign of the permutation σ. Notice that s_n is multilinear (i.e.

multihomogeneous and linear in every variable), and skew symmetric. Hence every PI algebra over a field of positive characteristic satisfies some identities of a finite dimensional algebra, since s_n is an identity for every algebra of dimension $n-1$. This fact has no analog in characteristic 0. For example, the Grassmann algebra G of an infinite dimensional vector space over a field of characteristic 0 satisfies no standard identities. (The situation in characteristic p is different since G satisfies s_{p+1} and $p+1$ is the minimal degree of a standard identity that holds in G. Therefore G satisfies the identity s_t if and only if $t > p$.)

Let f and g be two polynomials in $K(X)$. Then g follows from f (or is a consequence of f) as a PI if $g \in (f)^T$, the least T-ideal that contains f. Thus g follows from f if and only if in every algebra A where f is a PI, g also is. The system of polynomials $\{f_i \mid i \in I\}$ forms a basis of the T-ideal $T(A)$ if $T(A)$ coincides with the least T-ideal that contains $\{f_i \mid i \in I\}$. Such a basis is called minimal if no polynomial can be excluded from it. Another extremely important result of Kemer states that when char $K = 0$, every nontrivial T-ideal is finitely based i.e. has a finite basis, see [22, Theorem 2.4]. This theorem answers in affirmative the famous *Specht Problem*. The original problem was stated as to whether the T-ideal of any associative algebra over a field of characteristic 0 is finitely based; see below other (equivalent) reformulations of this problem. But one may ask an analogous problem for algebras over any field, or even for Lie, Jordan, alternative algebras etc. The answer is negative in the case of Lie algebras over a field of positive characteristic, as shown in [49] and [13], and positive for finitely generated Jordan algebras under certain restrictions [42], finitely generated alternmative algebras [19], and finite dimensional Lie algebras [20], over fields of characteristic 0.

It is not difficult to show that the positive answer of the Specht problem is equivalent to any one of the following assertions.

- There does not exist an infinite sequence of polynomials $f_1, f_2, \ldots$, in the free associative (Lie, Jordan etc.) algebra such that for every i, f_i does not belong to the T-ideal generated by $\{f_1, f_2, \ldots, f_{i-1}\}$.

- Every strictly ascending chain of T-ideals is finite.

- Every strictly descending chain of varieties is finite.

- Every nonempty set of T-ideals possesses a maximal element.

A variety (or a T-ideal) is Spechtian if every strictly ascending chain of T-ideals that contain the given ideal is finite. One modifies without difficulties the above four assertions in this case.

An important problem in PI theory is the description of the identities satisfied by "important" algebras. Such algebras include the T-prime ones due to evident reasons.

The following notation will be used throughout. If A is an associative algebra then introducing the commutator product $[a, b] = ab - ba$ one obtains a Lie algebra

denoted by A^-, and by means of the circle product $a \circ b = (ab + ba)/2$ one obtains a Jordan algebra denoted by A^+; here $a, b \in A$. We suppose the commutators left-normed i.e., $[a, b, c] = [[a, b], c]$. If $L(X)$ is the Lie subalgebra of $K(X)^-$ generated by X then $L(X)$ is canonically isomorphic to the free Lie algebra freely generated by X over K.

1 Identities in matrix algebras

Here we survey results concerning the polynomial identities satisfied by the matrix algebras $M_n = M_n(K)$ of order n over the field K. The first deep result about the T-ideals of these algebras was the famous Amitsur–Levitzki theorem (1950). It states that the lowest degree of a polynomial identity satisfied by M_n is $2n$, and the standard polynomial $s_{2n}(x_1, x_2, \ldots, x_{2n})$ is an identity for M_n. One can see various proofs of this important result in [14]. We recommend the one given by S. Rosset in [41]. This proof is very elegant, and it requires virtually no knowledge except for some notion of tensor product and Grassmann algebras. And it has one further advantage, it is only two pages long (without omitting details "to the reader"). Probably the most general result about the identities satisfied by M_n is given in the not less famous theorem due to Procesi [35] and Razmyslov [38], see [39] as well, that describes the *trace identities* in M_n. Recall that the algebra of the polynomials with trace is defined, roughly speaking, in the following manner. One lets some (or all) variables appear in traces. Here the trace is a symmetric bilinear function that satisfies all properties of the usual trace of matrices, see the above references for details.

