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Symmetries of Riemann Ellipsoids

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Abstract: The results of Dirichlet, Dedekind and Riemann on 'ellipsoidal figures of equilibrium' of rotating self-gravitating fluids are reviewed in the context of the geometric theory of Hamiltonian systems with symmetry. In particular Riemann's classification is derived using only the existence of physically natural rotational symmetries, and so is shown to be applicable to models of liquid drops, atomic nuclei and elastic bodies as well as self-gravitating fluids. Similarly Dedekind's transposition symmetry is obtained as a simple consequence of the rotational symmetries. A detailed description is given of a generalization of 'self-adjoint' ellipsoids. The symmetry groups of the different types of ellipsoidal figures of equilibrium are also computed.

Key words: Riemann ellipsoids, conserved quantities, momentum maps, affine rigid body, relative equilibrium.

1 Introduction

In the *Principia* Newton used his theory of gravitational attraction to show that an axisymmetric self-gravitating body of fluid that is rotating slowly about its axis of symmetry will be oblate, ie flattened at the poles. This result initiated a chain of mathematical discoveries spanning more than two centuries. We briefly recall some of the highlights, referring the reader to the Introduction of Chandrasekhar's book (1987) for further details.

In 1742 Maclaurin showed that rotating oblate spheroids are possible for a much larger range of angular velocities than the slow rotations considered by Newton. Then in the first half of the 19th century Jacobi (1834) extended Newton's method to show that a self-gravitating fluid can also take on ellipsoidal shapes which rotate about one of the principal axes and have all three principal axes of different lengths.

The spheroidal and ellipsoidal motions of Newton, Maclaurin and Jacobi are all rigid. In a frame rotating with the body the fluid is stationary. A new direction was opened up when Dirichlet (1860) derived equations for a self-gravitating fluid mass that allowed for non-rigid motions. He retained the assumption that the body remained an ellipsoid at all times, but allowed the fluid velocity to be a general linear function of the coordinates. Dirichlet's work was edited for publication by Dedekind who, in an addendum (Dedekind 1860), noted that transposing the matrix giving the fluid velocity of one solution generates another solution. In particular transposing the matrix of one of Jacobi's ellipsoids results in an ellipsoid that is stationary in space, but in which the fluid particles follow elliptical paths in planes orthogonal to a principal axis of the ellipsoid.

Dirichlet's paper inspired Riemann to turn his attention to the problem. He gave a classification of the solutions of Dirichlet's equations for which the ellipsoidal shape of the body remains constant, ie the 'ellipsoidal figures of equilibrium' (Riemann 1860). At the heart of Riemann's work lies the result that the angular velocity and vorticity vectors of such a solution must either (i) be parallel to the same principal axis of the ellipsoid, or (ii) lie in the same principal plane. Ellipsoids satisfying the first condition are said to be of *type S*, while Riemann subdivided those satisfying the second condition into *types I, II* and *III*, distinguished by relationships between the lengths of their principal axes. It seems that Riemann's work in this area was highly regarded by his contemporaries. When Riemann was elected to membership of the Berlin Academy just before he died in 1866, his promoters chose to describe just three aspects of his work, ranking his results on Riemann ellipsoids alongside those on prime numbers (the 'Riemann hypothesis') and shock waves (Monastyrsky 1987).

In the 1960's the earlier work on self-gravitating fluids was consolidated and extended using modern analytical methods in a series of papers by Chandrasekhar and Lebovitz, culminating in the book by Chandrasekhar. Since then similar ideas have been applied to models of other systems, including rotating liquid drops (Chandrasekhar 1965), spinning gas clouds (Dyson 1968), atomic nuclei (Rosensteel 1987) and elastic bodies (Sławianowski 1974, Cohen & Muncaster 1988, Lewis & Simo 1990). In each of these cases there are forces that hold the body together and oppose the centrifugal forces when the body is spinning. They differ in the nature of those forces and how they depend on the deformed state of the body. However, in each case a modification of Dirichlet's model can be used for motions during which the body is always ellipsoidal and has a velocity field which is a linear function of the coordinates. In the case of elastic bodies this model is equivalent to considering only homogeneous deformations of the body.

One of the main aims of this paper is to give an account of Riemann's classification of the possible ellipsoidal figures of equilibrium using the geometric theory of Hamiltonian systems (see eg Marsden and Ratiu 1994). The translation of Dirichlet's model into this setting has been discussed by a number of authors (see particularly the papers of Sławianowski (1974, 1975a, 1975b, 1982, 1988)) and is described here in section 2. In this the space of configurations is the group $GL^+(3)$ of all 3×3 matrices with positive determinant. To some extent the theory can be developed along the same lines as that for rigid bodies, for which the configuration space is taken to be $SO(3)$, the group of all rotations of $\mathbb{R}^3$. For this reason we follow Sławianowski (1974) in referring to such models as *affine rigid bodies*. In the elasticity literature they are also called *pseudo-rigid bodies*.

Strictly speaking we do not make quite the same assumptions as many of the authors who have worked on self-gravitating fluid masses in that we do not assume that the body is incompressible. In fact this makes very little difference to the analysis presented here. The results all go through essentially unchanged if the configuration space $GL^+(3)$ is replaced by $SL(3)$, the group of all 3×3 matrices with determinant equal to 1.

This geometric setting is particularly appropriate for discussing symmetry properties of Hamiltonian systems and their consequences and our account emphasizes these. In particular we show that Riemann's classification holds for any system which is invariant under both rotations of space and of a spherically symmetric reference body (see section 3 for details). This means that the potential energy of the body depends only on its shape, ie the lengths of the principal axes of the ellipsoid. Physically these are very natural symmetries. In elasticity theory they are known as *material frame-indifference* and *isotropy.*

By Noether's theorem these symmetries generate conserved quantities that are described in section 3. Invariance under spatial rotations implies, as usual, that angular momentum is conserved. Invariance under rotations of the reference body generates a conserved quantity that is essentially the fluid dynamical circulation. These quantities have natural symmetry properties. Rotation of an affine rigid body in space rotates the angular momentum vector by the same amount, while rotation of the reference body has a similar effect on the circulation vector. In section 5 we show that the conservation of angular momentum and circulation, together with their symmetry properties, leads directly to Riemann's theorem and the constraints on the lengths of the principal axes that differentiate between the ellipsoids of types *I, II* and *III.* This approach differs slightly from those of Riemann (1860) and Chandrasekhar (1987), both of whom worked from the equations of motion of the system, rather than directly from the conserved quantities.

In section 4 we consider Dedekind's transposition symmetry. This looks completely unphysical, for example it maps Jacobi ellipsoids to the physically very different Dedekind ellipsoids. Remarkably, however, a very simple argument shows that it must always be present whenever the system is invariant under rotations of both space and the reference body. Certain types of motions can be distinguished by their behaviour under transposition. A motion that is mapped to itself by transposition is said to be *self-adjoint.* More generally a motion that is mapped to another that can also be obtained by applying some combination of rotations of space and the reference body to the first, we define to be *symmetric-adjoint.* In section 5.2 we show that self-adjoint ellipsoids lie on the boundary of the space of ellipsoids of type I while another type of symmetric-adjoint solution, the *skew-adjoint* ellipsoids, are on the boundary of the spaces of ellipsoids of types *II* and *III.* This distinction between self-adjoint and skew-adjoint ellipsoids is new and clarifies a number of remarks made by Chandrasekhar (1987).

The final section of this paper is devoted to determining the *symmetry groups* of the ellipsoidal figures of equilibrium. The symmetry group of a trajectory of a dynamical system with symmetry is the set of symmetry operations which leave the system invariant and map that trajectory to itself. Thus the symmetry group of a self-adjoint motion of an affine rigid body will contain the transposition operator. The symmetry group of a relative equilibrium strongly influences its stability, possible bifurcations and nearby dynamics (Golubitsky & Stewart 1987, Montaldi *et al* 1988, Marsden & Ratiu 1994). For example the symmetry groups of members of a bifurcating family of relative equilibria must be subgroups of the symmetry group of the relative equilibrium they are bifurcating from.

In sections 6.1 and 6.2 we compute all possible symmetry groups of motions of affine rigid bodies and in section 6.3 we match these symmetry types to Riemann's classification of equilibrium ellipsoids and draw some consequences concerning the possible symmetry changes that may occur in bifurcations. All the results of section 6 are new. The effects of the symmetry groups on the stability and nearby dynamics of relative equilibria will be investigated elsewhere.

2 Affine Rigid Bodies

2.1 Geometric Formulation

The configuration of an affine rigid body at time t is an orientation preserving linear transformation of $\mathbb{R}^3$ denoted by $Q(t)$. Physically we think of the body as being the image under $Q(t)$ of a fixed reference body $\mathcal{B}$. The space of configurations is the group, $GL^+(3)$, of all linear transformations of $\mathbb{R}^3$ which are orientation preserving (ie have positive determinant). We will identify these transformations with their matrices with respect to some fixed basis of $\mathbb{R}^3$. A motion of the body is a family of linear transformations, smoothly parametrized by time, giving a curve in $GL^+(3)$. We will use 'X' to denote a coordinate system in the reference body $\mathcal{B}$ and 'x' a coordinate system for $\mathbb{R}^3$. Thus X labels the 'particles' making up the body while x gives their positions in space. The position of particle X at time t is given by $x(t) = Q(t)X$.

Any configuration Q of an affine rigid body can be decomposed as a product:

$$Q = R^T AS$$

where R and S are orthogonal and A is diagonal. It follows that a unit sphere in the reference body $\mathcal{B}$, centred at the origin, is mapped to an ellipsoid in space with principal axis half-lengths equal to the eigenvalues of A.

