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The First Lyapunov Method for Strongly Non-linear Systems of Differential Equations [1]

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Abstract: The article is aimed to give a brief review of works published by authors during at least last 10 years and devoted to the construction of solutions of systems of ordinary differential equations in a neighbourhood of a non-elementary critical point. It is assumed that those solutions have non-exponential asymptotics. The main idea of the proposed technique is closely connected with the so-called first Lyapunov method. On the first stage one should cut the original system of equations in an appropriate way, then find a particular solution of the obtained cut system and, finally, complete it up to a particular solution of the entire system by means of series. The authors show how the above scenario works for different classes of dynamical objects.

Key words: strongly non-linear systems, quasi-homogeneous cut, normal forms, formal invariant manifolds, asymptotic expansions.

1°. We will consider a certain class of dynamical systems which can be described by means of a smooth vector field $v(x)$ and which has an equilibrium at the origin $x = 0$

$$\dot{x} = v(x),\ x \in \mathbf{R}^n,\ v(0) = 0 \tag{1}$$

Using the first Lyapunov method, one can explicitly construct families of solutions of (1) entering the equilibrium position $x = 0$ as $t \to +\infty$ or $t \to -\infty$ in a form of series. The behavior of the above trajectories of dynamical systems contains a lot of important information about the structure of the phase portrait of the system in a small neighborhood of $x = 0$. In particular, the existence of trajectories entering the equilibrium position as $t \to -\infty$ implies instability of the latter equilibrium.

Let $\Lambda = \dfrac{\partial v}{\partial x}(0)$ be the Jacobian of the vector field $v(x)$ evaluated at the equilibrium. Let us assume that the characteristic equation

$$\det(\Lambda - \lambda I) = 0 \tag{2}$$

possesses p roots $\lambda_1, \ldots, \lambda_p$ with negative (positive) real parts. Then system (1)

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has a p-parametric family of solutions going to the origin as $t \to +\infty$ ($t \to -\infty$). Those solutions can be expanded into the following series [1]

$$x(t) = \sum_{j_1,\ldots,j_p=0}^{+\infty} x_{j_1,\ldots,j_p}(t) \exp((j_1\lambda_1 + \ldots + j_p\lambda_p)t) \tag{3}$$

Here $j_1 + \ldots + j_p \geq 1$ and coefficients $x_{j_1,\ldots,j_p}$ are polynomials in t and depend on p arbitrary parameters which have to be small enough to ensure convergence of series (3). The first partial sum of (3) ($j_1 + \ldots + j_p = 1$) is obviously a linear combination of particular solutions of the 'truncated' linear system

$$\dot{x} = \Lambda x \tag{4}$$

It is however worth noticing that in a general case formula (3) represents complex solutions of system (1). To construct real solutions, one needs more complicated and refined formulae.

The first Lyapunov method in its classical 'quasi-linear' setting actually consists of three main steps:

a) to simplify the original system by neglecting some terms and obtaining a cut system;

b) to construct a particular solution or a family of particular solutions of the cut system;

b) to build the above solutions of the cut system up to the particular solutions of the entire system in a form of series.

Before explaining the notion of strongly non-linear systems and advanced studying of the subject, let us consider a couple of examples on the classical first Lyapunov method. If the characteristic equation (2) has roots which do not lie on the imaginary axis, system (1) has particular solutions with specific asymptotic properties. Hence, it is very useful to know whether there are such roots of equation (2) or not without calculating them. We formulate now several corollaries which are almost trivial in the quasi-linear case.

Example I. There is a hypothesis suggested by V.Ten [2]. Let $x = 0$ be an isolated equilibrium position, the dimension of the phase space n is odd and the vector field under consideration has an invariant measure with a smooth density $\rho(x)(\mathrm{div}(\rho v) = 0, \rho > 0)$. Under those assumptions the origin $x = 0$ is unstable 'in the future' and 'in the past'. It is worth mentioning that the above hypothesis is not true for infinitely differentiable vector fields and invariant measures (there is a counter-example). But for the analytic case it seems to be true.

If $\det \Lambda \neq 0$, the origin $x = 0$ is isolated. In this case we can simply prove that $\mathrm{tr}\Lambda = 0$ and equation (2) must have roots with positive and negative real parts.

Example II. Let us consider a gradient system

$$\dot{x} = \frac{\partial \varphi}{\partial x}(x) \tag{5}$$

with a harmonic potential: $\Delta\varphi(x) \equiv 0$ for which $x = 0$ is a critical point.

Then the equilibrium position of (5) $x = 0$ is unstable in the future and in the past. Let us first notice that if we expand the potential $\varphi(x)$ into the Maclaurin series with respect to homogeneous forms of x $\varphi(x) = \varphi_2(x) + \varphi_3(x) + \ldots$, all of those forms admit both positive and negative values since they are also harmonic ($\Delta\varphi_k(x) \equiv 0, k = 2, 3, \ldots$). Presenting the quadratic part of the potential as $\varphi_2(x) = \frac{1}{2}(\Lambda x, x), \Lambda^T = \Lambda$ ($(\cdot,\cdot)$ is a standard scalar product in $\mathbf{R}^n$), we can simply prove that $\mathrm{tr}\Lambda = 0$, and if $\Lambda \neq 0$, there are positive and negative roots of equation (2).

2°. Let us pass now to the main subject. A system is usually said to be strongly non-linear [3,4] if the behavior of trajectories in a small neighborhood of an equilibrium position cannot be determined by using only the linear approximation. According to this terminology, systems with non-exponential asymptotic trajectories, i.e. those going to the equilibrium position as $t \to +\infty$ or $t \to -\infty$ but not exponentially with respect to time, may be called strongly non-linear ones. On the other hand, the phase portrait of a smooth system of differential equations in a small neighborhood of an equilibrium is topologically conjugate to the phase portrait of its linearization, if there are no roots of the characteristic equation with zero real parts (the Grobman-Hartmann theorem). Therefore, systems for which the characteristic equation has zero or purely imaginary roots can be also called strongly non-linear ones.

Let us first concentrate our attention on the 'supercritical' case when Λ is a nilpotent matrix. The central notion we introduce in this section is the notion of semiquasi-homogeneous vector fields.

Definition I. Let $S = \mathrm{diag}(s_1, \ldots, s_2)$ be a diagonal matrix with positive integer elements. Let us denote the following diagonal matrix $\lambda^S = \mathrm{diag}(\lambda^{s_1}, \ldots, \lambda^{s_n})$. Let further m be a positive integer number, $m \neq 1$. We say that vector field $v(x) = v_m(x)$ is quasi-homogeneous of degree m with exponents $s_1, \ldots, s_n$ if for any $x \in \mathbf{R}^n, \lambda > 0$ the following equality holds

$$v_m(\lambda^S x) = \lambda^{S+(m-1)I} v_m(x) \tag{6}$$

It is useful to note that if $v(x)$ is a quasi-homogeneous vector field, system (1) is invariant under the transformation $t \mapsto \mu^{-1}t$, $x \mapsto \mu^G x$ where $G = \alpha S$, $\alpha = 1/(m-1)$. Hence, (6) can be rewritten as

$$v_m(\mu^G x) = \mu^{G+I} v_m(x) \tag{7}$$

Equality (7) can be treated as a generalized definition of quasi-homogeneous vector fields for an arbitrary real matrix G with eigen-values in the right half-plane where μ^G is meant as the following matrix exponential $\mu^G = \exp(G \log \mu)$. In this case the degree m plays a formal role.