Notably, the only case where there exists a "good" description of the identities satisfied by M_n is the case $n = 2$, and in particular, char $K = 0$. Thus in [37] it was proved that when char $K = 0$ the T-ideal $T(M_2)$ is finitely based. In fact it follows from the proof that a basis of this T-ideal consists of 9 identities. Later in [12] it was shown that the T-ideal of $M_2(K)$, char $K = 0$, is generated by the identities

$$s_4 = s_4(x_1, x_2, x_3, x_4), \qquad h_5 = [[x_1, x_2] \circ [x_3, x_4], x_5].$$

The second identity is the Hall identity. Furthermore, these two identities are independent, that is, neither of them implies the other. The proof of the finite basability of $T(M_2)$ in [37] depended on two facts that are of independent interest and deserve attention. First it was shown in [37] that the identities of the Lie algebra $sl_2 = sl_2(K)$ of all traceless matrices of order two are finitely based. In fact it was shown that these follow from 3 identities. Furthermore, a basis consisting of single identity, namely $[x_1 \circ x_2, x_3]$, of the *weak polynomial identities* for the pair (M_2, sl_2) over a field of characteristic 0, was obtained as well.

Let us recall what a weak identity is. Suppose A is an associative algebra and L is a Lie subalgebra of A^-. The polynomial $f(x_1, x_2, \ldots, x_n) \in K(X)$ is a weak identity for the pair (A, L) if $f(b_1, b_2, \ldots, b_n) = 0$ in A for every $b_i \in L$.

The weak identities of (A, L) form an ideal in $K(X)$ that is closed under Lie substitutions, and is called *weak T-ideal*. One defines analogously varieties of pairs, consequences etc. for weak identities. The weak identities were introduced by Razmyslov (see for example [39]) and are an important tool in the study of identities in associative, Lie and Jordan algebras. See for generalisations and further reference [26, 27]. We note only that sometimes the weak identities are called *identities of representations* of Lie algebras.

The T-ideal of $M_2(K)$ was tightly described in characteristic 0, see for example [36], and [14] for further reference. At least in the case of characteristic 0 it is known almost anything concerning the identities satisfied by M_2. When the base field is of positive characteristic, the situation is less satisfactory but still we know something about the T-ideals $T(M_n)$. Thus for example, in the papers [32, 16, 17] finite bases for the identities of the matrix algebra $M_n(K)$ over a finite field K were described for $n = 2$, 3, 4, respectively. Note that the methods of proof when K is finite differ significantly from those used in characteristic 0. When char $K = 0$ one may consider multilinear polynomial identities, and employing methods of the representation theory of the symmetric and of the general linear group and/or invariant theory one obtains quite complete results. In the case of a finite field neither of these works but group theoretical methods, combinatorics and structure theory of rings work fine. The case of an infinite field K of positive characteristic falls somewhere among these two "extreme" cases. We shall survey the known results in this direction, when the characteristic of the field K is an odd prime p.

In [8, 7] it was developed the invariant theory of the classical groups in a characteristic-free way. The results of [8, 7] yielded various descriptions of the invariants of the T-ideal of M_2, some of them characteristic-independent, see for example [36], [15]. In [45], S. Vasilovsky proved the following remarkable result, using essentially methods of the invariant theory of the orthogonal group.

Theorem 2 *Let K be an infinite field of characteristic $p \neq 2$. Then the identities of the Lie algebra $sl_2(K)$ follow from the identity*

$$v_5 = [x_1, x_2, [x_3, x_4], x_4] + [x_1, x_4, [x_2, x_4], x_3].$$

Furthermore, in [26] it was shown that the weak identities for the pair (M_2, sl_2) follow from the identity $[x_1 \circ x_2, x_3]$ whenever K is an infinite field of characteristic $p \neq 2$. The methods employed in [26] were based again on the invariants of the orthogonal group. See [27] for further generalisations and applications of these methods.