Examples: Jacobi and Dedekind Ellipsoids

Define $Q(t) = R(t)^T A(t)S(t)$ by:

$$\begin{aligned} S(t) &= I \\ A(t) &= diag(a_1, a_2, a_3) \\ R(t) &= \begin{pmatrix} \cos\omega_L t & \sin\omega_L t & 0 \\ -\sin\omega_L t & \cos\omega_L t & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where I denotes the identity matrix and $diag(a_1, a_2, a_3)$ the diagonal matrix with the constants a_1, a_2 and a_3 on the diagonal. If the reference body $\mathcal{B}$ is the unit sphere then $Q(t)\mathcal{B}$ is an ellipsoid, with principal axes of lengths $2a_1, 2a_2$ and $2a_3$, rotating rigidly about a principal axis which is parallel to the z-axis in space. The particles in the body describe circular paths about the z-axis. Motions of this form were first investigated by Jacobi in the context of self-gravitating fluids (Jacobi 1834) and are known as *Jacobi ellipsoids.*

We can similarly define an affine rigid body motion $Q(t)$ by:

$$\begin{aligned} S(t) &= \begin{pmatrix} \cos\omega_R t & \sin\omega_R t & 0 \\ -\sin\omega_R t & \cos\omega_R t & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ A(t) &= diag(a_1, a_2, a_3) \\ R(t) &= I. \end{aligned}$$

In this case $Q(t)$ maps the unit sphere to an ellipsoid which remains stationary in space. However the body has an 'internal' motion with the particles describing elliptical paths about the z-axis. The existence of motions of this form for self gravitating fluids was first noted by Dedekind (using the 'transposition' described below in section 4). They are called *Dedekind ellipsoids.*

The velocity at time t of an affine rigid body with motion $Q(t)$ is the tangent, $\dot{Q}(t)$, to the curve at that point. The velocity of particle X is $\dot{x}(t) = \dot{Q}(t)X$. The kinetic energy of the body at time t is given by:

$$\begin{aligned} K(t) &= \frac{1}{2}\int_{\mathcal{B}} |\dot{Q}(t)X|^2 \rho(X)dX \\ &= \frac{1}{2}tr(\dot{Q}(t)\mathcal{E}_0\dot{Q}(t)^T) \end{aligned} \tag{1}$$

where the superscript T denotes the transposed matrix, $tr(M)$ is the trace of the matrix M, $\rho(X)$ is the density function of $\mathcal{B}$ and $\mathcal{E}_0$ is the inertia dyadic of the reference body:

$$\mathcal{E}_0 = \int_{\mathcal{B}} XX^T \rho(X)dX.$$

Note that $\mathcal{E}_0$ is related to the usual inertia tensor $\mathcal{I}_0$ of $\mathcal{B}$ by:

$$\mathcal{I}_0 = tr(\mathcal{E}_0)I - \mathcal{E}_0.$$

If $\mathcal{B}$ is spherically symmetric then $\mathcal{E}_0$ and $\mathcal{I}_0$ are scalar matrices, $\mathcal{E}_0 = \mu I$ and $\mathcal{I}_0 = 2\mu I$ for some $\mu > 0$. In general the momentum of the body at time t is given by:

$$P(t) = \dot{Q}(t)\mathcal{E}_0$$

and so the kinetic energy can also be written as:

$$K(t) = \frac{1}{2} tr(P(t)\mathcal{E}_0^{-1} P(t)^T).$$

We assume that the potential energy, V, is a function only of the configuration, $V = V(Q)$. Different applications will have different potential energy functions. In this paper, however, we will be concerned only with results that are independent of the precise form of the function.

In Hamiltonian mechanics the state of an affine rigid body at time t is described by the pair $(Q(t), P(t))$, a point in the cotangent bundle $T^*GL^+(3)$. The cotangent bundle can be identified with the tangent bundle using the pairing:

$$\alpha(v) = \langle \alpha, v \rangle = tr(\alpha^T v) \tag{2}$$

for α in $T^*GL^+(3)$ and v in $TGL^+(3)$.

The group $GL^+(3)$ acts on itself by both a left and a right action:

$$\begin{aligned} L_g Q &= gQ \\ R_g Q &= Qg \end{aligned}$$

for g and Q in $GL^+(3)$. These induce actions on $T^*GL^+(3)$:

$$T^*L_g(Q, P) = (g^{-1}Q, g^T P) \tag{3}$$

$$T^*R_g(Q, P) = (Qg^{-1}, Pg^T). \tag{4}$$

where the pairing (2) has been used to identify $T^*GL^+(3)$ with $TGL^+(3)$.

Let $gl(3)$ denote the Lie algebra of $GL^+(3)$, identified with $T_I GL^+(3)$. Let $gl(3)^*$ denote the dual of $gl(3)$, identified with $T_I^* GL^+(3)$. The left and right actions of $GL^+(3)$ on $T^*GL^+(3)$ each give an identification of this space with $GL^+(3) \times gl(3)^*$. The identification given by the left action defines *body coordinates* on $T^*GL^+(3)$ while the right action gives *space coordinates*.

In this paper we will use body coordinates when they are convenient. In this coordinate system the left and right actions of $GL^+(3)$ on the Hamiltonian phase space $T^*GL^+(3)$ are:

$$\lambda_g^*(Q, \Pi) = (g^{-1}Q, \Pi) \tag{5}$$

$$\rho_g^*(Q, \Pi) = (Qg^{-1}, g^{-T}\Pi g^T) \tag{6}$$

for g and Q in $GL^+(3)$ and Π in $gl(3)^*$. Here and elsewhere in this paper g^{-T} is short for $(g^{-1})^T$.

3 Spherical Symmetry and Conserved Quantities

For many applications of affine rigid bodies it is reasonable to assume that the system is invariant under orientation preserving orthogonal transformations of $\mathbb{R}^3$. This is equivalent to the invariance of the Hamiltonian under the restriction to the special orthogonal group $SO(3)$ of the left action of $GL^+(3)$ on $T^*GL^+(3)$. For notational convenience later on we make elements, g, of $SO(3)$ act on the left of $GL(3)$ via their transposes.

Symmetries of the reference body $\mathcal{B}$ may also be inherited by the system through the potential energy function. In this paper we will assume that $\mathcal{B}$ is spherically symmetric and that the Hamiltonian is invariant under the action of $SO(3)$ on $T^*GL^+(3)$ obtained by restricting the right action of $GL^+(3)$.

A system which is invariant under both left and right actions of $SO(3)$ is invariant under an action of the product group $SO(3) \times SO(3)$ which by equations (3) and (4) is given by:

$$(g,h).(Q,P) = (gQh^T, gPh^T). \tag{7}$$

In body coordinates this action is obtained restricting the actions (5) and (6) to $SO(3)$:

$$(g,h).(Q,\Pi) = (gQh^T, h\Pi h^T) \tag{8}$$

If the Hamiltonian H is invariant under either the left or right $SO(3)$ action then, by Noether's Theorem, there is a corresponding conserved quantity. In this section we calculate these conserved quantities and show that they are the angular momentum and circulation of the body. Note that the invariance of the kinetic energy under both actions is clear from equation (1). However the requirement that the potential energy V is invariant under one or both of the actions imposes restrictions on the form it can take. Assuming that it is invariant under both actions is equivalent to assuming that the potential energy of a configuration depends only on its 'shape', ie the lengths of the principal axes of the corresponding ellipsoid.

The results in this section are not new - indeed the conservation of angular momentum and circulation is discussed in Dirichlet (1860). The material on angular velocities in subsection 3.3 is adapted from Chandrasekhar (1987).

3.1 Momentum Maps

Let $so(3)$ denote the Lie algebra of $SO(3)$, identified with the space of skew-symmetric matrices. Its dual space $so(3)^*$ can be identified with $so(3)$ through the pairing on $so(3)$ given by:

$$\langle \xi, \nu \rangle = tr(\xi^T \nu). \tag{9}$$

For any ξ in $so(3)$ we define the infinitesimal generators for the left and right actions, denoted ξ_L and ξ_R, to be the vector fields on $GL^+(3)$ given by:

$$\begin{aligned} \xi_L(Q) &= \tfrac{d}{ds}(e^{s\xi}Q)\big|_{s=0} &= \xi Q \\ \xi_R(Q) &= \tfrac{d}{ds}(Qe^{-s\xi})\big|_{s=0} &= -Q\xi. \end{aligned}$$

Note that, as in equation (7), an element h of $SO(3)$ acts on the right of $GL^+(3)$ by multiplication by h^T.

The momentum maps, J_L and J_R, for the left and right actions of $SO(3)$ on $T^*GL^+(3)$ are maps:

$$J_{L,R} : T^*GL^+(3) \to so(3)^*$$

defined by:

$$\begin{aligned} \langle J_L(Q,P), \xi\rangle &= \langle P, \xi_L(Q)\rangle \\ \langle J_R(Q,P), \xi\rangle &= \langle P, \xi_R(Q)\rangle \end{aligned}$$

(see, for example, (Marsden 1992 section 2.7)). Thus we get:

$$\begin{aligned} \langle J_L(Q,P), \xi\rangle &= tr(P^T\xi Q) \\ &= tr(\frac{1}{2}(QP^T - PQ^T)\xi) \end{aligned} \tag{10}$$

and so:

$$J_L(Q,P) = \frac{1}{2}(PQ^T - QP^T). \tag{11}$$

Note that in equation (10) the anti-symmetric part of QP^T is taken to ensure that J_L lies in $so(3)^*$. A similar calculation shows that:

$$J_R(Q,P) = \frac{1}{2}(P^TQ - Q^TP). \tag{12}$$

If the potential energy function V (and hence also the Hamiltonian H) is invariant under the left action of $SO(3)$ then J_L is constant along the solutions of Hamilton's equations. Similarly if V is invariant under the right action then J_R is constant.

The momentum map J_L commutes with the left action of $SO(3)$ on $T^*GL^+(3)$ and its coadjoint action on $so(3)^*$, but is invariant under the right action. Similarly J_R commutes with the right action and coadjoint action, and is invariant under the left action. The momentum map for the action of the product group $SO(3) \times SO(3)$ is just the product of the momentum maps for the left and right actions:

$$J = (J_L, J_R) : T^*GL^+(3) \to so(3)^* \times so(3)^*.$$

This is equivariant with respect to the action given by equation (7) on phase space and the action:

$$(g,h).(\xi,\zeta) = (g\xi g^T, h\zeta h^T) \tag{13}$$

on the Lie algebra dual $so(3)^* \times so(3)^*$.

3.2 Angular Momentum and Circulation

First recall that we can identify skew-symmetric matrices with vectors in $\mathbb{R}^3$ by the mapping:

$$v = (v_1, v_2, v_3) \mapsto \hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \tag{14}$$

The following results are obtained by straightforward calculations.

Lemma 3.1

1. For any vectors v and w in $\mathbb{R}^3$:

$$\begin{aligned} \hat{v}w &= v \times w \\ \hat{v}\hat{w} - \hat{w}\hat{v} &= \widehat{v \times w} \end{aligned}$$

2. For any invertible matrix L and vector v:

$$\widehat{Lv} = (detL)L^{-T}\hat{v}L^{-1}.$$

3. If L is a symmetric matrix and v a vector:

$$L\hat{v} + \hat{v}L = (tr\widehat{(L)I - L})v$$

□

The total angular momentum of an affine rigid body with motion $Q(t)$ is the vector:

$$\mathcal{A} = \int_{\mathcal{B}} (QX \times \dot{Q}X)\rho(X)dX \tag{15}$$

This can be calculated as follows:

$$\begin{aligned} \widehat{\mathcal{A}} &= \dot{Q}\mathcal{E}_0 Q^T - Q\mathcal{E}_0\dot{Q}^T \\ &= PQ^T - QP^T \\ &= 2J_L(Q, P). \end{aligned} \tag{16}$$

Thus the conserved quantity generated by invariance of the Hamiltonian under the left action of $SO(3)$ is the angular momentum of the body.

