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General superalgebras of vector type and (γ, δ)-superalgebras[1]

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Abstract: A general superalgebra of vector type is a superalgebra obtained by a certain double process from an associative and commutative algebra A with fixed derivation D and elements λ, μ, ν. We prove that any such a superalgebra is a superalgebra of (γ, δ) type. Conversely, any simple finite dimensional nonassociative (γ, δ) superalgebra with $(\gamma, \delta) \neq (1,1)$ or (-1,0) is isomorphic to a certain general superalgebra of vector type.

Let A be an associative and commutative algebra over a ring of scalars Φ, with fixed nonzero derivation $D \in Der(A)$, and elements $\lambda, \mu, \nu \in A$. Denote by $\bar{A}$ an isomorphic copy of a Φ-module A, with the isomorphism mapping $a \mapsto \bar{a}$. Consider the direct sum of Φ-modules $B = A + \bar{A}$ and define multiplication on it by the rules

$$\begin{aligned} a \cdot b &= ab, \\ a \cdot \bar{b} &= \bar{a} \cdot b = \overline{ab}, \\ \bar{a} \cdot \bar{b} &= \lambda\, ab + \mu\, D(a)b + \nu\, aD(b), \end{aligned}$$

where $a, b \in A$ and ab is the product in A. Define a Z_2-grading on B by setting $B_0 = A,\ B_1 = \bar{A}$; then B becomes a superalgebra, which we will denote by $B(A, D, \lambda, \mu, \nu)$ and call a *general superalgebra of vector type*.

Various partial cases of this construction have been considered before: the superalgebras $B(A, D, 0, 1, -1)$ are just the *Jordan superalgebras of vector type* [4, 5, 7, 8]; the superalgebras $B(A, D, \lambda, 2, 1)$ in case $char\,\Phi = 3$ are *alternative* [9], and in case of arbitrary characteristic are *(-1,1) superalgebras* [9, 10].

Conversely, it was proved in [9] that any simple nontrivial nonassociative alternative superalgebra of dimension more than six is isomorphic to a superalgebra $B(A, D, \lambda, 2, 1)$, with A being a D-simple algebra of characteristic 3. Similarly, any simple nonassociative (-1,1) superalgebra of positive characteristic $p > 3$ is isomorphic to a superalgebra $B(A, D, \lambda, 2, 1)$ [10]. In particular, any simple finite dimensional nonassociative (-1,1) superalgebra always has a positive characteristic and so is isomorphic to $B(A, D, \lambda, 2, 1)$.

In this paper we give a similar characterization for a general superalgebra of vector type $B(A, D, \lambda, \mu, \nu)$ with $\mu \neq \pm\nu$. We first show that any such a superalgebra is a so called (γ, δ) *superalgebra* (see below), and then we prove that, under certain conditions, a simple nonassociative (γ, δ) superalgebra is isomorphic to $B(A, D, \lambda, \mu, \nu)$.

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Let us start with the definitions. Throughout the paper, if otherwise is not stating, the word "(super)algebra" means a (super)algebra over an associative and commutative ring of scalars Φ with $1/6 \in \Phi$.

An algebra A is called a (γ, δ) *algebra* if it satisfies the identities:

$$\begin{aligned} (x,y,z) + \gamma(y,x,z) - \delta(z,x,y) &= 0, \\ (x,y,z) + (y,z,x) + (z,x,y) &= 0, \end{aligned}$$

where $(x,y,z) = (xy)z - x(yz)$ denotes the *associator* of elements x, y, z, and γ, δ are some elements from Φ, satisfying the equality $\gamma^2 - \delta^2 + \delta - 1 = 0$.

These algebras were introduced in 1949 by A.Albert [1] in the study of *2-varieties* of algebras, that is, the varieties in which for any ideal I its square I^2 is again an ideal. Together with alternative algebras, the varieties of (γ, δ) algebras for different γ, δ give all the possible examples of homogeneous 2-varieties of algebras that contain strictly the class of associative algebras.