In this way all the "prerequisites" were obtained, and in [28] it was established the following theorem.

Theorem 3 *Let K be an infinite field of characteristic $\neq 2$. Then the polynomial identities for the algebra $M_2(K)$ admit basis consisting of the identities*

$$s_4 = 2([x_1, x_2] \circ [x_3, x_4] - [x_1, x_3] \circ [x_2, x_4] + [x_1, x_4] \circ [x_2, x_3])$$

$$h_5 = [x_1, x_2, x_3] \circ [x_4, x_5] + [x_1, x_2] \circ [x_4, x_5, x_3]$$
$$v_5' = [x_1, x_2, [x_3, x_4]] \circ [x_5, x_6] + [x_1, x_2, [x_3, x_5]] \circ [x_4, x_6]$$
$$+ \ [x_1, x_4, [x_2, x_5]] \circ [x_3, x_6] + [x_1, x_5, [x_2, x_4]] \circ [x_3, x_6]$$
$$r_6 = [x_1, x_2] \circ ([x_3, x_4] \circ [x_5, x_6]) - (1/8)([x_1, [x_3, x_4], [x_5, x_6], x_2]$$
$$- \ [x_1, [x_5, x_6], [x_3, x_4], x_2] - [x_2, [x_3, x_4], x_1, [x_5, x_6]]$$
$$- \ [x_2, [x_5, x_6], x_1, [x_3, x_4]])$$

Furthermore,
  (i) When char $K \geq 7$ the identities s_4 and h_5 form a minimal basis;
  (ii) When char $K = 3$ the identities s_4, h_5 and r_6 are independent.

Note that the form of the identity h_5 here is different from the one defined earlier; an easy verification shows that these two forms are equivalent.

This means that when char $K = 3$ one needs at least three identities for generators of the T-ideal $T(M_2)$. The last fact was established in [18] where it gave a negative answer to a conjecture due to A. Kemer. The conjecture asked whether the infinite dimensional Grassmann algebra G over a field K of characteristic 3 satisfies all identities of $M_2(K)$. (Observe that it does satisfy s_4 and the Hall identity h_5.) Of course there still remain open questions concerning the T-ideal $T(M_2(K))$. Some of them follow.

1. Describe minimal bases of the identities for M_2 in the cases of char $K = 3$ and 5.

2. Describe basis of the identities for M_2 in the case char $K = 2$.

3. Describe the central polynomials for $M_2(K)$ when char $K = p > 2$.

4. Describe the numerical invariants of $T(M_2(K))$ when char $K = p > 2$.

5. Describe the subvarieties of the variety generated by $M_2(K)$ when char $K = p > 2$. In particular, is it true that every proper subvariety lies in the variety $N_k A$ of all nilpotent of class k-by-commutative algebras, for some k? (This is the case in characteristic 0.)

We conjecture that when char $K = 3$ the minimal basis of identities for M_2 should consist of the identities s_4, h_5 and r_6. The case char $K = 5$ should yield the same basis as that when char $K > 5$ (and char $K = 0$). Note that the standard identity s_4 implies the identity v_5 hence all Lie identities of sl_2 are (associative) consequences of s_4. The proof of this fact is quite straightforward but lengthy and technical, see for example [28]. Probably some computer algebra methods would help in obtaining, in the cases of characteristic 3 and 5, minimal bases of the identities of $M_2(K)$.

Concerning the second question it seems difficult to make any guess. The Lie algebra sl_2 in this case is nilpotent (as Lie algebra), and the identity matrix is

traceless as well. We mention that the Lie algebra $M_2(K)^-$ has no finite basis of its identities, see for example [49].