Let γ be a closed curve in the reference body $\mathcal{B}$. Then $Q(t)\gamma$ is a closed curve in the body which 'moves with the fluid'. Let l be the arc-length parameter for γ and l' that for $Q\gamma$. Then $dl' = Qdl$ and the circulation of the velocity field $\dot{Q}X$ around $Q\gamma$ is given by:

$$\begin{aligned} \mathcal{C}_\gamma &= \int_{Q\gamma} \dot{Q}X.dl' \\ &= \int_\gamma Q^T\dot{Q}X.dl. \end{aligned} \tag{17}$$

Lemma 3.2 *If γ is the boundary of a surface S in $\mathcal{B}$ then for any matrix L:*

$$\int_\gamma LX.dl = \int_\gamma L^A X.dl$$

where $L^A = \frac{1}{2}(L - L^T)$.

Proof: Write $L = L^S + L^A$ where $L^S = \frac{1}{2}(L + L^T)$. Then:

$$\int_\gamma LX.dl = \int_\gamma L^S X.dl + \int_\gamma L^A X.dl.$$

By Stokes Theorem:

$$\int_\gamma L^S X.dl = \int_S (\nabla \times (L^S X)).dS = 0$$

since $\nabla \times (L^S X) = 0$ for a symmetric matrix L^S.

□

It follows from equation (17) and the lemma that:

$$\mathcal{C}_\gamma = \int_\gamma \frac{1}{2}(Q^T \dot{Q} - \dot{Q}^T Q) X.dl$$

Let $\mathcal{C}$ denote the vector defined by:

$$\widehat{\mathcal{C}} = \frac{1}{2}(Q^T \dot{Q} - \dot{Q}^T Q). \tag{18}$$

Then:

$$\mathcal{C}_\gamma \;=\; \int_\gamma \widehat{\mathcal{C}} X.dl \;=\; \int_\gamma (\mathcal{C} \times X).dl \;=\; \mathcal{C} \int_\gamma X \times dl.$$

Note that the vector $n_\gamma = \int_\gamma X \times dl$ depends only on γ and not on the motion. The vector $\mathcal{C}$ is Kelvin's circulation vector for the motion, see (Landau & Lifshitz 1959 page 14). If $\mathcal{B}$ is spherically symmetric then $\mathcal{E}_0 = \mu I$, $\dot{Q} = \frac{1}{\mu} P$ and equations (12) and (18) give $\widehat{\mathcal{C}} = -\frac{1}{\mu} J_R$.

We summarize the results of this subsection as follows:

Proposition 3.3

1. For any affine rigid body:

$$J_L(Q, P) = \frac{1}{2} \widehat{\mathcal{A}}$$

2. If $\mathcal{B}$ is spherically symmetric:

$$J_R(Q, P) = -\mu \widehat{\mathcal{C}}$$

□

It follows that if the potential energy function is invariant under spatial rotations then its angular momentum is conserved, while if it is invariant under rotations of a spherically symmetric reference body then Kelvin's circulation vector is conserved.

3.3 Angular Velocities

Let $Q(t)$ be the motion of an affine rigid body given by:

$$Q(t) = R(t)^T A(t) S(t) \tag{19}$$

where $R(t)$ and $S(t)$ are smooth families of orthogonal matrices and $A(t)$ is a smooth family of diagonal matrices. We define two moving coordinate frames, in addition to the coordinate system X in the body and the inertial system x, by:

$$\begin{aligned} Y(t) &= S(t)X \\ y(t) &= R(t)x(t). \end{aligned}$$

We refer to these as the moving reference frame and the moving space frame, respectively. The mappings between the four coordinate systems are summarized in figure 1.

moving reference frame　　　　moving space frame

$$\begin{tikzcd} Y \arrow[r, "A"'] & y \\ X \arrow[u, "S"] \arrow[r, "Q"] & x \arrow[u, "R"'] \end{tikzcd}$$

reference frame　　　　inertial frame

Figure 1: Coordinate systems for an affine rigid body.

The moving reference and space frames rotate rigidly with respect to the reference and inertial frames, respectively. Moreover the transition from the moving reference frame to the moving space frame is obtained by simply stretching the coordinate axes. Thus an affine rigid body is a coupled system of two rigid bodies and a three degree of freedom system of 'shape oscillations'. The coordinate axes of the moving space frame are aligned with the principal axes of the ellipsoidal image of the unit sphere in the reference frame and so $R(t)$ denotes the rotation of the principal axes with respect to the inertial frame while $S(t)$ can be regarded as their motion with respect to the fixed reference frame. Note that the axes of the moving reference frame (resp. moving space frame) are parallel to the principal axes of Q^TQ (resp. QQ^T).

The instantaneous angular velocities ω_L and ω_R of the moving reference and space frames with respect to the fixed reference and inertial frames, respectively, are defined by:

$$\begin{aligned}\hat{\omega}_L(t) &= \Omega_L(t) = \dot{R}(t)^T R(t)\\ \hat{\omega}_R(t) &= \Omega_R(t) = \dot{S}(t)^T S(t)\end{aligned}$$

where the hat denotes the isomorphism between vectors in $\mathbb{R}^3$ and skew-symmetric matrices defined by equation (14). These angular velocities are related to the angular momentum $\mathcal{A}$ and circulation $\mathcal{C}$, and hence to J_L and J_R, through the inertia tensor of the body. For any configuration Q of an affine rigid body define the inertia dyadic $\mathcal{E}$ and inertia tensor $\mathcal{I}$ by:

$$\begin{aligned}\mathcal{E} &= \int_{\mathcal{B}} (QX)(QX)^T \rho(X)dX\\ \mathcal{I} &= \int_{\mathcal{B}} (\widehat{QX})(\widehat{QX})^T \rho(X)dX.\end{aligned}$$

These are related to each other by:

$$\mathcal{I} = tr(\mathcal{E})I - \mathcal{E}$$

and to the inertia of the reference body by:

$$\mathcal{E} = Q\mathcal{E}_0 Q^T.$$

Note that if $\mathcal{B}$ is spherically symmetric and $\mathcal{E}_0 = \mu I$ then $\mathcal{E} = \mu QQ^T$ and $\mathcal{I} = \mu(tr(QQ^T)I - QQ^T)$.

In the next proposition the notation $[A, B]$ is used for the commutator of A and B, that is $[A, B] = AB - BA$. We also use vectors j_L and j_R defined by $J_L = \hat{j}_L$ and $J_R = \hat{j}_R$.

Proposition 3.4

1. $\widehat{\mathcal{A}} = \Omega_L\mathcal{E} + \mathcal{E}\Omega_L - Q(\Omega_R\mathcal{E}_0 + \mathcal{E}_0\Omega_R)Q^T + [R^T\dot{A}AR, \mathcal{E}]$

2. $-2\widehat{\mathcal{C}} = \Omega_R Q^T Q + Q^T Q\Omega_R - 2Q^T\Omega_L Q$

If $\mathcal{B}$ is spherically symmetric and $\mathcal{E}_0 = \mu I$ then:

3. $j_L = \frac{1}{2}\mathcal{A} = \frac{1}{2}\mathcal{I}\omega_L - \mu(detQ)Q^{-T}\omega_R$

4. $j_R = -\mu\mathcal{C} = \frac{1}{2}Q^{-1}\mathcal{I}Q\omega_R - \mu(detQ)Q^{-1}\omega_L$

Proof: The results in 1 and 2 are obtained by substituting $Q = R^TAS$ and $\dot{Q} = \dot{R}^TAS + R^T\dot{A}S + R^TA\dot{S}$ in equation (16) and equation (18) for $\mathcal{A}$ and $\mathcal{C}$ respectively. For 3 a calculation shows that $[R^T\dot{A}AR, \mathcal{E}] = [S^T\dot{A}AS, \mathcal{E}_0]$ and so is zero if $\mathcal{E}_0$ is a scalar matrix. Equation 3 then follows from 1 and the identities of Lemma 3.1. Equation 4 follows similarly from 2 and Lemma 3.1.

□

Remarks

1. Montaldi (1995) has noted that the existence of the smooth decomposition (19) for solutions of the equations of motion is not obvious, but that for analytic Hamiltonians it follows from a result of Kato (1984 Chapter II, section 6). For more general smooth Hamiltonians it will also exist for the relative equilibria considered in section 5.

2. Even if $\mathcal{B}$ is not spherically symmetric the commutator in the equation for $\widehat{\mathcal{A}}$ vanishes if either $\dot{A} = 0$ or $\mathcal{E}$ is a scalar matrix. Note that $\dot{A}$ is identically zero if the 'shape' of the body is not changing, ie the lengths of the principal axes are constant.

3. In the moving space frame the path of a particle at position X in the reference body is given by $y(t) = R(t)x(t)$, and so its velocity is given by:

$$\begin{aligned} \dot{y} &= \dot{R}x + R\dot{x} \\ &= -R\Omega_L R^T y + R\dot{Q}S^T A^{-1} y \\ &= (\dot{A}A^{-1} - A\widetilde{\Omega}_R A^{-1})y. \end{aligned} \tag{20}$$

where $\dot{Q} = \dot{R}^T AS + R^T \dot{A}S + R^T A\dot{S}$ has been used to obtain the third equality and $\widetilde{\Omega}_R = S\Omega_R S^T$. Define $\widetilde{\omega}_R$ to be the vector corresponding to $\widetilde{\Omega}_R$ under the mapping (14). In the moving space frame the vorticity, denoted ξ, is given by:

$$\xi = \nabla \times \dot{y} = -\nabla \times (A\widetilde{\Omega}_R A^{-1})y,$$

since $\nabla \times \dot{A}A^{-1}y = 0$. In coordinates:

$$\xi_i = -\frac{(a_j^2 + a_k^2)}{a_j a_k}\widetilde{\omega}_{R,i}$$

for cyclic permutations (i, j, k) of $(1, 2, 3)$. Thus the 'internal' motion of the body is the sum of the anisotropic radial motion given by $\dot{A}A^{-1}y$ and a vorticial motion with uniform vorticity ξ, see (Chandrasekhar 1987 section 27). Note that ξ depends on the lengths of the principal axes and will be time dependent if $\dot{A}$ is non-zero.