Definition II. A vector field $v(x)$ is said to be a semiquasi-homogeneous one if it can be presented as a formal sum of quasi-homogeneous polynomials $v(x) = \sum_{k=0}^{\infty} v_{m+k}(x)$ so that

$$v_{m+k}(\mu^G x) = \mu^{G+(1+\alpha k)I} v_{m+k}(x) \tag{8}$$

In principle α can be any real positive number.

Any smooth vector field with a 'trivial' linear part can be represented in a semiquasi-homogeneous form. This can be done by means of Newton's polyhedron technique. And one should bear in mind that such a representation is not unique. Let us consider two simple examples.

Example III. The following system of equations

$$\dot{x}_1 = x_2^2, \quad \dot{x}_2 = x_1^3$$

is quasi-homogeneous of degree $m = 3$ with exponents $s_1 = 2$, $s_2 = 1$. On the other hand, it is semi-homogeneous, i.e. semiquasi-homogeneous for equal $s_1 = s_2 = 1$ and $m = 2$. In this case the cut system reads as follows

$$\dot{x}_1 = x_2^2, \quad \dot{x}_2 = 0$$

Example IV. The system of equations

$$\dot{x}_1 = (x_1^2 + x_2^2)(ax_1 - bx_2), \quad \dot{x}_2 = (x_1^2 + x_2^2)(ax_2 + bx_1)$$

is of course homogeneous of degree $m = 3$. On the other hand, it is quasi-homogeneous with respect to the structure associated with a non-diagonal matrix

$$\begin{pmatrix} 1/2 & \delta \\ -\delta & 1/2 \end{pmatrix}$$

where δ is a free parameter.

Furthermore, any system of the form

$$\dot{x}_1 = (\rho + \psi(\rho))(ax_1 - bx_2), \quad \dot{x}_2 = (\rho + \psi(\rho))(ax_2 + bx_1)$$

where $\rho = x_1^2 + x_2^2$ and $\psi(\rho) = o(\rho)$ as $\rho \to \infty$, is semiquasi-homogeneous.

Systems of equations for which the matrix of the linearization is nilpotent can be studied using the program of the first Lyapunov method described in Section 1:

a) to represent an original system of equations as a semiquasi-homogeneous one and to cut it up to the quasi-homogeneous system corresponding to the introduced structure;

b) to construct a particular solution of the quasi-homogeneous cut system in the form

$$x_{(0)}(t) = (\pm t)^{-G}c, \quad c \in \mathbf{R}^n; \tag{9}$$

to build the particular solution (9) of the cut system up to the particular solutions of the entire system in the form

$$x(t) = (\pm t)^{-G} \sum_{k=0}^{+\infty} x_k(\log(\pm t))(\pm t)^{-\alpha k}, \quad x_0 = c, \tag{10}$$

where x_k are polynomial vector functions. Here the sign '+' is used if we are interested in asymptotics of the solution of the system under consideration (1) as $t \to +\infty$ and '−' if $t \to -\infty$. Below we confine ourselves mostly to the case $t \to +\infty$ if the opposite is not assumed.

Let us consequently consider those three steps.

In fact, the first step has been already described. Using Newton's polyhedron technique [5], we can obtain a quasi-homogeneous cut system

$$\dot{x} = v_m(x) \tag{11}$$

The second step consists of finding a real vector c in (9). Obviously, that vector has to satisfy the following equality

$$v_m(x) = \mp Gc \tag{12}$$

The problem of finding the above vector looks like the linear algebraic problem of eigen vectors. In the linear case such a vector can be found explicitly. In the non-linear situation it is not so. However, the next proposition shows that under quite general restrictions such a vector c really exists.

Lemma I. Let $v_m(x) \neq 0$ for any $x \neq 0$. Then

a) if the index of the vector field $G^{-1}v_m(x)$ is even (for example, if $v_m(x)$ is a homogeneous vector field of even degree m), there exist both 'positive' and 'negative' eigen-vectors.

b) if the dimension of the phase space n is odd, there exists either a 'positive' eigen vector or a 'negative' one.

We will use the last statement discussing later the hypothesis on instability for vector fields with an invariant measure formulated previously.

The proof of the above lemma can be found in [3,4].

If $v(x)$ is a gradient vector field with a harmonic potential φ, for which $x = 0$ is a degenerate critical point, we can fulfill those two steps in the following way. Let the Maclaurin expansion of $\varphi(x)$ start at order $m+1$, $m \geq 2$, $\varphi(x) = \varphi_{m+1}(x) + \varphi_{m+2}(x) + \ldots$.

The cut system reads as follows.

$$\dot{x} = \frac{\partial \varphi_{m+1}}{\partial x}(x) \tag{13}$$

That homogeneous form $\varphi_{m+1}(x)$ is also harmonic and can admit both negative and positive values. System (13) possesses a rectilinear particular solution

$$x_{(0)}(t) = ct^{-\alpha}, \ \ \alpha = \frac{1}{m-1} \tag{14}$$

which enters the equilibrium position $x = 0$ as $t \to +\infty$.

The vector c satisfies the equality $c = |c|e$ where e provides the homogeneous form $\varphi_{m+1}(x)$ with the absolute minimum $-a$ on the unit sphere S^{n-1} and $|c| = \left(\dfrac{\alpha}{(m+1)a}\right)^{\alpha}$. In a similar way we can find a solution of (13) entering the origin as $t \to -\infty$. To do so, we have to use a vector which provides $\varphi_{m+1}(x)$ with the absolute maximum on the unit sphere S^{n-1}.

As for the third step, if the desired particular solution of the cut system (11) exists, we can always complete the above solution up to a particular solution of the entire system (1) in a form of series. To completely construct series (10), we should perform the following recurrent procedure. We consider only the case with the sign '+'. Let us first do the exponential change of time $\tau = \log t$. Then by substituting series (10) into system of equations (1), we obtain an infinite chain of linear differential equations with constant coefficients and polynomial right-hand sides.

$$\frac{dx_k}{d\tau} - K_k x_k = \phi_k(\tau) \tag{15}$$

Here ϕ_k, $k = 1, 2, \ldots$ are, in fact, polynomial functions of 'previous' coefficients $x_0, \ldots, x_{k-1}$, and $K_k = \alpha k I + K$, where K is the so-called Kovalevsky matrix

$$K = G + \frac{\partial v_m}{\partial x}(c) \tag{16}$$

System (15) always has a polynomial particular solution. Hence, series (10) can be completely constructed.