A partial answer to the fourth question is announced in [6], where a finite set of generators of the T-space of the central polynomials for $M_2(K)$ was described (over an infinite field of characteristic $p > 2$; in characteristic 0 the same was done in [33]).

It seems plausible that using results and methods from [45, 26, 27, 28, 6] one may describe the numerical invariants of the T-ideal $T(M_2(K))$ when char $K = p > 2$. Some results in this direction can be found in [36].

A natural question arises. We know quite a lot about the identities satisfied by M_2, what about these of M_n, $n > 2$? Unfortunately it is known very little about them. There arise technical difficulties that seem formidable. But there exist problems of principal nature. It seems that the main one is that for M_n, $n > 2$, one lacks the good structure of the subvarieties of the variety generated by M_2. But it is still unclear which identities in addition to s_6 can participate in a finite basis of $T(M_3)$ even in characteristic 0. It is only known that s_6 and the identity of algebraicity (see [14], Chapter 7) do not generate the T-ideal $T(M_3)$. This means that one needs more identities in order to generate $T(M_3)$.

That is why one is led to consider other types of polynomials and somewhat "weaker" kinds of identities. We already mentioned the Procesi–Razmyslov theorem that describes the trace identities in M_n. Later we consider graded polynomial identities in matrix and other T-prime algebras.

2   Finite bases of identities in concrete algebras

We considered in detail the identities satisfied by the matrix algebra of order two. Here we give a brief account about the identities in some other important algebras such as the upper triangular matrices, the Grassmann algebra and its tensor square. In fact little is known about the concrete identities satisfied by other algebras.

Suppose K is an infinite field. The polynomial identities of the Grassmann algebra follow from the identity $[x_1, x_2, x_3]$ as it was shown in [31] when char $K = 0$. In fact, in [31] it was obtained detailed information about the structure of the T-ideal of G. The identities in G are described in every characteristic and even over finite fields, see [5, 50]. (If the latter are not available, see for very brief account [18].)

Another class of algebras where the polynomial identities are described in detail are the algebras of upper triangular matrices of any order. See for example [14, Chapter 5] for reference and further information.

Concerning the T-prime algebras, in [34] it was obtained an explicit basis of the identities satisfied by the algebra $G \otimes G$ over a field of characteristic 0. It follows from the structure theory of Kemer that this algebra satisfies the same polynomial identities as $M_{1,1}$, see [22, p. 26]. We shall discuss this fact later.

In the case of Lie algebras, we already mentioned the result of Vasilovsky [45] about the identities for sl_2. No idea how the identities even of sl_3 look like. In [51] the Lie T-ideals containing $[[x_1, x_2, x_3], [x_4, x_5, x_6]]$ in characteristic 0 were described. Note that such T-ideals played a crucial role in the paper [49]. See for example [2, 39] for the current situation in the case of Lie algebras. Note that knowing the identities in certain Lie algebras yields a lot of information about the identities in related associative algebras, see various examples in [39].

That is why one is led to study other kinds of identical relations. We already discussed some applications of the weak identities. Another kind of identities are these with involution, trace identities, and most important, graded identities.

3 Graded identities

Let H be an additive abelian group (or semigroup), and A an algebra. Suppose that $A = \oplus_{g \in H} A_g$ is a direct sum of vector subspaces such that $A_g A_h \subseteq A_{g+h}$ for every g, $h \in H$. Then A is H-graded algebra. When $H = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ is the cyclic group of order 2 we call it simply graded algebra, and when H is the cyclic group of order n we speak about n-graded algebras. Let $X = \cup_{g \in H} X_g$ be a disjoint union of infinite sets, then one defines in a standard way H-grading on the free associative algebra $K(X)$. A polynomial $f \in K(X)$ is H-graded identity for the H-graded algebra A if f vanishes whenever one substitutes the variables of X_g for elements of the component A_g of A. The simplest (and one of the most important) example of a graded algebra is the Grassmann algebra $G = G_0 \oplus G_1$. If $X = Y \cup Z, Y \cap Z = \emptyset$ for Y and Z being the even and odd variables, respectively, then the polynomial $[y_1, y_2]$ is a graded identity of G, $z_1 \circ z_2$ is another. Here $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$. The Grassmann algebra is involved in the structure theory of T-ideals in the following way.