4. Chandrasekhar (1987) and Rosensteel (1987) work with $R\mathcal{A}$, the angular momentum resolved instantaneously along the principal axes of the deformed body. Using the formula for $\mathcal{A}$ in part 3 of Proposition 3.4 and $\mathcal{I} = \mu\left(tr(QQ^T)I - QQ^T\right)$ we find:

$$(R\mathcal{A})_i = \mu\left((a_j^2 + a_k^2)\widetilde{\omega}_{L,i} - 2a_j a_k \widetilde{\omega}_{R,i}\right)$$

for cyclic permutations (i,j,k) of $(1,2,3)$, where $\widetilde{\omega}_L = R\omega_L$ and $\widetilde{\omega}_R = S\omega_L$ are the angular velocities in their respective rotating frames. For rigid motions, ie those with $\omega_R = 0$, this reduces to the usual relationship between the angular momentum and angular velocity in 'body coordinates'. A similar calculation shows that:

$$\begin{aligned}(S\mathcal{C})_i &= a_j a_k \widetilde{\omega}_{L,i} - \frac{1}{2}(a_j^2 + a_k^2)\widetilde{\omega}_{R,i} \\ &= a_j a_k(\widetilde{\omega}_{L,i} + \frac{1}{2}\xi_i)\end{aligned}$$

where ξ is the vorticity calculated in the previous remark.

3.4 Examples: Ellipsoids of Type S

The Jacobi and Dedekind ellipsoids described in section 2 are special cases of motions given by $Q(t) = R(t)^T AS(t)$ with:

$$\begin{aligned}R(t) &= \begin{pmatrix} \cos\omega_L t & \sin\omega_L t & 0 \\ -\sin\omega_L t & \cos\omega_L t & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ A(t) &= diag(a_1, a_2, a_3) \\ S(t) &= \begin{pmatrix} \cos\omega_R t & \sin\omega_R t & 0 \\ -\sin\omega_R t & \cos\omega_R t & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned} \qquad (21)\ (22)$$

If the reference body is a sphere the deformed bodies are ellipsoids with constant principal axis lengths which are rotating in space and have internal motion. They have $\Omega_L = (0,\widehat{0,\omega_L})$, $\Omega_R = (0,\widehat{0,\omega_R})$ and:

$$\begin{aligned}j_L &= \tfrac{1}{2}\mathcal{A} &= \tfrac{\mu}{2}\left(0,0,(a_1^2+a_2^2)\omega_L - 2a_1a_2\omega_R\right) \\ j_R &= -\mu\mathcal{C} &= \tfrac{\mu}{2}\left(0,0,(a_1^2+a_2^2)\omega_R - 2a_1a_2\omega_L\right).\end{aligned}$$

Thus the angular velocities and conserved quantities are parallel to one of the principal axes of the deformed body. Motions with this property are known as ellipsoids of *type S*. As we shall see in section 5, they form just one of the possible families of relative equilibria of an affine rigid body.

The angular velocities in the moving frames, $\widetilde{\omega}_L$ and $\widetilde{\omega}_R$, are equal to ω_L and ω_R and so by (20) the particle paths in moving space coordinates are the solutions of:

$$\dot{y} = -A\Omega_R A^{-1}y = \left(\tfrac{a_1}{a_2}y_2, -\tfrac{a_2}{a_1}y_1, 0\right)\Omega_R$$

and so are elliptical. In inertial coordinates the paths are superpositions of these elliptical paths and circular paths corresponding to the rotation of the ellipsoid. The vorticity in moving space coordinates is given by:

$$\xi = \left(0, 0, -\frac{(a_1^2 + a_2^2)}{a_1 a_2}\omega_R\right).$$

If $a_1 = a_2$ then the ellipsoid becomes a spheroid rotating about its axis of symmetry. In this case $R(t)$ and $S(t)$ commute with A and the motion is given by $Q(t) = T(t)A$ where:

$$T(t) = R(t)^T S(t) = \begin{pmatrix} \cos(\omega_L - \omega_R)t & \sin(\omega_L - \omega_R)t & 0 \\ -\sin(\omega_L - \omega_R)t & \cos(\omega_L - \omega_R)t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is a rigid rotation of the sphere which depends only on the difference $\omega_L - \omega_R$. The motions of self-gravitating fluid masses described by Newton and Maclaurin were of this form and they are referred to as *Maclaurin spheroids*.

Following Riemann, Chandrasekhar (1987) defines a *Riemann sequence* to be a family of relative equilibria of type S with the same ratio of vorticity ξ to angular velocity ω_L, ie the same value of:

$$f \;=\; \frac{\xi_3}{\omega_{L,3}} \;=\; -\frac{(a_1^2 + a_2^2)}{a_1 a_2}\frac{\omega_R}{\omega_L} \;=\; -\left(e + \frac{1}{e}\right)\frac{\omega_R}{\omega_L}$$

where $e = \frac{a_1}{a_2}$. Without loss of generality we may assume that $a_1 \geq a_2$ and so $e \geq 1$. The rigid rotations (Jacobi ellipsoids) make up the $f = 0$ sequence while the stationary Dedekind ellipsoids correspond to $f = \pm\infty$. The *irrotational* ellipsoids, ie those with zero vorticity in the inertial frame, form the Riemann sequence with $f = -2$. Those with zero angular momentum give a family which is not a Riemann sequence, but which intersects all the Riemann sequences with $-\infty < f \leq -2$ precisely once and the remaining Riemann sequences not at all. See figure 2. Note also that all the Riemann sequences contain a single Maclaurin spheroid and that the families of ellipsoids with zero angular momentum and with zero circulation both bifurcate from the stationary spheroid. See also section 48 of (Chandrasekhar 1987). The self and skew-adjoint ellipsoids marked in figure 2 are discussed in the next section.

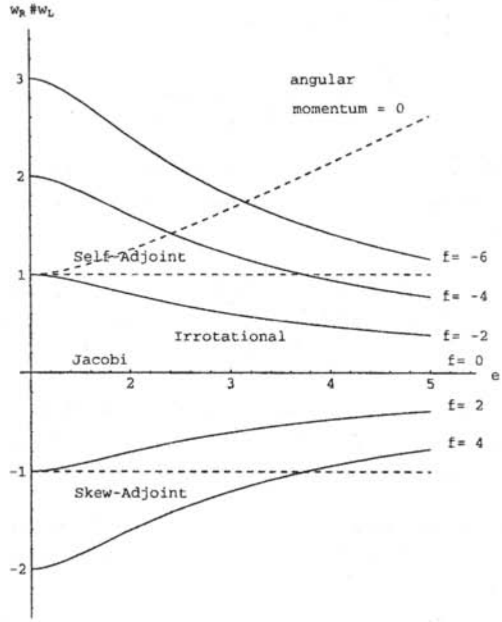

Figure 2: The ratio of the angular velocities as a function of $e = a_1/a_2$ for families of ellipsoids of type S. The solid lines show Riemann sequences and the broken lines the families of zero angular momentum, self-adjoint and skew–adjoint ellipsoids. The Dedekind ellipsoids are at "infinity" and the MacLaurin spheroids are along the line $e = 1$.

4 Transposition Symmetry

In this section we describe some of the consequences of a symmetry operation generated by transposition of configuration matrices. We restrict discussion to the case when $\mathcal{B}$ is spherically symmetric. In this case transposition of matrices induces the involution on $T^*GL^+(3)$ given by:

$$\tau : (Q, P) \mapsto (Q^T, P^T). \tag{23}$$

In body coordinates for $T^*GL^+(3)$ we have:

$$\tau : (Q, \Pi) \mapsto (Q^T, Q\Pi^T Q^{-1}). \tag{24}$$

This operation is a symmetry of an affine rigid body if the Hamiltonian $H = K + V$ is invariant under its action. Equation (1) show that the kinetic energy is always invariant and so the invariance of H is equivalent to that of V. Unlike the left and right $SO(3)$ symmetries, invariance of V under τ does not appear to have any obvious physical interpretation. However the next result, which does not appear to have been noted before, shows that it is always present whenever both $SO(3)$ symmetries are.

Proposition 4.1 *If V is invariant under both left and right actions of $SO(3)$ on $GL^+(3)$ then it is also invariant under the action of τ.*

Proof: Invariance of V under both actions of $SO(3)$ implies that $V(g^TQh) = V(Q)$ for all Q in $GL^+(3)$ and (g,h) in $SO(3) \times SO(3)$. So if $Q = R^TAS$, where R and S are in $SO(3)$ and A is diagonal, then:

$$V(Q) = V(R^TAS) = V(A) = V(S^TAR) = V(Q^T).$$

□

We also define an action of τ on the product group $SO(3) \times SO(3)$ by:

$$\tau.(g,h) = (h,g). \tag{25}$$

This induces an action on the Lie algebra dual $so(3)^* \times so(3)^*$ given by:

$$\tau.(\xi,\zeta) = (\zeta,\xi). \tag{26}$$

The two actions of $SO(3)$ together with that of τ generate actions of the semi-direct product group $Z_2 \times_s (SO(3) \times SO(3))$ on $GL^+(3)$, $T^*GL^+(3)$ and $so(3)^* \times so(3)^*$. Note that the momentum map $J = (J_L, J_R)$ for the action of $SO(3) \times SO(3)$ on the phase space commutes with the actions of τ given by equation (23) and equation (26).

An operation on $GL^+(3)$ inducing an operation on $T^*GL^+(3)$ leaving invariant the Hamiltonian H maps any solution of the equations of motion to another. For the transposition operator this was first noted by Dedekind:

Theorem 4.2 (Dedekind) *Let $(Q(t), P(t))$ be a solution of the equations of motion of an affine rigid body for which the Hamiltonian is invariant under the action of transposition on $T^*GL^+(3)$. Then $(Q(t)^T, P(t)^T)$ is also a solution.*

In the case of the $SO(3)$ actions the new solutions are simply rotated versions of the old and they have identical physical properties. This is not true for transposition. The transposes of the rigidly rotating Jacobi ellipsoids are the Dedekind ellipsoids, which are stationary in space, but have internal motion. In general motions which are related by transposition have their values of J_L and J_R interchanged. Thus the angular momentum for one solution becomes the circulation for the other (up to a factor of -2μ) and vice versa. Since transposition interchanges R and S in the decomposition $Q = R^T AS$, it also interchanges the angular velocities ω_L and ω_R.