As for convergence of series (10), in general, we can only affirm that there is always a particular solution of (1), infinitely smooth on an interval $[T, +\infty)$, for which (10) is an asymptotic expansion [3,4,6]. The proof of the above fact is based on a version of the abstract implicit function theorem. Nevertheless, if the vector field $v(x)$ is analytic and if -1 is the only eigen value of the Kovalevsky matrix of the type $-\alpha k$, $k = 1, 2, \dots$ the above series converges on an interval $[T, +\infty)$ and we can explicitly construct an infinitely sheeted Riemann surface on which the corresponding particular solution of (1) is holomorphic [6]. Precisely speaking, that means that the desired solution can be obtained in the form of series

$$x(t) = t^{-G} \sum_{k=0}^{+\infty} y_k s^k, \quad y_0 = c, \tag{17}$$

which converges on a small complex disk $s \leq s_0$, where $s = s(t)$ is a function inverse to the function $t(s) = s^{1-m} - \sigma\alpha^{-1} \log s$, σ is a certain real parameter.

It is worth noticing that -1 is always an eigen value of the Kovalevsky matrix K in accordance with the lemma from [7].

So, the following result holds.

Theorem I. If the cut system (11) has a particular solution (9), then the entire system (1) has a particular solution $x(t) \to 0$ as $t \to +\infty$ or $t \to -\infty$ for which (10) is an asymptotic expansion.

Let us consider now several examples.

Example V. We can partially prove the hypothesis mentioned in Section 1.

Theorem II. Let the dimension of the phase space n of system (1) be odd and $v(x)$ be an analytic vector field with a nilpotent linear part possessing an invariant measure with a smooth density. If there exists a quasi- homogeneous structure such that the origin $x = 0$ is the only critical point of the cut vector field $v_m(x)$, the equilibrium position $x = 0$ of the entire system (1) is unstable both 'in the future' and 'in the past'.

Using lemma I, we can simply prove the above theorem. Indeed, according to the lemma I, there exists either 'positive' or 'negative' eigen vector of problem (12). Consequently, there is a particular solution $x(t)$ of the entire system (1) going to the origin $x = 0$ either as $t \to +\infty$ or as $t \to -\infty$. That means that $x = 0$ is unstable either 'in the future' or 'in the past'. But since system (1) possesses an invariant measure, instability 'in the future' results in instability 'in the past' and vice versa instability 'in the past' results in instability 'in the future'.

Example VI. Let us now consider a gradient system (5) with a harmonic potential. The following statement holds.

Theorem III. The equilibrium position $x = 0$ of every gradient system with a harmonic non-constant potential $\varphi(x)$ is unstable both 'in the future' and 'in the past'.

Proof. Let us consider the Maclaurin expansion of the potential $\varphi(x) = \varphi_2(x) + \dots$. If that expansion starts at a second order form, the statement can be proved only by means of the linear approximation, as was shown in Section 1. That is why, we confine ourselves only to the case when the Maclaurin expansion starts at a form of order $m+1$, $m \geq 2$. Then the above expansion reads $\varphi(x) = \varphi_{m+1}(x) + \dots$.

As we have already shown, the cut system (13) possesses two rectilinear solutions going to zero as $t \to +\infty$ and as $t \to -\infty$ which can be completed to particular solutions of the entire system (5) with the same asymptotic properties which leads to both instabilities.

Example VII. A particular case of the inversion of the Lagrange-Dirichlet theorem on stability of an equilibrium position of a mechanical system and the Earnshaw theorem on instability of a point charge in an electrostatic field.

Let us consider a mechanical system described by a Hamiltonian system of differential equations

$$\dot{q} = \frac{\partial H}{\partial p}(q,p), \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p), \quad (q,p) \in \mathbf{R}^n \times \mathbf{R}^n, \tag{18}$$

where $H(q,p)$ is an analytic function of the kind $H(q,p) = \frac{1}{2}(K(q)p,p) + U(q)$.

Here $K(q)$ is a positive definite symmetric matrix of coefficients of the kinetic energy of the system (without loss of generality we can assume that $K(0) = I$, $U(q)$ is the potential energy of the system. If $q = 0$ is a critical point of the potential energy, $q = p = 0$ is an equilibrium position of the system. The Lagrange-Dirichlet theorem [1] states that the above equilibrium position is stable if $q = 0$ provides $U(q)$ with a strict minimum. The following question arises. Is the trivial equilibrium unstable if $U(q)$ does not have a minimum at the point $q = 0$? The positive answer was quite recently given by V.Palamodov [8] who completely proved the above statement. We give a simple proof of a weaker theorem.

Theorem IV. Let us consider the Maclaurin expansion of the potential energy at the equilibrium position $U(q) = U_{m+1}(q) + \ldots$, $m \geq 1$. Then if the first nontrivial form $U_{m+1}(q)$ does not have a minimum, the equilibrium position $q = p = 0$ is unstable [4,9].

It is worth noticing that via the time-reversibility of equations (18) the above instability is 'two-sided'.

Proof. The case $m = 1$ can be simply studied by means of the linear approximation. Let the Maclaurin expansion of $U(q)$ start at order greater than two. Then introducing a quasi-homogeneous structure by means of the following positive definite diagonal matrix $G = \mathrm{diag}(2\alpha, \ldots, 2\alpha, (m+1)\alpha, \ldots, (m+1)\alpha)$, $\alpha = 1/(m-1)$ we can obtain a cut system

$$\dot{q} = p, \quad \dot{p} = -\frac{\partial U_{m+1}}{\partial q}(q), \tag{19}$$

for the entire system (18).

If conditions of theorem IV hold, the quasi-homogeneous cut system (19) admits a particular solution $q_{(0)}(t) = ct^{-2\alpha}$, $p_{(0)}(t) = -2\alpha c t^{-(m+1)\alpha}$, where the vector $c \in \mathbf{R}^n$ is parallel to a unit vector e providing $U_{m+1}(q)$ with a minimum $-a$, $a > 0$ on the unit sphere S_{n-1}. The length of the vector c can be calculated as $|c| = \left(\frac{\alpha}{(m+1)a}\right)^{\alpha}$. Therefore, the entire system (18) has a particular solution

$(q(t), p(t)) \to (0, 0)$ as $t \to +\infty$. This leads to instability 'in the past' and consequently 'in the future'.

The equations of motion of a point charge in an electrostatic field also have the form of (18) where $n = 3$, $K(q) = I$, and $U(q)$ is a harmonic function. All the forms in the expansion of the potential $U(q)$ into the Maclaurin series are alternating. Hence, to prove instability (the Earnshaw theorem), we can simply refer to theorem IV.

Example VIII. At the end of this Section we consider an extension of the Lyapunov criterion on instability of a trivial equilibrium position of a system of differential equations with a nilpotent linear part [3,4]. Lyapunov studied only the case $n = 2$. We consider the case of a higher co-dimension when $n > 2$.

Let us write the system of equations for that critical case as follows

$$\dot{x}_i = x_{i+1} + \ldots, \quad i = 1, \ldots, n-1; \quad \dot{x}_n = ax_1^2 + \ldots, \tag{20}$$

where the dots present non-linear terms for which the monomial ax_1^2 in the last equation is singled out.

Those dots show the manner how system (20) can be truncated. In fact, the cut system is quasi-homogeneous with respect to the diagonal matrix $G = \mathrm{diag}(n, n+1, \ldots, 2n-1)$.