According to the general definition of a superalgebra in a given homogeneous variety of algebras (see [11]), a superalgebra $R = R_0 + R_1$ is a (γ, δ) *superalgebra* if and only if it satisfies the (super)identities:

$$(x,y,z) + (-1)^{p(x)p(y)}\gamma(y,x,z) - (-1)^{(p(x)+p(y))p(z)}\delta(z,x,y) = 0, \quad (1)$$
$$(x,y,z) + (-1)^{p(x)(p(y)+p(z))}(y,z,x) + (-1)^{(p(x)+p(y))p(z)}(z,x,y) = 0, \quad (2)$$

where $x, y, z \in R_0 \cup R_1$ and $p(r) \in \{0,1\}$ denotes a parity index of a homogeneous element $r : p(r) = i$ if $r \in R_i$.

In the sequel $B = A + M$ will denote a (γ, δ) superalgebra with $A = B_0,\ M = B_1$. Note that A is a (γ, δ) subalgebra of B, and M is a (γ, δ) bimodule over A.

It was proved in [2] that any simple (γ, δ) algebra of characteristic $\neq 2, 3$, with $(\gamma, \delta) \neq$ (1,1),(-1,0), is associative. We will see now that this statement is not true any more in the case of (γ, δ) superalgebras.

Theorem 1 *Any general superalgebra of vector type $B(A, D, \lambda, \mu, \nu)$ with $\mu \neq \pm\nu$ is a (γ, δ) superalgebra for $\gamma = \frac{-\mu^2+\mu\nu-\nu^2}{\mu^2-\nu^2}$, $\delta = \frac{2\mu\nu-\nu^2}{\mu^2-\nu^2}$. This superalgebra is simple if and only if the algebra A is D-simple; and if $D(A)A^2 \neq 0$, then $B(A, D, \lambda, \mu, \nu)$ is not associative.*

Proof. Since $\bar{A}$ is an associative bimodule over A, it suffices to consider only the associators that contain at least two elements from $\bar{A}$. For any $a, b, c \in A$ we have

$$\begin{aligned} (a, \bar{b}, \bar{c}) &= \mu D(a)bc, & (3) \\ (\bar{a}, b, \bar{c}) &= (\mu - \nu)aD(b)c, & (4) \\ (\bar{a}, \bar{b}, c) &= -\nu abD(c), & (5) \\ (\bar{a}, \bar{b}, \bar{c}) &= \mu = \overline{D(a)bc} + (\nu - \mu)\overline{aD(b)c} - \nu\overline{abD(c)}. & (6) \end{aligned}$$

It follows easily from (3)–(6) that the identity (2) holds in $B(A, D, \lambda, \mu, \nu)$. Furthermore, let

$$\gamma = \frac{-\mu^2 + \mu\nu - \nu^2}{\mu^2 - \nu^2}, \quad \delta = \frac{2\mu\nu - \nu^2}{\mu^2 - \nu^2},$$

then the equality $\gamma^2 - \delta^2 + \delta - 1 = 0$ is straightforward, and we have by (3)–(6)

$$\begin{aligned}
(a, \bar{b}, \bar{c}) + \gamma(\bar{b}, a, \bar{c}) - \delta(\bar{c}, a, \bar{b}) &= \mu D(a)bc + \gamma(\mu - \nu)bD(a)c \\
&\quad - \delta(\mu - \nu)cD(a)b \\
&= (\mu + (\gamma - \delta)(\mu - \nu))D(a)bc = 0, \\
(\bar{a}, b, \bar{c}) + \gamma(b, \bar{a}, \bar{c}) - \delta(\bar{c}, \bar{a}, b) &= (\mu - \nu)aD(b)c + \gamma\mu D(b)ac + \delta\nu caD(b) \\
&= (\mu - \nu + \gamma\mu + \delta\nu)aD(b)c = 0, \\
(\bar{a}, \bar{b}, c) - \gamma(\bar{b}, \bar{a}, c) + \delta(c, \bar{a}, \bar{b}) &= - - \nu abD(c) + \gamma\nu baD(c) + \delta\mu D(c)ab \\
&= (-\nu + \gamma\nu + \delta\mu)abD(c) = 0, \\
(\bar{a}, \bar{b}, \bar{c}) - \gamma(\bar{b}, \bar{a}, \bar{c}) + \delta(\bar{c}, \bar{a}, \bar{\bar{b}}) &= \mu\overline{D(a)bc} + (\nu - \mu)\overline{aD(b)c} - \nu\overline{abD(c)} \\
&\quad - \gamma(\mu\overline{D(b)ac} + (\nu - \mu)\overline{bD(a)c} - \nu\overline{baD(c)}) \\
&\quad + \delta = (\mu\overline{D(c)ab} + (\nu - \mu)\overline{cD(a)b} \\
&\quad - \nu\overline{caD(b)}) = 0.
\end{aligned}$$