Previous authors have defined a motion to be self-adjoint if $Q(t)^T = Q(t)$ for all t (see Chandrasekhar 1987 p.73). We generalise this as follows.

Definition A motion $Q(t)$ is said to be *symmetric-adjoint* if there exists a pair of orthogonal transformations (g, h) in $SO(3) \times SO(3)$ such that for all t:

$$\tau.Q(t) = Q(t)^T = g^T Q(t) h. \tag{27}$$

If equation (27) holds, then for all t:

$$\begin{aligned} Q^T Q &= g^T Q Q^T g && (28) \\ Q Q^T &= h^T Q^T Q h. && (29) \end{aligned}$$

Thus g and h rotate the principal axes of $Q^T Q$ and QQ^T into each other. It follows that a motion is symmetric-adjoint if and only if the moving reference frame and moving space frame are related to each other by rotations which are independent of time. This implies that the pair of orthogonal transformations (g, h) can not be arbitrary. In fact we have the following result.

Lemma 4.3 *If $Q \in GL^+(3)$ and $(g, h) \in SO(3) \times SO(3)$ satisfy $Q^T = g^T Q h$ then:*

1. *The product gh (resp. hg) is a rotation about an eigenvector of QQ^T (resp. $Q^T Q$);*

2. *If QQ^T, or equivalently $Q^T Q$, has distinct eigenvalues then $g = h^T$.*

Proof

1. If $Q^T = g^T Q h$ then gh commutes with QQ^T and hg commutes with $Q^T Q$. The result follows.

2. Let $Q = R^T AS$ with R and S in $SO(3)$ and A diagonal. From equation (29) we obtain

$$A^2 = (ShR^T)^T A^2 (ShR^T).$$

If QQ^T has distinct eigenvalues then so does A^2 and ShR^T must be either the identity or a rotation by π about a coordinate axis. In particular ShR^T commutes with A and satifies $ShR^T = (ShR^T)^T$. From $Q^T = g^T Q h$ we also obtain

$$A = (RgS^T)^T A (ShR^T).$$

Hence $RgS^T = ShR^T = Rh^T S^T$ and so $g = h^T$. □

If $Q(t)$ is symmetric-adjoint and $Q(0) = A$ is diagonal with distinct eigenvalues, then $A = g^T A h$ implies that $g = h$ must be either the identity or a rotation by π about a coordinate axis. This motivates the following definition.

Definition A motion $Q(t)$ is said to be:

1. *Self-adjoint* if $Q(t)^T = Q(t)$ for all t;
2. *Skew-adjoint* if $Q(t)^T = g_\pi Q(t) g_\pi$ for all t, where g_π is a rotation by π about a principal axis of $Q^T Q$.

Note that self and skew-adjoint motions satisfy $QQ^T = Q^T Q$. Every symmetric-adjoint $Q(t)$ with $Q(0)$ diagonal is either self-adjoint or skew-adjoint.

The symmetry properties of symmetric adjoint motions are reflected in the following straightforward relationship between the left and right momentum maps:

Proposition 4.4 *For a symmetric adjoint motion with $Q(t)^T = g^T Q(t) h$:*

$$\begin{aligned} J_L &= h^T J_R h \\ J_R &= g^T J_L g. \end{aligned}$$

□

Examples: Ellipsoids of Type S

We return to the ellipsoids of type S that were described in subsection 3.4. Those that are symmetric-adjoint are either self-adjoint or skew-adjoint. Since transposition interchanges ω_L and ω_R, the self-adjoint ellipsoids are those with $\omega_L = \omega_R$. The skew-adjoint ellipsoids satisfy:

$$(R^T A S)^T = g_\pi R^T A S g_\pi.$$

The rotation axis of g_π can be the same as the rotation axis of $R(t)$ and $S(t)$, or it can be a perpendicular axis. In the first case g_π commutes with $R(t)$ and $S(t)$, which implies that $Q(t)^T = Q(t)$, ie the motion is also self-adjoint. In the second case Proposition 4.4 implies that $j_L = -j_R$. From the equations for j_L and j_R in subsection 3.4 we obtain:

$$(a_1 - a_2)^2 (\omega_L + \omega_R) = 0$$

and so either $a_1 = a_2$ or $\omega_L = -\omega_R$. The motions with $a_1 = a_2$ are the Maclaurin spheroids,which can be regarded as a special type of skew-adjoint ellipsoid of type S. Maclaurin spheroids are only self-adjoint if $\omega_L - \omega_R = 0$, ie they are stationary.

As can been seen from figure 2, if we assume $a_1 \geq a_2$, then the family of self-adjoint ellipsoids of type S intersects each Riemann sequence with $-\infty \leq f \leq -2$ precisely once. Similarly the family of skew-adjoint ellipsoids intersects each Riemann sequence with $2 \leq f \leq \infty$ once. Note that in section 48 of (Chandrasekhar 1987) the skew-adjoint ellipsoids are referred to as a second family of self-adjoint ellipsoids.

5 Classification of Relative Equilibria

Throughout this section we work with affine rigid bodies with spherically symmetric reference bodies and potential functions which are invariant under both left and right actions of $SO(3)$, and hence also under transposition. For brevity we will summarize these conditions by saying that the affine rigid body is spherically symmetric.

5.1 Riemann's Theorem

A *relative equilibrium* of an affine rigid body is a motion of the form:

$$Q(t) = R(t)^T Q_0 S(t)$$

where Q_0 is a constant matrix which we can assume to be diagonal, and:

$$\begin{aligned} R(t) &= e^{-t\Omega_L} \\ S(t) &= e^{-t\Omega_R} \end{aligned}$$

for a pair of constant skew-symmetric matrices Ω_L and Ω_R. The 'shape' of the body, ie the lengths of the principal axes of the ellipsoid, is constant during the motion. However in general there is both an internal rotation of particles and a rotation of the ellipsoid in space. Examples of relative equilibria include the ellipsoids of type S discussed in sections 3 and 4. Note that $\dot{S}^T S = \Omega_R$ and $\dot{R}^T R = \Omega_L$ and so Ω_R and Ω_L are, respectively, the angular velocities of the moving reference and space frames with respect to the stationary reference and inertial frames. Define vectors ω_R and ω_L by $\widehat{\omega}_R = \Omega_R$ and $\widehat{\omega}_L = \Omega_L$. We also define $j_R(Q,P)$ and $j_L(Q,P)$ by $\widehat{j}_R(Q,P) = J_R(Q,P)$ and $\widehat{j}_L(Q,P) = J_L(Q,P)$. Finally we say that a relative equilibrium is *spherical* if all the principal axes have the same length, *spheroidal* if precisely two are the same, and *ellipsoidal* if all three are different.

Theorem 5.1 (Riemann) *Let $Q(t) = e^{t\hat{\omega}_L} Q_0 e^{-t\hat{\omega}_R}$ be a relative equilibrium of a spherically symmetric affine rigid body. Assume that Q_0 is diagonal.*

1. *If ω_L and ω_R are both non-zero, then:*

(a) *j_L is parallel to ω_L and j_R is parallel to ω_R;*

(b) *If the relative equilibrium is ellipsoidal then ω_L and ω_R (and hence also j_L and j_R) must satisfy one of the following conditions:*

(i) ω_L and ω_R are parallel to the same principal axis of the ellipsoid,

(ii) ω_L and ω_R lie in the same principal plane of the ellipsoid;

(c) *If the relative equilibrium is spheroidal then ω_L and ω_R lie in a plane which contains the axis of rotational symmetry of the body;*

(d) *If the relative equilibrium is spherical then ω_L and ω_R are parallel to each other.*

2. *If $\omega_R = 0$ and $\omega_L \neq 0$ then j_R, j_L and ω_L are all parallel. The vector j_R points in the opposite direction to j_L and ω_L. If the relative equilibrium is ellipsoidal then they are parallel to a principal axis, while if it is spheroidal they are either parallel or orthogonal to the axis of symmetry. A corresponding statement holds if $\omega_L = 0$ and $\omega_R \neq 0$*

3. *If $\omega_R = 0 = \omega_L$ then $j_R = 0 = j_L$.*

Remark

Condition (i) of $1(b)$ may, of course, be regarded as a special case of condition (ii). However we shall see that relative equilibria fall naturally into families. In one of these, all the members satisfy condition (i), while most of the members of the others do not.

Proof:
$1(a)$. For the relative equilibrium $Q(t)$ conservation of angular momentum and circulation imply that:

$$\begin{aligned} J_L(Q(t),P(t)) &= J_L(Q(0),P(0)) \\ J_R(Q(t),P(t)) &= J_R(Q(0),P(0)) \end{aligned} \tag{30}$$

while equivariance with respect to the left and right $SO(3)$ actions implies:

$$\begin{aligned} J_L(Q(t),P(t)) &= e^{t\Omega_L} J_L(Q(0),P(0)) e^{-t\Omega_L} \\ J_R(Q(t),P(t)) &= e^{t\Omega_R} J_R(Q(0),P(0)) e^{-t\Omega_R}. \end{aligned} \tag{31}$$

Equating right hand sides of equations (30) and (31), differentiating with respect to t and evaluating at $t = 0$ gives:

$$\begin{aligned} \Omega_L J_L - J_L \Omega_L &= 0 \\ \Omega_R J_R - J_R \Omega_R &= 0. \end{aligned}$$

By lemma 3.1 this is equivalent to:

$$\begin{aligned} j_L \times \omega_L &= 0 \\ j_R \times \omega_R &= 0 \end{aligned}$$

and so j_L is parallel to ω_L and j_R is parallel to ω_R.