That cut system has a particular solution $x_{(0)i}(t) = c_i t^{-n+i-1}$, $i = 1, \ldots, n$ if $a \neq 0$. Here $c_i = (-1)^{n+i-1} \dfrac{(2n-1)!(n+i-2)!}{a((n-1)!)^2}$. That solution can be completed to a particular solution of the entire system (20) entering the equilibrium position as $t \to +\infty$. This means instability 'in the past'. Analogously, a solution of (20) which enters the equilibrium position as $t \to -\infty$ can also be constructed. And instability 'in the future' also takes place. This holds if $n \geq 2$.

If, nevertheless, $a = 0$, the initial system of equations should be rewritten as follows

$$\dot{x}_i = x_{i+1} + \ldots, \quad i = 1, \ldots, n-2; \quad \dot{x}_{n-1} = x_n + bx_1^2 + \ldots, \quad x_n = 2cx_1x_2 + \ldots \tag{21}$$

If we neglect all the non-linear terms denoted by dots, we obtain a quasi-homogeneous system of equations associated with the matrix $G = \mathrm{diag}(n-1, n, \ldots, 2n-2)$ (here we assume that $n \geq 3$). That system has a particular solution $x_{(0)i}(t) = c_i t^{-n+i-2}$, $i = 1, \ldots, n$ if $b + c \neq 0$, $c_i = (-1)^{n+i} \dfrac{(2n-3)!(n+i-3)!}{(b+c)((n-2)!)^2}$, $i = 1, \ldots, n-1$, $c_n = c \left(\dfrac{(2n-3)!}{(b+c)(n-2)!} \right)^2$. As previously, that solution can be built up to a particular solution of the entire system (21), which goes to the origin as $t \to +\infty$. In this case the origin is unstable 'in the past'. On the other hand, it is also possible to construct a particular solution of system (21) entering the origin as $t \to -\infty$ which results in instability 'in the future'.

3°. Let us pass to the case when the characteristic equation (2) has non-zero roots. We mainly confine ourselves to the case of purely imaginary roots of equation (2). The matrix Λ of the linear approximation to system (1) can be expressed as a sum $\Lambda = D + J$, where D is diagonalizable and J is nilpotent. Then (1) reads

$$\dot{x} = Dx + u(x) \tag{22}$$

where $u(x) = Jx + \ldots$, dots represent all the non-linear terms.

The next step is to transform system (22) into a normal form. Let us carry out a formal power transformation $x = y + \sum Y_p(y)$, where $Y_p(y)$ are homogeneous vector forms of order $p \geq 2$. After that system (21) gets the form

$$\dot{y} = Dy + w(y) \tag{23}$$

where $w(y) = Jy + \ldots$.

Let us remind of the following definition. System (23) is said to be written in Poincare's normal form if $\exp(Dt)w(y) = w(\exp(Dt)y)$ (see, for example, [10]). Thus, we can use the following construction. After the following linear non-autonomous bounded invertible transformation $y = \exp(Dt)z$ system (23) becomes

$$\dot{z} = w(z), \tag{24}$$

where formal vector field $w(z)$ has a nilpotent linear part.

Thus, we find ourselves in the situation of the previous section and can perform the whole scheme described above. But though, in general, the eigen vector problem (12) is solvable for diagonal matrices G which the Newton polyhedron method provides us with, in concrete critical cases it is not so and we have to use a more refined technique.

Let there exist $n_0 < n$ independent linear semi-simple fields of symmetry for system (24) $D_j z$, $j = 1, \ldots, n_0$ with diagonalizable matrices D_j. Let also G be a diagonal matrix defining a quasi-homogeneous structure obtained by means of the Newton polyhedron technique such that it commutes with all matrices of linear fields of symmetry ($GD_j = D_jG, \;\; j = 1, \ldots, n_0$). Let the system of equations

$$\dot{z} = w_m(z), \tag{25}$$

be a quasi-homogeneous cut system for (24).

The following statement holds.

Lemma III [4]. The set of matrices $G_\delta = G + \sum_{j=1}^{n_0} \delta_j D_j$, where $\delta = (\delta_1, \ldots, \delta_{n_0})$ is a set of arbitrary real parameters, defines a quasi-homogeneous structure under which (25) is quasi-homogeneous and (24) is semiquasi-homogeneous and (25) is the corresponding cut system.

For instance, we can use the matrix $G_\delta = G + \delta D$, $\delta \in \mathbf{R}^n$ as a matrix of the desired structure. Using that approach, we obtain not only solutions of the cut system of the 'ray' type but also solutions looking like curled rays.

Opposite to (9) particular solutions of (25) may have the following form in general

$$z_{(0)}(t) = (\pm t)^{-G_\delta} c = \exp(-\delta \log(\pm t) D)(\pm t)^{-G} c \tag{26}$$

The further construction is the same. Hence, the desired formal particular solution of (23) can be obtained in a form of series

$$y(t) = \exp(Dt - G_\delta \log(\pm t)) \sum_{k=0}^{+\infty} y_k(\log(\pm t))(\pm t)^{-\alpha k}, \ \ y_0 = c \tag{27}$$

However, we should bear in mind that in general a normalizing transformation diverges and since the normalized system (23) is only a formal system of differential equations, series (27) are only formal series. But using a partial normalizing transformation and a kind of the implicit function theorem technique we can prove that a partial sum of (27) of high order approximates a real smooth solution of a partially normalized system which goes to the equilibrium position as $t \to +\infty$ or $t \to -\infty$.

The following general result takes place.

Theorem V. If the cut system (25) has a particular solution (26), then the entire system (22) possesses a particular solution $x(t) \to 0$ as $t \to +\infty$ or $t \to -\infty$.

Example IX. Non-exponential asymptotic solutions of general systems of differential equations with additional 1:1 frequency resonance and non- simple elementary divisors.

Let us consider a 4D system of differential equations for which the characteristic equation has purely imaginary roots $\lambda_{1,3} = \pm\sqrt{-1}\,\omega$, $\lambda_{2,4} = \pm\sqrt{-1}\,\omega$, with a non-diagonalizable matrix of the first approximation.

Stability of the trivial equilibrium position of the system under consideration was investigated in [11].

Let $y = (y_1, y_2, y_3, y_4)$ be a phase vector of the normalized system. The diagonalizable part of the matrix of the linear approximation may be represented as

$$D = \begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{pmatrix} \tag{28}$$

The cut system analogous to (25) reads [11]

$$\begin{aligned} \dot{z}_1 = z_2, \quad \dot{z}_2 = (az_1 - bz_3)(z_1^2 + z_3^2), \\ \dot{z}_3 = z_4, \quad \dot{z}_2 = (bz_1 + az_3)(z_1^2 + z_3^2), \end{aligned} \tag{29}$$

a, b are real parameters of the system. Here the diagonal matrix $G = \text{diag}(1, 2, 1, 2)$ is used to perform the necessary quasi-homogeneous truncation.