Therefore, (1) holds in $B(A, D, \lambda, \mu, \nu)$ too, and $B(A, D, \lambda, \mu, \nu)$ is a (γ, δ) superalgebra.

It is clear that for any D-ideal I of A the set $I + \bar{I}$ is an ideal of $B(A, D, \lambda, \mu, \nu)$, so the D - simplicity of A is a necessary condition for the simplicity of $B(A, D, \lambda, \mu, \nu)$. On the other hand, if A is D-simple, then the Jordan superalgebra of vector type $B(A, D, 0, \alpha, -\alpha)$ is simple for any $0 \neq \alpha \in \Phi$ (see [4, 8]). Therefore, the supersymmetrized superalgebra $B(A, D, \lambda, \mu, \nu)^+ \cong B(A, D, 0, \mu - \nu, \nu - \mu)$ is simple, which yields immediately the simplicity of $B(A, D, \lambda, \mu, \nu)$. □

Let now $B = A + M$ be a (γ, δ) superalgebra with $(\gamma, \delta) \neq$ (1,1),(-1,0). (Note that any (1,1) superalgebra is antiisomorphic to a (-1,0) superalgebra.)

Lemma 1 *If B is simple and not associative, then it satisfies the superidentity*

$$\langle\langle x, y\rangle, z\rangle = 0, \tag{7}$$

where x, y, z are homogeneous and $\langle x, y\rangle = xy - (-1)^{p(x)p(y)}yx$.

Proof. Since B is simple and not associative, it coincides with its associator ideal $D(B)$. Therefore, it suffices to prove that the associator ideal of any (γ, δ) superalgebra R satisfies (7). Let $G = G_0 + G_1$ be a Grassmann algebra, consider the *Grassmann envelope* $G(R) = G_0 \otimes R_0 + G_1 \otimes R_1$ of the superalgebra R. The algebra $G(R)$ is an ordinary (γ, δ) algebra, with $\gamma - 2\delta + 1 \neq 0$, so by [3] its associator ideal $D(G(R))$ satisfies the identity $[[x, y], z] = 0$. From here, by standard arguments on Grassmann envelope, we conclude that $D(R)$ satisfies (7). □

The following lemma shows that, in the presence of identity (7), the study of (γ,δ) (super)algebras is reduced to (-1,1) (super)algebras. This fact, in the algebra case, was observed by the author in the beginning of seventies (see [6, Proposition 4]); we used the modification of this fact given in [3, lemma 6].

Lemma 2 *Let B be a (γ,δ) superalgebra that satisfies identity (7). For any $\alpha \in \Phi$ denote by $B(\alpha)$ the superalgebra, obtained from B by introducing the new multiplication*

$$x \cdot_\alpha y = \alpha xy + (1-\alpha)(-1)^{p(x)p(y)} yx.$$

Then, the superalgebra $B' = B(1-\gamma-\delta)$ is a (-1,1) superalgebra, and $B = B'(\beta)$ for $\beta = \frac{1-\gamma+\delta}{3}$.