Suppose that K is a field of characteristic 0. If $A = A_0 \oplus A_1$ is graded algebra then $G(A) = A_0 \otimes G_0 \oplus A_1 \otimes G_1$ is the *Grassmann hull* of A. It is known that every nontrivial T-ideal coincides with the T-ideal of the Grassmann hull of some finitely generated graded algebra, see [22, Chapter 1], [4]. Furthermore the T-ideal of a finitely generated superalgebra coincides with the T-ideal of some finite dimensional superalgebra. Hence every nontrivial T-ideal equals the T-ideal of some finite dimensional superalgebra, considered as an ordinary algebra. This extremely important result of Kemer (see [22, Chapter 2]) leads directly to the positive solution of the Specht problem in characteristic 0. All this comes to show that graded identities are of interest. That is why they have become object of independent study. A good deal of new results and reference concerning graded (and H-graded) identities can be found in the survey [3].

We mention some important results about various graded identities related to T-prime algebras, and their applications. In [9] the ideals of graded identities of the algebras $M_2(K)$, $M_{1,1}$ and $G \otimes G$ were described when the base field is of

characteristic 0. The gradings on these algebras are the standard ones, namely

$$M_2(K) = A_0 \oplus A_1, \quad A_0 = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\}, \quad A_1 = \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\},$$

for $a, b, c, d \in K$;

$$M_{1,1} = B_0 \oplus B_1, \quad B_0 = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\}, \quad B_1 = \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\},$$

for $a, d \in G_0$, $b, c \in G_1$;

$$G \otimes G = (G_0 \otimes G_0 \oplus G_1 \otimes G_1) \oplus (G_0 \otimes G_1 \oplus G_1 \otimes G_0).$$

Di Vincenzo used these descriptions in order to obtain, in the same paper, a new proof of the coincidence of the T-ideals of $M_{1,1}$ and $G \otimes G$. It was mentioned earlier that this coincidence follows from the theory of Kemer. Another proof of it was given by Regev in [40]. Note that the proof in [9] makes use, to certain extent, of the structure theory of T-ideals, and the proof of Regev in [40] is direct one. In [29] the three algebras $M_2(K)$, $M_{1,1}$, and $G \otimes G$ were considered over infinite fields K of characteristic $p \neq 2$. It was proved that the graded identities of $M_2(K)$ have exactly the same basis as in the case $\operatorname{char} K = 0$, namely the polynomials $[y_1, y_2]$ and $z_1 z_2 z_3 - z_3 z_2 z_1$. Furthermore, constructing appropriate models for the corresponding relatively free graded algebras it was shown in [29] that the graded identities of $M_{1,1}$ admit a basis of two identities. These are $[y_1, y_2]$ and $z_1 z_2 z_3 + z_3 z_2 z_1$. Here and in the sequel y_i and z_i are even, respectively odd variables. Note that these two identities are exactly the same as in characteristic 0. Still further it was proved that the graded identities of $G \otimes G$ follow from the two from the basis of $M_{1,1}$, and if $\operatorname{char} K = p > 2$ then one adds the identity $[y_1^p, z_1]$. Observe that the last is not a graded identity for $M_{1,1}$ since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in M_{1,1}$, the matrix a^p is not central. Hence one obtains once again

Theorem 4 *If* $\operatorname{char} K = 0$ *then* $T(M_{1,1}) = T(G \otimes G)$.

We notice that the proof of this theorem in [29] is completely elementary one. It uses an appropriate explicit construction for the relatively free graded algebra. In addition, one needs a version of the so-called Specht reduction to commutator polynomials that we discuss shortly.