Obviously, the matrix G commutes with the matrix D and we can apply the procedure described above. The mentioned algorithm makes us to search for particular solutions of system (29) in the following form

$$\begin{aligned} z_{(0)1}(t) &= t^{-1}(c_1 \cos \delta\omega \log t - c_3 \sin \delta\omega \log t) \\ z_{(0)2}(t) &= t^{-2}(c_2 \cos \delta\omega \log t - c_4 \sin \delta\omega \log t) \\ z_{(0)3}(t) &= t^{-1}(c_1 \sin \delta\omega \log t + c_3 \cos \delta\omega \log t) \\ z_{(0)4}(t) &= t^{-2}(c_2 \sin \delta\omega \log t + c_4 \cos \delta\omega \log t) \end{aligned} \tag{30}$$

Those solutions (30) form a one-parameter family $c_1 = \rho \cos \theta$, $c_2 = \rho(\delta \sin \theta - \cos \theta)$, $c_3 = \rho \sin \theta$, $c_4 = -\rho(\delta \cos \theta + \sin \theta)$.

Here θ is the parameter of the above family and the magnitudes ρ, δ have to be determined. They satisfy the following algebraic system of equations

$$a\rho^2 + (\delta\omega)^2 - 2 = 0, \quad 3\delta\omega - b\rho^2 = 0 \tag{31}$$

Conditions of solvability of system (31) can be presented in a complex form which coincides with conditions to instability found in [11]

$$w \neq -|w|, \quad w = a + \sqrt{-1}\, b \tag{32}$$

If inequality (32) holds, the initial system of equations possesses a one-parameter family of solutions which goes to the equilibrium position as $t \to +\infty$. We can also prove the existence of solutions going to the equilibrium position as $t \to -\infty$.

It is worth pointing out some obstacles we meet while investigating the case when some of roots of the characteristic equations (2) do not lie on the imaginary axis. In this case we can reduce the system under consideration onto a center manifold. After that we can apply the procedure described above. But the problem is that the center manifold may have only finite order of smoothness. That is why, series like (27) remain only formal series and the question of persistence of non-exponential asymptotic solutions requires a more careful analysis.

4°. Now we stop at the situation which can be characterized as singular. This means that we can construct series like (10) but they diverge in most cases even for analytic right-hand sides of the system under consideration. A classical example has been already given by Euler.

Example X. Let us consider the following 2D system of differential equations.

$$\dot{x}_1 = -x_1 + x_2, \quad \dot{x}_2 = -x_2^2 \tag{33}$$

System (33) has a particular solution $x_1(t) = e^{-t} \int_{-\infty}^{t} s^{-1} e^s \, ds$, $x_2(t) = t^{-1} \to (0, 0)$ as $t \to +\infty$ which means, of course, instability. The function $x_1(t)$ can be developed into power series with respect to inverse powers of t. That series takes the form $x_1(t) = \sum_{k=1}^{+\infty}(k-1)!t^{-k}$, and, of course, diverges.

The reason for such a phenomenon can be explained from two different points of view. First, such non-exponential asymptotic solutions lie on a center manifold which in most cases is not analytic. Second, if we insert a small parameter into the system under consideration which corresponds to the quasi-homogeneous scale associated with first non-trivial terms of the above series, the system looses some of derivatives while the small parameter vanishes. Anyway, such a phenomenon is connected with a crucial interaction between variables corresponding to zero and non-zero roots of the characteristic equation. The obtained series are asymptotic series to the desired particular solutions but the direct technique of the implicit function theorem is not applicable here. We have to use a more refined result of A.Kuznetsov [12,13]. Roughly speaking, he managed to prove that if a smooth system of equations possessed a formal solution in the form of series (10) then it had a real smooth solution for which (10) was an asymptotic expansion. That powerful tool fails, however, in the case when the matrix G is non-diagonal. Therefore, it does not work for the case of 'curled' solutions which normally appears if there are some purely imaginary roots of the characteristic equation.

Now let us discuss the first reason for the divergence of series (10) connected with non-analyticity of the center manifold. We can formulate the following statement.

Lemma IV [4]. Let the characteristic equation for system (1) have $n_0 < n$ zero roots. (It is not assumed here that other roots have non-zero real parts. They only have to be unequal to zero.) Then system (1) can be rewritten as follows

$$\dot{y} = Ay + Bz + f(y,z), \quad \dot{z} = Jz + g(y,z), \quad (y,z) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_0}, \tag{34}$$

where y, z are n_1- and n_0- vectors respectively $(n_1 = n - n_0)$, A is a non-degenerate $n_1 \times n_1-$ matrix, J is a nilpotent $n_0 \times n_0-$ matrix, B is a $n_1 \times n_0-$ matrix, the vector-functions f, g denote non-linear terms.

There exists a formal invariant manifold $y = \varphi(z)$ where φ can be presented in a form of formal power series, such that on that manifold system (34) reads

$$\dot{z} = Jz + h(z), \tag{35}$$

$h(z)$ means power series representing a set of non-linear terms.

Now we can apply the procedure described in Section 2 to system (35) and, therefore, construct a formal solution of system (34). The application of Kuznetsov's theorem mentioned above gives us a real solution.

We can also apply that result to systems which are not solved with respect to derivatives

$$F(x, \dot{x}) = 0, \quad x \in \mathbf{R}^n, \quad F(0,0) = 0 \tag{36}$$

Let there be a quasi-homogeneous structure associated with a matrix G. This structure can be lifted to the following quasi-homogeneous structure associated with the block matrix block(G, G+I). It is assumed that after the transformation $x \mapsto \mu^G x$, $\dot{x} \mapsto \mu^{G+I}\dot{x}$ the system can be rewritten as follows $F_m(x,\dot{x}) + O(\mu^\alpha) = 0$, $\alpha = 1/(m-1)$ as $\mu \to 0$. Then system (36) is said to be semiquasi-homogeneous. The cut system

$$F_m(x,\dot{x}) = 0 \tag{37}$$

may loose some derivatives and become a system of differential-algebraic or even algebraic equations.

For simplicity let us consider the case of asymptotic behavior of solutions of system (36) at $+\infty$. This situation corresponds to the sign '+' in (10). We can search for a particular solution of (37) in a 'traditional' form of (9) and then try to continue it up to the series (10). But to find all the coefficients of (10), we have to solve a linear system

$$A\frac{dx_k}{d\tau} + B_k x_k = \phi_k(\tau) \tag{38}$$

on every k-th step instead of system (15).

Here $A = \dfrac{\partial F_m}{\partial \dot{x}}(c, -Gc)$, $B_k = -A(\alpha k I + G) + \dfrac{\partial F_m}{\partial x}(c, -Gc)$. If $\det A \neq 0$, that procedure does not differ from the procedure described previously in Section 2. But in the case of irregularity of the matrix we should examine solvability of systems (38) much more carefully. If nevertheless the whole chain of systems (38) is solvable, we can construct series (10) and apply Kuznetsov's theory.