Proof. Consider the Grassmann envelope $G(B)$, which is an ordinary (γ,δ) algebra. It is easy to check that $G(B)(\alpha) = G(B(\alpha))$ for any $\alpha \in \Phi$. Therefore, by [3, lemma 6], the algebra $G(B') = G(B)(1-\gamma-\delta)$ is a (-1,1) algebra, which proves that B' is a (-1,1) superalgebra. Moreover, by the same lemma we have the equality $(G(B)(1-\gamma-\delta))(\beta) = G(B)$ for $\beta = \frac{1-\gamma+\delta}{3}$, which proves that $B'(\beta) = B$. □

We can give now the description of simple (γ,δ) superalgebras.

Theorem 2 *Let $B = A + M$ be a simple nonassociative (γ,δ) superalgebra of characteristic $\neq 2,3$, with $(\gamma,\delta) \neq$ (1,1),(-1,0). Then $(B,A,A) = (A,B,A) = [A,B] = 0$, and there exist $x_1,\dots,x_n \in M$ such that $M = Ax_1 + \dots + Ax_n$ and the product in M is defined by*

$$ax_i \cdot bx_j = \lambda_{ij} \cdot ab + (-\gamma+\delta)D_{ij}(a)b + (-1-\gamma+\delta)D_{ij}(b)a,\ i,j = 1,\dots,n,$$

where $\lambda_{ij} \in A, D_{ij} = D_{ji} \in Der\, A$. In particular, if $n = 1$ then B is isomorphic to a superalgebra $B(A, D, \lambda, -\gamma+\delta, -1-\gamma+\delta)$, where A is a (unital) commutative and associative D–simple algebra with $0 \neq D \in Der\, A$, $\lambda \in A$.

Proof. Let $\alpha = 1-\gamma-\delta$, $\beta = \frac{1-\gamma+\delta}{3}$, then by lemmas 1 and 2 we have that $B' = B(\alpha)$ is a (-1,1) superalgebra and $B = B'(\beta)$. It is obvious that the two-sided ideals of B and B' are the same; hence B' is simple. Furthermore, since B is not associative, neither is B'. Therefore, by [10], B' has the following properties:

(i) A is a commutative and associative subalgebra of B', and B' is an associative and commutative A-bimodule;

(ii) there exist $x_1,\dots,x_n \in M$ such that $M = Ax_1+\dots+Ax_n$ and the product of odd elements in B' is defined by

$$ax_i \cdot bx_j = \lambda_{ij} \cdot ab + 2D_{ij}(a)b + D_{ij}(b)a,\ i,j = 1,\dots,n,$$

where $\lambda_{ij} \in A, D_{ij} = D_{ji} \in Der A$.

It follows immediately that B also satisfies (i) and the first part of (ii). As for the product of the elements of M in B is concerned, it is given by

$$ax_i \cdot bx_j = (2\beta - 1)\lambda_{ij} \cdot ab + (3\beta - 1)D_{ij}(a)b + (3\beta - 2)D_{ij}(b)a, \ i, j = 1, \ldots, n.$$

The theorem now is obvious. □

Corollary 1 *Let $B = A + M$ be a simple nonassociative (γ, δ) superalgebra of characteristic $\neq 2, 3$, with $(\gamma, \delta) \neq$ (1,1),(-1,0). Assume that one of the following conditions is satisfied:*

(i) B is of positive characteristic;

(ii) B is finite dimensional;

(iii) A is a polynomial algebra on a finite number of variables;

(iv) A is a local algebra.

Then B is isomorphic to $B(A, D, \lambda, -\gamma + \delta, -1 - \gamma + \delta)$.

The proof follows easily from [10] in view of the fact that the condition $n = 1$ in the theorem is satisfied by B if and only if it is satisfied by the (-1,1) superalgebra B'. □

As in the case of (-1,1) superalgebras [10], we could not find any example of a simple nonassociative (γ, δ) superalgebra which would not be isomorphic to a superalgebra of the type $B(A, D, \lambda, \mu, \nu)$. So it is still an open question whether such superalgebras exist. Notice that in case a new simple (γ, δ) superalgebra B exists, its attached superalgebra B^+ would give a new example of a simple Jordan superalgebra.

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