It is well known fact that every T-ideal over an infinite field is generated by its multihomogeneous (i.e. homogeneous in every variable) polynomials. If $\operatorname{char} K = 0$ one may consider multilinear polynomials only. Suppose $|K| = \infty$, and denote as $B(X)$ the subalgebra of $K(X)$ generated by all commutators $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$, $n = 2, 3, \ldots$ Hence $[x_1, x_2, x_1] \in B(X)$ and $[x_1 x_2, x_3] \notin B(X)$.

This means that $B(X)$ is generated as an algebra by all homogeneous elements of the free Lie algebra $L(X)$ whose degrees are at least 2. Another well known

fact states that every T-ideal T is generated by its polynomials from $B(X)$ i.e. by the intersection $T \cap B(X)$, as long as one considers unitary algebras. See for proofs of these basic assertions, for example [14, Chapter 4]. The proof of the first assertion uses the standard Vandermonde argument, while the second is based on the Poincaré, Birkhoff and Witt theorem for the universal enveloping of a Lie algebra. Note that the universal enveloping of $L(X)$ is exactly $K(X)$. The polynomials of $B(X)$ are called *proper* or *commutator* polynomials. Their usage may simplify quite a lot the computations, and may even turn them "realistic". In the case of graded identities one uses somewhat weaker version of this reduction. Namely it can be proved that every ideal of graded identities over an infinite field is generated by its polynomials such that every even variable y_i appears in commutators only.

Furthermore, in order to prove the above theorem one needs some elementary combinatorics. The centre of the algebra $G \otimes G$ equals $G_0 \otimes G_0$. Since one may consider only the graded identities where every even variable appears in commutator, the elements from $G_0 \otimes G_0$ make the commutators vanish. Hence it is sufficient to substitute the variables in a polynomial for elements of $G_1 \otimes G_1$ (for the even variables), and of $G_0 \otimes G_1 \oplus G_1 \otimes G_0$ (for the odd variables). But the elements of $G_0 \otimes G_1$ anticommute, and the same holds for those of $G_1 \otimes G_0$. On the other hand every element of $G_0 \otimes G_1$ commutes with the elements of $G_1 \otimes G_0$. Hence one is led to consider two groups of odd variables such that variables in any group anticommute, and variables of different groups commute. Given the set $\{1, 2, \ldots, n\}$, suppose it being partitioned into two disjoint subsets (colours) A and B. Let $i = (i_1, i_2, \ldots, i_n)$ be a permutation of $(1, 2, \ldots, n)$ and let $1 \leq \alpha < \beta \leq n$. The entries i_α and i_β of i form a *coloured* inversion in i if $i_\alpha > i_\beta$ and either i_α, $i_\beta \in A$ or i_α, $i_\beta \in B$. The coloured sign of i is $(-1)^{s(i)}$ where $s(i)$ is the number of coloured inversions in i. In other words we "forget" the inversions formed by two entries of different colours. The following fact is used to prove the theorem about the coincidence of $T(M_{1,1})$ and $T(G \otimes G)$.

Proposition 5 *Suppose that i is fixed permutation of $(1, 2, \ldots, n)$ and let (A, B) run over all possible 2^n colourings of $(1, 2, \ldots, n)$. Then the coloured sign of i is either always 1, or always -1, or is 1 for 2^{n-1} colourings, and -1 for the rest 2^{n-1} colourings.*

The proof if this proposition consists of an elementary induction on n.

We note that one may use this fact in order to obtain an alternative description of the linear structure of the so-called *meson algebras*, see [21, pp. 115, 264–272].

Modifying accordingly the proof one can establish the coincidence of the T-ideals $T(M_{1,1})$ and $T(G \otimes G)$ over any infinite field K, char $K \neq 2$ when one considers nonunitary algebras.

Lots of open questions remain in this area, especially when the base field is of positive characteristic. Note that the graded identities of $M_2(K)$ when K is finite were described in [30], for all possible gradings. It turns out that there exist, up to graded isomorphism, only two nontrivial gradings, and their graded identities

are different. Here we list some open problems that we think are of importance for the theory.