Example X may be examined from those two different points of view. First, system (33) has a center manifold $x_1 = \varphi(x_2) = -\exp\left(-\dfrac{1}{x_2}\right)\displaystyle\int_0^{x_2}\frac{\exp(1/u)}{u}du = \sum_{k=1}^{\infty}(k-1)!x_2^k$. Hence, the particular solution of the second equation of system (33) $x_2(t) = t^{-1}$ can be lifted up to the particular solution of the whole system as follows $x_1(t) = \varphi(x_2(t))$. Second, by using the following quasi-homogeneous scale $x_1 \mapsto \mu x_1$, $x_2 \mapsto \mu x_2$, $\dot{x}_1 \mapsto \mu^2\dot{x}_1$, $\dot{x}_2 \mapsto \mu^2\dot{x}_2$ and putting $\mu = 0$, we obtain the cut system of differential-algebraic equations

$$x_2 - x_1 = 0, \quad \dot{x}_2 = -x_2^2, \tag{39}$$

possessing a particular solution $x_{(0)1}(t) = x_{(0)2}(t) = t^{-1}$ which can be continued up to the solution of the entire system (33).

As we can see, the derivative $\dot{x}_1$ in (39) is lost. The above construction remains valid if we perturb system (33) as follows

$$\dot{x}_1 = -x_1 + x_2 + f_1(x_1, x_2), \quad \dot{x}_2 = -x_2^2 + x_2 + f_2(x_1, x_2), \tag{40}$$

where f_1 starts at terms of second order while f_2 starts at terms of third order.

Coefficients of the corresponding series for system (40) can be found from the following chain of equations

$$x_{1k} - x_{2k} = \phi_{1k}(\tau), \quad \frac{dx_{2k}}{d\tau} + (1-k)x_{2k} = \phi_{2k}(\tau) \tag{41}$$

System (41) is, of course, solvable for any polynomials $\phi_{1k}(\tau)$, $\phi_{2k}(\tau)$.

A less trivial example of using the above technique is connected with the problem of inversion of the Lagrange-Dirichlet theorem formulated in Section 2.

Example XI.

Let us consider a Hamiltonian system of equations (18) where $U(q)$ does not have a minimum at the origin $q = 0$. As usual, we consider the Maclaurin expansion of the potential energy $U(q) = U_2(q) + U_{m+1}(q) + \ldots$, $m \geq 2$. Let the second variation of the potential energy at the critical point $q = 0$ be positive semi-definite, i.e. the vector q can be presented as follows $q = (q^{(0)}, q^{(1)})$, $q^{(0)} \in \mathbf{R}^{n_0}$, $q^{(1)} \in \mathbf{R}^{n_1}$, $n_0 < n$, $n_1 = n - n_0$, so that $U_2(q^{(0)}, q^{(1)}) = \frac{1}{2}(Aq^{(1)}, q^{(1)})$, where A is a positive definite symmetric $n_1 \times n_1-$ matrix. That matrix can be diagonalized as follows $A \sim \operatorname{diag}\left(\omega_1^2, \ldots, \omega_{n_1}^2\right)$, where $\omega_1, \ldots, \omega_{n_1}$ are frequencies of small vibrations. We also denote the restriction of the form $U_{m+1}(q)$ onto the plane $q^{(1)} = 0$ as $V_{m+1}(q^{(0)})$.

The following statement holds.

Theorem VI [4,14]. If the form V_{m+1} does not have a minimum, the equilibrium position $q = p = 0$ of system (18) is unstable.

Proof. We only outline the main stages of the proclaimed proof. System (18) can be formally treated as a system non-solved with respect to derivatives. Using the following quasi-homogeneous scaling $q \mapsto \mu^{2\alpha} q$, $p \mapsto \mu^{(m+1)\alpha} p$, $\alpha = 1/(m-1)$ and putting $\mu = 0$, we obtain a cut system

$$\dot{q}^{(0)} = p^{(0)}, \quad \dot{q}^{(1)} = p^{(1)}, \quad \dot{p}^{(0)} = -\frac{\partial U_{m+1}}{\partial q^{(0)}}(q^{(0)}, q^{(1)}), \quad Aq^{(1)} = 0 \tag{42}$$

If conditions of theorem VI hold, system (42) has an evident particular solution $q_{(0)}^{(0)}(t) = ct^{-2\alpha}$, $q_{(0)}^{(1)}(t) = 0$, $p_{(0)}^{(0)}(t) = -2\alpha c t^{-(m+1)\alpha}$, $p_{(0)}^{(1)}(t) = 0$, $c \in \mathbf{R}^{n_0}$.The vector c is parallel to a unit vector e providing the form V_{m+1} with a minimum on the unit sphere S^{n_0-1}. It is easy to prove that the system for higher coefficients of the corresponding series is solvable. Applying the cited results of [12,13], we can prove the existence of a particular solution of (18) $(q(t), (p(t)) \to (0,0)$ as $t \to +\infty$. This means instability 'in the past'. Due to the usual time-reversibility of system (18) the trivial equilibrium position is unstable 'in the future'.

5°. We show now a quite unexpectable application of the first Lyapunov method described above for a class of systems of differential equations with delays. Let us consider the following system of functional differential equations

$$\dot{x}(t) = f\left(x(t), x(t-t_1), \ldots, x(t-t_s)\right), \tag{43}$$

such that $x(t) = 0$ is a trivial particular solution ($f(0,0,\ldots,0) = 0$). Here $t_1, \ldots, t_s$ are real positive constants.

Moreover, let us assume that the characteristic equation for (43) has only zero roots. For this very special case we find conditions for instability of the trivial solution by constructing a particular solution of (43) $x(t) \to 0$ as $t \to -\infty$. Since there are no roots with either positive or negative real parts, it is expected that there is a center manifold attracting solutions of (43) superexponentially (Prof. Jack Hale has drawn the authors' attention to that fact, see also [15]). On the other hand, after the reduction onto the above center manifold the reduced finite dimensional system has only zero roots and we can expect that the desired solution possesses power asymptotic properties. But we try to find conditions for the existence of the above particular solutions without appealing to the center manifold arguments.

Let us introduce the following notations $x^{(0)}(t) = x(t)$, $x^{(1)}(t) = x(t-t_1)$, ..., $x^{(s)}(t) = x(t-t_s)$. We say that the system of equations (43) is quasi-homogeneous with respect to the structure associated with a matrix G and denote it as

$$\dot{x}(t) = f_m\left(x(t), x(t-t_1), \ldots, x(t-t_s)\right), \tag{44}$$

if after the following formal transformation $t \mapsto \mu^{-1}t$, $x^{(0)} \mapsto \mu^G x^{(0)}$ $x^{(1)} \mapsto \mu^G x^{(1)}$, ..., $x^{(s)} \mapsto \mu^G x^{(s)}$ it remains invariant.

If after the above transformation it takes the form of formal power series with respect to μ^α $\dot{x}(t) = f_m\left(x(t), x(t-t_1), \ldots, x(t-t_s)\right) + O(\mu^\alpha)$, where $\alpha = 1/(m-1)$, we say that (43) is semiquasi-homogeneous. System (44) is said to be the corresponding quasi-homogeneous cut for the entire system (43).

The problem is that the cut system (44) does not generally have any particular solutions like (9) or (26). That is why we have to carry out additional simplifications. By putting all the delays equal to zero, we obtain another cut system of ordinary differential equations

$$\dot{x} = g_m(x) = f_m(x, x, \ldots, x) \tag{45}$$

Roughly speaking, it is possible to prove that the entire system of FDE (43) inherits instability properties of the cut system of ODE (45). More precisely, the following result takes place.