$1(b)$. Let $Q_0 = A = diag(a_1, a_2, a_3)$. Then:

$$\begin{aligned} J_L &= \frac{1}{2}(PQ^T - QP^T) \\ &= \frac{\mu}{2}(\dot{Q}Q^T - Q\dot{Q}^T) \\ &= \frac{\mu}{2}(\hat{\omega}_L A^2 + A^2 \hat{\omega}_L - 2A\hat{\omega}_R A). \end{aligned}$$

This gives:

$$j_L = \mu(M\omega_L - N\omega_R) \tag{32}$$

where:

$$\begin{aligned} M &= diag\left(\frac{1}{2}(a_2^2+a_3^2), \frac{1}{2}(a_3^2+a_1^2), \frac{1}{2}(a_1^2+a_2^2)\right) \\ N &= diag\,(a_2a_3, a_3a_1, a_1a_2)\,. \end{aligned}$$

Similarly:

$$J_R = \frac{\mu}{2}(\hat{\omega}_R A^2 + A^2\hat{\omega}_R - 2A\hat{\omega}_L A)$$

and:

$$j_R = \mu(-N\omega_L + M\omega_R) \tag{33}$$

By part $1(a)$ of the theorem, if $\omega_L \neq 0 \neq \omega_R$, then $j_L = \mu\alpha_L\omega_L$ and $j_R = \mu\alpha_R\omega_R$ for a pair of real numbers α_L and α_R. So:

$$\begin{aligned} M\omega_L - N\omega_R &= \alpha_L\omega_L \\ -N\omega_L + M\omega_R &= \alpha_R\omega_R. \end{aligned} \tag{34}$$

Writing this out in coordinates gives for each cyclic permutation (i,j,k) of $(1,2,3)$:

$$\begin{aligned} \left(\frac{1}{2}(a_j^2+a_k^2) - \alpha_L\right)\omega_{L,i} - a_ja_k\omega_{R,i} &= 0 \\ -a_ja_k\omega_{L,i} + \left(\frac{1}{2}(a_j^2+a_k^2) - \alpha_R\right)\omega_{R,i} &= 0. \end{aligned} \tag{35}$$

These equations have a non-zero solution for $(\omega_{L,i}, \omega_{R,i})$ if and only if:

$$4\alpha_L\alpha_R - 2(a_j^2+a_k^2)(\alpha_L+\alpha_R) + (a_j^2-a_k^2)^2 = 0. \tag{36}$$

Lemma 5.2 *If $(\omega_{L,i}, \omega_{R,i})$ and $(\omega_{L,j}, \omega_{R,j})$ are non-zero solutions of equation (35) with $i \neq j$, then either $a_i = a_j$ or α_L and α_R are given by the following pair of equations:*

$$\begin{aligned} \alpha_L + \alpha_R &= \frac{1}{2}(a_i^2 + a_j^2 - 2a_k^2) \\ \alpha_L\alpha_R &= \frac{1}{4}\left(a_i^2a_j^2 + (a_i^2+a_j^2)a_k^2 - 3a_k^4\right) \end{aligned}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

Proof: Solve the two equations given by equation (36) with subscripts (i,j,k) and (j,k,i) for $\alpha_L + \alpha_R$ and $\alpha_L\alpha_R$.

□

To complete the proof of part 1(b) of Theorem 5.1 we note that it is equivalent to showing that for an ellipsoidal relative equilibrium at least one of the pairs $(\omega_{L,i}, \omega_{R,i})$ must be zero. If this is not true then by lemma 5.2:

$$a_1^2 + a_2^2 - 2a_3^2 \;=\; a_2^2 + a_3^2 - 2a_1^2 \;=\; a_3^2 + a_1^2 - 2a_2^2.$$

This implies that $a_1 = a_2 = a_3$, contradicting the assumption that the relative equilibrium is ellipsoidal.

1(c). If $a_2 = a_3$ then the equations (35) for $(\omega_{L,2}, \omega_{R,2})$ and $(\omega_{L,3}, \omega_{R,3})$ are the same and have one dimensional spaces of solutions. Thus $(\omega_{L,3}, \omega_{R,3}) = c(\omega_{L,2}, \omega_{R,2})$ for some constant c and so ω_L and ω_R lie in the span of the vectors $(1,0,0)$ and $(0,1,c)$.

1(d). If $A = aI$ then by equation (11) and equation (12) $j_L = -j_R$ and so by part 1 of the theorem ω_L and ω_R are parallel.

2. If $\omega_R = 0 \neq \omega_L$ then $j_R = \mu\alpha_R\omega_R$ need not hold in general. However we still have $j_L = \mu\alpha_L\omega_L$ and this implies that $M\omega_L = \alpha_L\omega_L$. It follows that j_L and ω_L are parallel to a principal axis in the ellipsoidal case, and either parallel or orthogonal to the axis of symmetry in the spheroidal case. The remaining statements now follow from equations (32) and (33), together with the fact that the entries in M and N are positive.

3. The final statement of Theorem 5.1 follows immediately from equations (32) and (33).

□

The following result of Riemann is also a corollary of lemma 5.2.

Proposition 5.3 *Let a_1, a_2 and a_3 be the half-lengths of the principal axes of a relative equilibrium for which $(\omega_{L,i}, \omega_{R,i})$ and $(\omega_{L,j}, \omega_{R,j})$ are non-zero. Then they satisfy one of the following conditions:*

1. $a_i = a_j$
2. $a_k \geq \frac{1}{2}(a_i + a_j)$
3. $a_k \leq \frac{1}{2}|a_i - a_j|$

where (i,j,k) is a permutation of $(1,2,3)$.

Proof: If $a_i \neq a_j$ then α_L and α_R must satisfy part (ii) of lemma 5.2. For them to be real we must have $(\alpha_L + \alpha_R)^2 \geq 4\alpha_L\alpha_R$ and so:

$$\left(a_i^2 + a_j^2 - 2a_k^2\right)^2 \geq 4\left(a_i^2a_j^2 + (a_i^2 + a_j^2)a_k^2 - 3a_k^2\right).$$

Rearranging and factorising this gives:

$$\left((a_i - a_j)^2 - 4a_k^2\right)\left((a_i + a_j)^2 - 4a_k^2\right) \geq 0$$

from which the result follows. □

Following Riemann (1860) and subsequent authors (see (Chandrasekhar 1987)) we define four classes of relative equilibria.

(1) Relative equilibria with ω_L and ω_R parallel to the same principal axis are said to be of *type S*.

Relative equilibria with ω_L and ω_R in the plane spanned by the principal axes of half-lengths a_i and a_j are said to be:

(2) of *type I* if $a_k \geq \frac{1}{2}(a_i + a_j)$,

(3) of *type II* if $a_k \leq \frac{1}{2}|a_i - a_j|$ and $min(a_i, a_j) < a_k < max(a_i, a_j)$,

(4) of *type III* if $a_k \leq \frac{1}{2}|a_i - a_j|$ and $a_k < min(a_i, a_j)$.

Figure 3 shows the regions in $(\frac{a_i}{a_k}, \frac{a_j}{a_k})$ - space within which the relative equilibria of types *I*, *II* and *III* must lie.

Remarks

1. The proofs given here of Riemann's theorem and the constraints on the shape of the relative equilibria in proposition 5.3 differ from those of Riemann (1860) and Chandrasekhar (1987). They first derived equations of motion for the system and deduced the results from these, whereas we have obtained the results directly from the conservation of angular momentum and circulation.

2. The constraints on the lengths of the principal axes given by proposition 5.3 are necessary conditions for the existence of relative equilibria. They are not, however, sufficient. The precise 'domain of occupancy' of relative equilibria in (a_1, a_2, a_3)-space will depend on the potential energy function V.

For example, for a self gravitating fluid Riemann showed that the type *I* ellipsoids always have a_k lying between a_i and a_j (in the notation above), although proposition 5.3 allows for the possibility that $a_k > max(a_i, a_j)$. This explains why Riemann did not split the ellipsoids satisfying inequality 2 of the proposition into two cases, as he did those satisfying inequality 3. For self-gravitating fluids there are also no relative equilibria with principal axis lengths near to the boundary between type *II* and type *III* ellipsoids. See Riemann (1860) and section 51 of Chandrasekhar (1987).

Riemann and Chandrasekhar do not explicitly distinguish between the constraints that depend on the precise form of the potential energy function and those that depend only on its symmetries. However it is clear from their accounts that the details of the potential energy for a self-gravitating fluid mass were not used to derive the constraints of proposition 5.3, although they were used for the other constraints that are not treated in this paper.

3. It follows from part 2 of Theorem 5.1 that rigidly rotating ellipsoids and their transposes, the stationary ellipsoids, must be of type S.

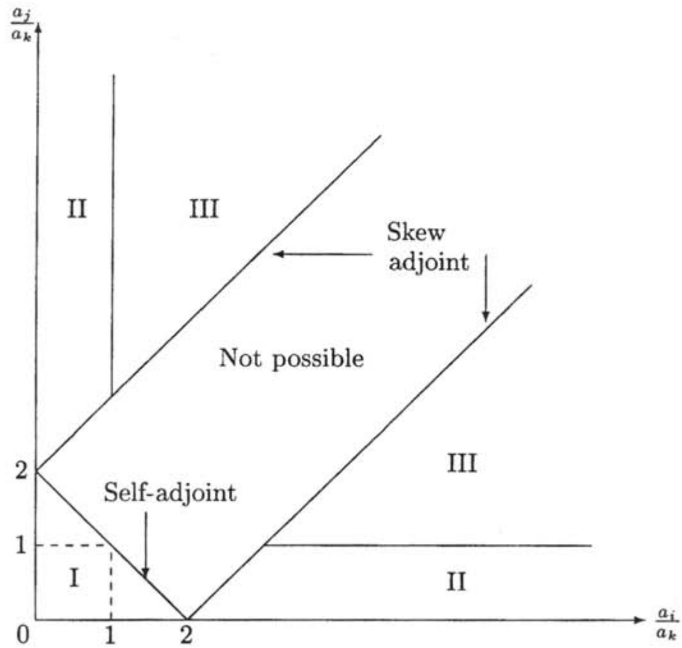

Figure 3: The shapes of relative equilibria of types *I*, *II* and *III*.

5.2 Symmetric-Adjoint Ellipsoids

In this subsection we describe the relationship between the symmetric-adjoint ellipsoids and the ellipsoids of types *I*, *II* and *III*.

Proposition 5.4 *Let $Q(t) = e^{t\hat{\omega}_L} Q_0 e^{-t\hat{\omega}_R}$ be a relative equilibrium of a spherically symmetric affine rigid body. Assume that Q_0 is diagonal.*

1. *If $Q(t)$ is self-adjoint then $\omega_L = \omega_R$ and $j_L = j_R$.*

2. *If $Q(t)$ is skew-adjoint then $\omega_L + \omega_R$ and $\omega_L - \omega_R$ are orthogonal to each other. If $Q(t)$ is ellipsoidal then $\omega_L + \omega_R$ and $\omega_L - \omega_R$ are parallel to principal axes. If $Q(t)$ is spheroidal then either both the vectors $\omega_L + \omega_R$ and $\omega_L - \omega_R$ lie in the plane orthogonal to the axis of rotational symmetry, or one of them is parallel to it.*

Proof: Part 1 follows immediately from the definitions of ω_L and ω_R and from Proposition 4.4.