1. Determine the ordinary identities satisfied by the algebras $M_{1,1}$ and $G \otimes G$ over an infinite field of characteristic $p > 2$. Or (weaker): determine the difference between the T-ideals of these two algebras.

2. Probably the determination of the weak identities for the algebra $M_{1,1}$ would help in finding a basis of the identities for this algebra. (Recall that $f \in K(X)$ is weak identity for $M_{1,1}$ if it vanishes under substitutions of matrices $f_i(e_{11} + e_{22}) + g_i e_{12} + h_i e_{21}$ where $f_i \in G_0$ and $g_i, h_i \in G_1$. Sometimes these are called matrices with supertrace zero.) In a recent paper [11] it was shown that the weak identities of $M_{1,1}$ follow from $[x_1, x_2, x_3] = 0$ and $[x_1, x_2][x_1, x_3][x_1, x_4] = 0$ if char $K = 0$.

3. Another interesting problem related to the algebras considered here is the following. Describe the possible 2-gradings of the algebras M_2, $M_{1,1}$, $G \otimes G$ in terms of the polynomial identities they satisfy. Note that this could help in describing the T-ideal $T(M_{1,1})$ and $T(G \otimes G)$ in positive characteristic.

4. What further information about the identities (ordinary and 2-graded) of $M_{1,1}$ and $G \otimes G$ can be deduced? In this direction, what information the codimension sequences and more important, the Hilbert series of the corresponding relatively free algebras, can yield? Note that computing the graded codimensions and Hilbert series of these relatively free algebras should be a simple technical question since we know linear bases of these algebras [29].

Now let us turn to more general graded identities. The algebra $M_n(K)$ possesses a natural $\mathbb{Z}_n$-grading. It is defined as follows: $M_n(K) = A_0 \oplus A_1 \oplus \ldots \oplus A_{n-1}$ where A_i is the span of the matrix units $\{e_{rs} \mid |r-s| = i\}$, $0 \le i \le n-1$. In [48], a finite basis of the $\mathbb{Z}_n$-graded identities of $M_n(K)$ was described when char $K = 0$, and in [1] this result was extended to an infinite field. The case of the $\mathbb{Z}$-graded identities for $M_n(K)$ (again in characteristic 0) was dealt with in [47]. In [43] the 2-graded identities of the upper triangular matrices of order two were described when char $K = 0$, and in [44], a basis for the $\mathbb{Z}_n$-graded identities in the upper triangular matrices of order n was described as well as some numerical invariants of the respective ideal of n-graded identities were computed.

There are lots of open questions in this direction. Let us state some of them.

1. Determine the $\mathbb{Z}_n$-gradings on the Lie algebra $sl_n(K)$ of the traceless $n \times n$ matrices. Determine in any one of the gradings the graded identities they satisfy. Or weaker: determine the gradings that satisfy different $\mathbb{Z}_n$-identities.

2. The same question(s) about the other classical Lie algebras.

3. It follows from the theory of Kemer that the algebras $M_{k,l} \otimes M_{p,q}$ and $M_{r,s}$, where $r = kp + lq$ and $s = kq + lp$ satisfy the same polynomial identities; the algebras $M_{k,l} \otimes G$ and $M_{k+l}(G)$ also satisfy the same identities (supposing the base field K is of characteristic 0). A direct proof of these results can be found in [40]. What about the coincidence of the above T-ideals when char K is an odd prime and K is infinite? It seems to us that there is no such coincidence in positive characteristic. Note that in a recent paper [10] the authors described the $\mathbb{Z}_n \times \mathbb{Z}_2$-identities of the algebras $M_n(G)$. As a corollary they obtained a new proof of the coincidence $T(M_2(G)) = T(M_{1,1} \otimes G)$ over a field of characteristic 0. Can this result be transferred to infinite fields of positive characteristic? Is it possible to establish the coincidence of the above T-ideals by using graded identities? That is, can one obtain the results of [10] proceeding in the spirit of [9, 29]?

4. Describe the subvarieties of the varieties of graded algebras determined by the above algebras.

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