Theorem VII [16]. If the cut system (45) has a particular solution (9) with the sign '+' in it, then the entire system (43) has a particular solution $x(t) \to 0$

as $t \to -\infty$ for which (10) with the sign '$-$' is an asymptotic expansion and the trivial solution is unstable.

The formal aspect of the proof is based on a simple fact that if a solution $x(t)$ can be expanded into the series (10), all the shifts $x(t-t_1), \ldots, x(t-t_s)$ can be re-expanded into the series like (10) with the same leading term. To prove that the above series is an asymptotic expansion for a real solution, one should use a kind of the implicit function technique. It is worth noticing that in the case of time delays of an arbitrary sign (for example, for advanced systems) series like (10) can be also formally constructed, but the implicit function theorem is not applicable here and we are not able to say whether they describe a real solution of the system under consideration.

Let us consider an interesting application of theorem VII.

Example XII [16]. Explosive instability in logistics equations.

The following system of equations

$$\dot{N}_i(t) = N_i(t)\left(k_i + \beta_i^{-1}\sum_{j=1}^{n} a_{ij}N_j(t-t_{ij})\right), \quad i = 1, \ldots, n \tag{46}$$

is a very popular model describing different processes in various fields such as biology, ecology, economics etc. For instance, by means of (46) we can describe the interaction of populations in a certain ecosystem, where $N_i(t)$ is the number of individuals in a population of the i-th species at the time t, (a_{ij}) is a constant matrix which, as a rule, is assumed to be skew-symmetric in this problem ($a_{ij} > 0$ means that the i-th species increases at the cost of the j-th species while in the opposite case the i-th species is reduced in favor of the j-th species), k_i is the difference between the birth rate and the death rate of the i-th species if it is left to its own resources, and the coefficients $\beta_i > 0$ are parameters characterizing the fact that the reproduction of one 'predator' is usually connected with the disappearance of more than one 'prey'. Since the number of individuals in the populations affects the birth rate in each species with a delay, the constants t_{ij} are positive. In reality it makes sense to consider systems of the form (46) only for non-negative N_i, $i = 1, \ldots, n$. The properties of solutions of system (46) were first studied by Volterra without taking delays into account [17], and then he later considered the question of the influence of the delay effect on the birth rate in a population, but in a form somewhat different from that in (46). It should be mentioned that the case $k_i = 0$, $i = 1, \ldots, n$ for system (46) has hardly been studied at all. An exception is formed by a very few papers, for example, [18], where (46) is used under the condition $k_i = 0$, $i = 1, \ldots, n$, $t_{ij} = 0$, $i, j = 1, \ldots n$ to analyze the dynamics of employment in different branches of production. In [18] the authors assumed that the components of the matrix (a_{ij}) satisfy the conditions $a_{ij} > 0$, $i, j = 1, \ldots, n$, $i \neq j$, $a_{ii} < 0$, $i = 1, \ldots, n$. It is obvious that in other important applied problems it is possible to impose different restrictions on the coefficients a_{ij}.

Thus, we are interested in conditions of the existence of positive solutions $N_i(t) \to 0$, $i = 1, \ldots, n$ as $t \to -\infty$ of the following system of equations

$$\dot{N}_i(t) = \beta_i^{-1} N_i(t) \sum_{j=1}^{n} a_{ij} N_j(t - t_{ij}), \quad i = 1, \ldots, n \tag{47}$$

for $t_{ij} > 0$, $i, j = 1, \ldots, n$.

The corresponding cut system of ODE takes the following simple form

$$\dot{N}_i = \beta_i^{-1} N_i \sum_{j=1}^{n} a_{ij} N_j, \quad i = 1, \ldots, n \tag{48}$$

Let us find conditions for the existence of a positive particular solution of (48) of the form $N_{(0)i} = c_i(-t)^{-1}$, $c_i > 0$, $i = 1, \ldots, n$, $t < 0$. It exists iff the following system of linear algebraic equations

$$\sum_{j=1}^{n} a_{ij} c_j = \beta_i, \quad i = 1, \ldots, n \tag{49}$$

has a positive solution.

According to theorem VII this means that system (47) has a positive solution $N_i(t) \to 0$ as $t \to -\infty$. Since equations (47) are invariant under the time shifts $t \mapsto t - T$, the existence of a positive solution of (49) results in instability of the trivial solution of (47). This instability is of explosive type. Solutions stop to exist or go to infinity in a finite time. This means that there exists a regime in the ecosystem under consideration such that all the species are very small in number at an initial moment and then they begin to grow very rapidly.

6°. The last object we consider in this article is invariant curves of analytic maps. The problem can be described as follows. Let us consider a discrete dynamical system

$$x_{(p+1)} = f\left(x_{(p)}\right), \quad p \in \mathbf{Z} \tag{50}$$

associated with a map $f : \mathbf{R}^n \to \mathbf{R}^n$ analytic in a neighborhood of the origin $x = 0$.

Let $x = 0$ be a fixed point of that map and let f be invertible in a small neighborhood of the origin, i.e. $\det \Lambda \neq 0$, $\Lambda = \frac{\partial f}{\partial x}(0)$. To investigate the behavior of system (50) in a neighborhood of a fixed point, it is useful to find locally invariant curves of the above map, i.e. smooth vector functions $x(s)$, $x : (0, s_0) \to \mathbf{R}^n$, s_0 is small enough, $\lim_{s \to +\infty} x(s) = 0$ such that

$$f(x(s)) = x(\lambda(s)), \tag{51}$$

where $\lambda(s)$ is a smooth function, $\lambda : (0, s_0) \to (0, s_0)$ defining a reparametrisation of the curve $x(s)$, $\lim\limits_{s\to+\infty} \lambda(s) = 0$.

Then we can easily construct a trajectory of (50) on that invariant curve

$$x_{(p)} = x(s_{(p)}), \;\; s_{(p+1)} = \lambda(s_{(p)}), \;\; p \in \mathbf{Z} \tag{52}$$

Those curves can be obtained by means of the first Lyapunov method.

Let us first consider the situation when there are roots of the characteristic equation

$$\det(\Lambda - \lambda I) = 0 \tag{53}$$

lying strictly inside or outside of the unit circle which can be referred to non-critical cases.

For simplicity we consider only real roots. The following statement holds.

Theorem VIII [19]. Let there be a real root λ, $0 < |\lambda| < 1$ of the characteristic equation (53) such that for any other root μ of (53) the following non-resonance inequality holds

$$\mu \neq \lambda^k, \;\; k \in \mathbf{N} \setminus \{1\}, \tag{54}$$

then there is an analytic invariant curve of f which can be presented by means of converging power series

$$x(s) = s \sum_{k=0}^{\infty} x_k s^k \tag{55}$$

If, nevertheless, there are some roots μ of (53) for which

$$\mu = \lambda^k \tag{56}$$

then there is a C^{K-1}-curve which can be presented by means of asymptotic series

$$x(s) = s \sum_{k=0}^{\infty} x_k(\log|s|) s^k \tag{57}$$

where K is the minimal natural number for which resonance (56) holds and x_k are polynomials.