For 2 let $Q_0 = A = diag(a_1, a_2, a_3)$. From equation (32) and equation (33) we obtain:

$$\begin{aligned} j_L + j_R &= \mu(M-N)(\omega_L + \omega_R) \\ j_L - j_R &= \mu(M+N)(\omega_L - \omega_R) \end{aligned}$$

where:

$$\begin{aligned} M - N &= diag\left(\frac{1}{2}(a_2-a_3)^2, \frac{1}{2}(a_3-a_1)^2, \frac{1}{2}(a_1-a_2)^2\right) \\ M + N &= diag\left(\frac{1}{2}(a_2+a_3)^2, \frac{1}{2}(a_3+a_1)^2, \frac{1}{2}(a_1+a_2)^2\right). \end{aligned}$$

Suppose $Q(t)$ is skew-adjoint and let g_π be a rotation by π about a coordinate axis such that $A = g_\pi A g_\pi$. By proposition 4.4 $J_L = g_\pi J_R g_\pi$, which is equivalent to $j_L = g_\pi.j_R$. So:

$$\begin{gathered} g_\pi.(j_L + j_R) = (j_L + j_R) \\ g_\pi.(j_L - j_R) = -(j_L - j_R). \end{gathered}$$

Moreover g_π commutes with M and N. Equation (37) then gives:

$$\begin{aligned} (M-N)g_\pi.(\omega_L + \omega_R) &= (M-N)(\omega_L + \omega_R) \\ (M+N)g_\pi.(\omega_L - \omega_R) &= -(M+N)(\omega_L - \omega_R) \end{aligned}$$

Since $M + N$ is invertible the second of these equations implies that $g_\pi.(\omega_L - \omega_R) = -(\omega_L - \omega_R)$. If $Q(t)$ is ellipsoidal then $M - N$ is also invertible and so $g_\pi.(\omega_L + \omega_R) = (\omega_L + \omega_R)$. Thus $\omega_L + \omega_R$ must be parallel to the axis of rotation of g_π and $\omega_L - \omega_R$ perpendicular to it. It follows from Riemann's Theorem that $\omega_L - \omega_R$ must also be parallel to a principal axis. The proof of the spheroidal case is similar.

□

A skew-adjoint relative equilibrium does not have ω_L and ω_R parallel unless either $\omega_L = \omega_R$ or $\omega_L = -\omega_R$. In these cases ω_L and ω_R must be parallel to a principal axis (or orthogonal to the axis of rotational symmetry in the case of a spheroid). In the case of $\omega_L = \omega_R$ the relative equilibrium is also self-adjoint.

Proposition 5.5 *Let a_1, a_2 and a_3 be the half-lengths of the principal axes of an ellipsoidal relative equilibrium for which $(\omega_{L,i}, \omega_{R,i}) \neq (0,0)$ and $(\omega_{L,j}, \omega_{R,j}) \neq (0,0)$, with $i \neq j$.*

1. *If the relative equilibrium is self-adjoint then:*

$$a_k = \frac{1}{2}(a_i + a_j).$$

2. If the relative equilibrium is skew-adjoint then:

$$a_k = \frac{1}{2}|a_i - a_j|.$$

Proof:
1. If the relative equilibrium is self-adjoint then $\omega_L = \omega_R$. If $(\omega_{L,i}, \omega_{R,i}) \neq (0,0)$ then equation (35) implies:

$$\alpha_L = \alpha_R = \frac{1}{2}(a_j - a_k)^2.$$

Similarly if $(\omega_{L,j}, \omega_{R,j}) \neq (0,0)$ then:

$$\alpha_L = \alpha_R = \frac{1}{2}(a_i - a_k)^2.$$

So $(a_j - a_k)^2 = (a_i - a_k)^2$ which implies $a_k = \frac{1}{2}(a_i + a_j)$ for an ellipsoidal relative equilibrium.

2. If the relative equilibrium is skew-adjoint then there exists a rotation g_π such that $j_L = g_\pi.j_R$ and $\omega_L = g_\pi.\omega_R$. It follows that if $j_L = \mu\alpha_L\omega_L$ and $j_R = \mu\alpha_R\omega_R$ then $\alpha_L = \alpha_R = \alpha$, say. From equation (37) we have:

$$\begin{aligned} \alpha(\omega_L + \omega_R) &= (M - N)(\omega_L + \omega_R) \\ \alpha(\omega_L - \omega_R) &= (M + N)(\omega_L - \omega_R). \end{aligned}$$

If $(\omega_{L,i}, \omega_{R,i}) \neq (0,0)$ and $(\omega_{L,j}, \omega_{R,j}) \neq (0,0)$ then either:

$$\alpha = \frac{1}{2}(a_j - a_k)^2 = \frac{1}{2}(a_i + a_k)^2$$

or:

$$\alpha = \frac{1}{2}(a_i - a_k)^2 = \frac{1}{2}(a_j + a_k)^2.$$

The first case leads to $a_k = \frac{1}{2}(a_j - a_i)$ and the second to $a_k = \frac{1}{2}(a_i - a_j)$. □

Remarks

1. The self-adjoint relative equilibria which are not of type S occur on the boundary (in (a_1, a_2, a_3)-space) of the ellipsoids of type *I*, while the skew-adjoint relative equilibria which are not of type S occur on the boundaries of types *II* and *III*. See figure 3.

2. From the proof of part 2. of the proposition it follows that for skew-adjoint relative equilibria of types *II* and *III* the vector $\omega_L - \omega_R$ is parallel to the longest axis of the ellipsoid. For skew-adjoint type *II* relative equilibria $\omega_L + \omega_R$ is parallel to the shortest axis, while for skew-adjoint type *III* relative equilibria it is parallel to the principal axis of middle length.

3. In previous work the skew-adjoint relative equilibria appear to have been confused with self-adjoint relative equilibria. For example in section 51 of (Chandrasekhar 1987) it is stated that self-adjoint ellipsoids lie on the boundary of the ellipsoids of type III.

6 Symmetry Groups of Riemann Ellipsoids

The isotropy subgroup of a point s in a set S acted upon by a group G is the set of group elements which fix that point:

$$\Sigma_s = \{\gamma \in G : \gamma.s = s\}.$$

Isotropy subgroups are preserved by solutions of differential equations which are invariant under the action of a group on the phase space. In particular, for an affine rigid body invariant under the action of $Z_2^\tau \times_s (SO(3) \times SO(3))$ on $T^*GL^+(3)$, with:

$$\Sigma_{(Q(t),P(t))} = \{\gamma \in Z_2^\tau \times_s (SO(3) \times SO(3)) : \gamma.(Q(t), P(t)) = (Q(t), P(t))\}$$

we have:

$$\Sigma_{(Q(t),P(t))} = \Sigma_{(Q(0),P(0))} \tag{37}$$

for all t. We refer to this group as the *symmetry group* of the trajectory.

Examples

1. A trajectory of an affine rigid body has symmetry group equal to $Z_2^\tau \times_s (SO(3) \times SO(3))$ if and only if it is a stationary sphere, ie $Q(t) \equiv cI, P(t) \equiv 0$ for some c.

2. The symmetry group of a self-adjoint motion contains the subgroup Z_2^τ.

3. The symmetry group of a symmetric-adjoint motion contains the subgroup of $Z_2^\tau \times_s (SO(3) \times SO(3))$ generated by an element of the form $(\tau, (g, h))$.

Two subgroups, K and H, of a group G are said to be conjugate if there exists an element γ in G such that $\gamma K \gamma^{-1} = H$. If G acts on a set S and $\gamma s_1 = s_2$ (ie s_1 and s_2 belong to the same G-orbit), then $\Sigma_{s_1} = \gamma \Sigma_{s_2} \gamma^{-1}$. It follows that motions of an affine rigid body which are mapped to each other by a symmetry transformation have symmetry groups that are conjugated to each other by that transformation. For example the symmetry group of a Jacobi ellipsoid is mapped to the symmetry group of a Dedekind ellipsoid under conjugation by the transpose operator τ. We will refer to a conjugacy class of symmetry groups as a *symmetry type*.

In the next two subsections we compute the isotropy subgroups for the action of $Z_2^\tau \times_s (SO(3) \times SO(3))$ on $T^*GL^+(3)$, and hence the set of all possible symmetry types of motions of an affine rigid body. In the final subsection we match the classification of relative equilibria by symmetry type to that of Riemann and briefly describe some conclusions that can be drawn regarding possible bifurcations of relative equilibria.

6.1 Isotropy Subgroups of $Z_2^\tau \times SO(3)$ on $gl(3)^*$

In the next section we will show that the calculation of the isotropy subgroups of the $Z_2^\tau \times_s (SO(3) \times SO(3))$ action on $T^*GL^+(3)$ can be reduced to that of the isotropy subgroups of the action of the direct product group $Z_2^\tau \times SO(3)$ on $gl(3)^*$ given by:

$$\begin{aligned} (1, g).\pi &= g\pi g^T \\ (\tau, g).\pi &= g\pi^T g^T. \end{aligned} \tag{38}$$

In this subsection we compute the conjugacy classes of isotropy subgroups of this action.

First recall (eg from Golubitsky *et al* 1988) that every subgroup of $SO(3)$ is conjugate to one of $SO(3), O(2), SO(2), D_n$ or Z_n. Here $SO(2)$ consists of all rotations about the z-axis and Z_n is the subgroup of order n generated by rotation by $\frac{2\pi}{n}$. The subgroup $O(2)$ contains $SO(2)$ together with rotations by π about axes orthogonal to the z-axis. Similarly D_n consists of Z_n and rotations by π about n axes orthogonal to the z-axis with angle $\frac{2\pi}{n}$ between them. Notice in particular that D_2 is the group of rotations by π about each of the x, y and z axes, together with the identity element.

As an abstract group $Z_2^\tau \times SO(3)$ is isomorphic to $O(3)$. Its subgroups are of three types (Golubitsky *et al* 1988):

1. $I \times K$ where K is a subgroup of $SO(3)$;
2. $Z_2^\tau \times K$ where K is a subgroup of $SO(3)$;
3. subgroups L generated by a subgroup K of $SO(3)$ and an element (τ, g) in $Z_2^\tau \times SO(3)$ with g not in K.

In case (c) the subgroup L is isomorphic to the subgroup H of $SO(3)$ generated by K and the element g. We will denote such groups by $\widetilde{H}$. In particular $\widetilde{O(2)}$ will denote the group generated by $SO(2)$ and (τ, g) where g is rotation by π about an axis perpendicular to the z-axis, $\widetilde{D_n}$ will denote the group generated by Z_n and (τ, g) where g is one of the rotations by π in D_n, and $\widetilde{Z_{2n}}$ will denote the group generated by Z_n and (τ, g) with g being rotation by $\frac{\pi}{n}$ about the z-axis.