In both cases function $\lambda(s)$ has a very simple form $\lambda(s) = \lambda s$ independently of the fact whether (54) or (56) hold, and $x_0 = c$ where c is an eigen vector of the matrix Λ with the eigen value λ.

We omit the proof of the formulated theorem.

The series like (55) or (57) can be constructed also in the complex case ($\lambda \in \mathbf{C}$) but if we are interested in real curves, the corresponding series take more awkward form.

The opposite case $|\lambda| > 1$ can be also studied. To do so, we should simply invert the map f and apply the above theorem.

Let us pass to a more delicate situation. Let us assume that the reduction onto a center manifold has been already performed. That means that all the roots of (53) lie on the unit circle. Of course, that center manifold may have only finite class of smoothness [10] but we consider only a formal aspect of the problem. Precisely, we consider a procedure of constructing formal invariant curves in a form of formal series. For that purpose it is quite enough to assume that f can be expanded into formal Maclaurin series.

We can rewrite the map f as follows

$$f(x) = Dx + \tilde{f}(x), \tag{58}$$

where D is diagonalizable ($D \sim \operatorname{diag}(\exp(\sqrt{-1}\,\rho_1), \ldots, \exp(\sqrt{-1}\,\rho_n))$, $\rho_1, \ldots, \rho_n$ are some real numbers, $\tilde{f}(x) = Jx + \ldots$, J means a nilpotent matrix and dots present the set of non-linear terms.

Then the map in the form (22) should be transformed into a normal form by means of formal power transformation $x = y + \sum Y_p(y)$, where $Y_p(y)$ are homogeneous vector forms of order $p \geq 2$. Then we obtain a new map

$$g(y) = Dy + \tilde{g}(y), \tag{59}$$

$\tilde{g}(y) = Jy + \ldots$, such that for any real positive σ $\tilde{g}(D^\sigma y) = D^\sigma \tilde{g}(y)$

The next step is to consider the map

$$h(z) = z + \tilde{h}(z), \tag{60}$$

where $\tilde{h}(z) = D^{-1}\tilde{g}(z)$. This map can be obtained from (59) by substitution $y = D^{-1}z$.

The linear part of the vector field $\tilde{h}(z)$ is nilpotent, so we can try to represent the above map in a semiquasi-homogeneous form $\tilde{h}(z) = \sum_{k=0}^{\infty} \tilde{h}_{m+k}(z)$ so that $\tilde{h}_{m+k}(\lambda^S z) = \lambda^{S+(m+k-1)I}\tilde{h}_{m+k}(z)$, S is a diagonal matrix obtained by the Newton polyhedron approach. We introduce also the notation $\alpha = 1/(m-1)$.

Let the quasi-homogeneous cut $\tilde{h}_m(z)$ of the vector field $\tilde{h}(z)$ be such that there is a real non-zero vector c satisfying the following non-linear 'eigen vector problem'

$$\tilde{h}_m(c) = -Gc, \tag{61}$$

Theorem IX [20]. Let there be a non-zero real vector c such that (61) holds. Then the formal map (59) has an invariant curve which can be represented in a form of series

$$y(s) = D^{s^{1-m}} s^S \sum_{k=0}^{+\infty} y_k(\log s) s^k, \quad y_0 = c, \quad s > 0 \tag{62}$$

where y_k are polynomial vector functions and the function $\lambda(s)$ in (51) has the form

$$\lambda(s) = s\left(1 + s^{m-1}\right)^{-\alpha} \tag{63}$$

Here coefficients y_k can be also recurrently calculated as particular solutions of linear systems of ordinary differential equations with constant coefficients and polynomial right-hand sides. The question of convergence or asymptoticity of series (62) is less trivial. To prove that a particular sum of (62) is a good approximation for an invariant curve of the partially normalized map, we have to apply also the implicit function theorem technique, but a more refined one than in the case of ODE.

If we consider a formal dynamical system generated by the normalized map (59)

$$y_{(p+1)} = g\left(y_{(p)}\right), \quad p \in \mathbf{Z} \tag{64}$$

then it is quite easy to construct a formal trajectory of (64) going to the trivial fixed point $y = 0$ as $p \to -\infty$.

Indeed, since the reparametrization function $\lambda(s)$ has the quite simple form (63), we write $s_{(p)} = \left(p + \left(s_{(0)}\right)^{1-m}\right)^{-\alpha}$ and by substituting the last expression into (62), we obtain

$$y_{(p)} = D^p p^{-\alpha S} \sum_{k=0}^{+\infty} y_k^*(\log p) p^{-\alpha k}, \quad y_0^* = c, \tag{65}$$

It is also possible to obtain trajectories entering the trivial fixed point as $p \to -\infty$. To do so, we should invert the original map and apply the above results.

Example XII. Let us consider an automorphism of a two-dimensional plane for which the origin $x_1 = x_2 = 0$ is a fixed point and the eigen values of the linear part are equal to $\exp\left(\pm\frac{2\pi}{3}\sqrt{-1}\right)$. The partial normal form of that map reads as follows [11]

$$h(y_1, y_2) = \\ \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} a(y_1^2 - y_2^2) + 2by_1y_2 \\ -2ay_1y_2 + b(y_1^2 - y_2^2) \end{pmatrix} + \ldots, \tag{66}$$

a, b are real parameters.

Using theorem IX, it is possible to prove that if $a^2 + b^2 \neq 0$, the map (66) has an invariant curve

$$\begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi}{3s} & -\sin\frac{2\pi}{3s} \\ \sin\frac{2\pi}{3s} & \cos\frac{2\pi}{3s} \end{pmatrix} \begin{pmatrix} s(y_{01} + o(1)) \\ s(y_{02} + o(1)) \end{pmatrix}, \text{ as } s \to +0, \tag{67}$$

where $y_{01} = r\cos\left(\frac{\theta}{9} - \frac{2\pi}{9}\right)$, $y_{02} = r\sin\left(\frac{\theta}{9} - \frac{2\pi}{9}\right)$, $r = \sqrt{a^2+b^2}$, $\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$.

Asymptotic representation (67) shows that the corresponding trajectory of the dynamical system associated with the map (66) takes the form

$$\begin{pmatrix} y_{1(p)} \\ y_{2(p)} \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi p}{3} & -\sin\frac{2\pi p}{3} \\ \sin\frac{2\pi p}{3} & \cos\frac{2\pi p}{3} \end{pmatrix} \begin{pmatrix} p^{-1}(y_{01} + o(1)) \\ p^{-1}(y_{02} + o(1)) \end{pmatrix} \text{ as } p \to +\infty, \tag{68}$$

We can also prove that the dynamical system under consideration has a trajectory entering the fixed point as $p \to -\infty$ which means instability of the fixed point.

Thus, we have considered several objects providing us with very different types of dynamics. As we can see, main ideas of the first Lyapunov method allow us to study the behavior of trajectories in a neighborhood of a fixed point also in critical cases for all those objects.

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