Our strategy for finding the isotropy subgroups of the $Z_2^\tau \times SO(3)$ action on $gl(3)^*$ is to first find the isotropy subgroups K of the $SO(3)$ action, and then to find the elements (τ, g) which extend K to isotropy subgroups of the $Z_2^\tau \times SO(3)$ action.

To find the isotropy subgroups of the $SO(3)$ action we note that $gl(3)^*$, identified with the space of all 3×3 matrices, is the direct sum of three subspaces:

$$gl(3)^* \cong V_I \oplus V_a \oplus V_s$$

where V_I is the subspace of scalar matrices, V_a is the subspace is antisymmetric matrices and V_s is the subspace of traceless symmetric matrices. These subspaces are invariant under the action of $SO(3)$ and the restricted actions are the irreducible representations of dimensions 1, 3 and 5, respectively. On V_I the action is trivial while on V_a and V_s the isotropy subgroups are as shown in Figure 4 (a) and (b) (see (Golubitsky *et al* 1988) for details).

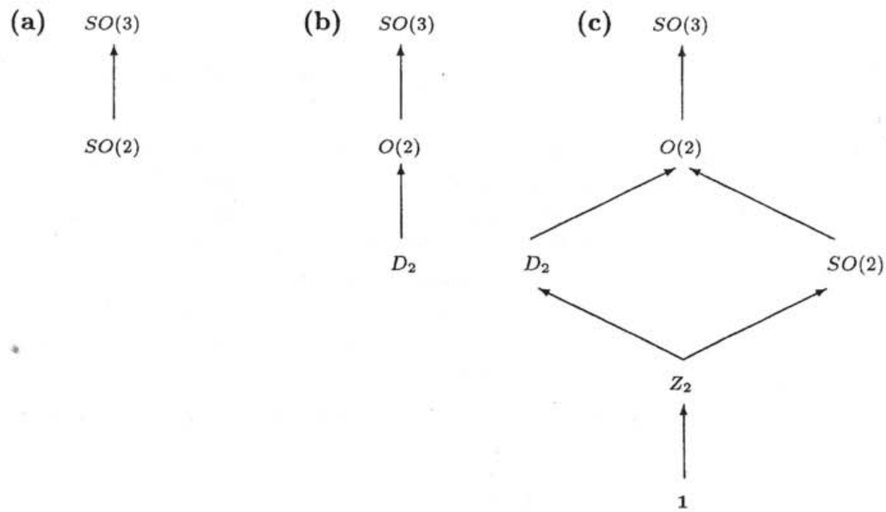

Figure 4: Conjugacy classes of isotropy subgroups of $SO(3)$ on (a) V_a, (b) V_s and (c) $gl(3)^*$.

The figures show a representative subgroup from each conjugacy class and the arrows indicate inclusions. The isotropy subgroups of the action of $SO(3)$ on $gl(3)^*$ are the intersections of pairs of isotropy subgroups of V_a and V_s. The conjugacy classes of these are shown in Figure 4 (c) and the fixed point set of a representative from each conjugacy class is given in Table 6.1.

Next we find the symmetries of the form (τ, g) that the elements of the fixed point sets in Table 6.1 can have. Note that a matrix is fixed by $(\tau, 1)$ if and only if it is symmetric, so the following are all isotropy subgroups:

$$Z_2^\tau \times SO(3), \quad Z_2^\tau \times O(2), \quad Z_2^\tau \times D_2, \quad Z_2^\tau \times Z_2, \quad Z_2^\tau.$$

Isotropy subgroup	Fixed point space
$SO(3)$	αI
$O(2)$	$\begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \beta \end{pmatrix}$
$SO(2)$	$\begin{pmatrix} \alpha & -\delta & \\ \delta & \alpha & \\ & & \beta \end{pmatrix}$
D_2	$\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}$
$\mathbf{Z}_2$	$\begin{pmatrix} \alpha & \delta & \\ \epsilon & \beta & \\ & & \gamma \end{pmatrix}$
$\mathbf{1}$	$gl(3)^*$

Table 1: Fixed point sets of representative isotropy subgroups of $SO(3)$ on $gl(3)^*$.

If

$$\pi = \begin{pmatrix} \alpha & -\delta & 0 \\ \delta & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

then $(\tau, g).\pi = \pi$ with $g = diag(1, -1, -1)$. It follows that there is an isotropy subgroup generated by $SO(2)$ and this (τ, g). The group generated by $SO(2)$ and g is $O(2)$, so the isotropy group is $\widetilde{O(2)}$.

The element τ acts on the fixed point space of Z_2 by interchanging the coordinates δ and ϵ shown in the entry for π in Table 6.1. If $\delta = -\epsilon$ then this is equivalent to the action of $g = diag(1, -1, -1)$. So π is fixed by (τ, g). The group generated by $Z_2 = \{I, diag(-1, -1, 1)\}$ and (τ, g) is $\widetilde{D_2}$.

Finally we note that if $g = diag(1, -1, -1)$ then (τ, g) fixes matrices of the form:

$$\begin{pmatrix} \alpha & -\delta & -\epsilon \\ \delta & \beta & \phi \\ \epsilon & \phi & \gamma \end{pmatrix}.$$

Thus $\widetilde{Z_2}$ is also an isotropy subgroup.

To summarise, the set of conjugacy classes of isotropy subgroups of the $Z_2^\tau \times SO(3)$ action on $gl(3)^*$ is shown in Figure 5. The fixed point sets of their representatives are the same as those shown in the column headed 'P' in Table 2. The reason for this will become clear in the next subsection.

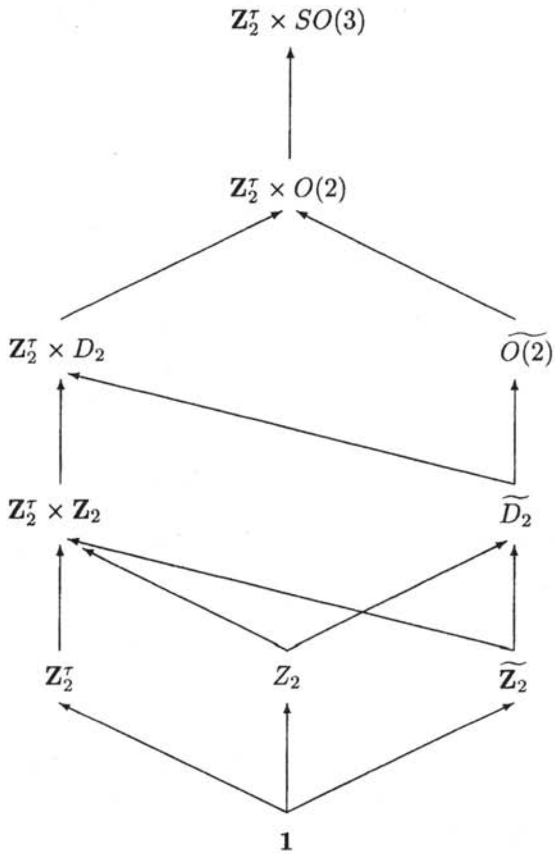

Figure 5: Conjugacy classes of isotropy subgroups of $\mathbf{Z}_2^\tau \times_s SO(3)$ on $gl(3)^*$.

6.2 Isotropy Subgroups of $Z_2^\tau \times_s (SO(3) \times SO(3))$ on $T^*GL^+(3)$

Let $SO(3)^D$ denote the diagonal subgroup of $SO(3) \times SO(3)$:

$$SO(3)^D = \{(g, h) \in SO(3) \times SO(3) : g = h\}$$

We show that the isotropy subgroups of the action of $Z_2^\tau \times_s (SO(3) \times SO(3))$ action on $T^*GL^+(3)$ are precisely the same as those of the action of $Z_2^\tau \times SO(3)^D$

on $gl(3)^*$ given by (38).

We will use body coordinates for $T^*GL^+(3)$. In these coordinates the action of $Z_2^\tau \times_s (SO(3) \times SO(3))$ is given by:

$$\begin{aligned}(I,(g,h)).(Q,\Pi) &= (gQh^T, h\Pi h^T)\\ (\tau,(g,h)).(Q,\Pi) &= (gQ^Th^T, hQ\Pi^TQ^{-1}h^T)\end{aligned} \tag{39}$$

Every point (Q,Π) is contained in the $SO(3)\times SO(3)$ orbit of one with Q diagonal and so to compute the conjugacy classes of isotropy subgroups it is sufficient to restrict to $Q = A = diag(a,b,c)$ where a, b and c are all strictly positive. Then, if $(I,(g,h))$ is an element of the isotropy subgroup of (A,Π), we have $gAh^T = A$ and, by Lemma 4.3, $g = h$. Similarly, if $(\tau,(g,h))$ is an element of the isotropy subgroup $gA^Th^T = gAh^T = A$ and so again $g = h$. Thus we obtain the following result.

Lemma 6.1 *The isotropy subgroup of (A,Π) is a subgroup of $Z_2^\tau \times SO(3)^D$.*

Restricting to $Z_2^\tau \times SO(3)^D$ the action (39) on (A,Π) is:

$$\begin{aligned}(1,g).(A,\Pi) &= (gAg^T, g\Pi g^T)\\ (\tau,g).(A,\Pi) &= (gAg^T, gA\Pi^TA^{-1}g^T).\end{aligned} \tag{40}$$

Define $A^{\frac{1}{2}} = diag(\sqrt{a},\sqrt{b},\sqrt{c})$ and $A^{-\frac{1}{2}} = (A^{\frac{1}{2}})^{-1}$.

Lemma 6.2 *Let σ be I or τ. Then (σ,g) fixes (A,Π) under the action (40) if and only if:*

1. *A commutes with g, and*
2. *(σ,g) fixes $\pi = A^{-\frac{1}{2}}\Pi A^{\frac{1}{2}}$ under the action (38).*

Proof: Part 1 follows immediately from (40). For part 2 we see that if A commutes with g, then:

$$g\Pi g^T = \Pi \iff g(A^{-\frac{1}{2}}\Pi A^{\frac{1}{2}})g^T = A^{-\frac{1}{2}}\Pi A^{\frac{1}{2}}$$

and:

$$gA\Pi^TA^{-1}g^T = \Pi \iff g(A^{\frac{1}{2}}\Pi^TA^{-\frac{1}{2}})g^T = A^{-\frac{1}{2}}\Pi A^{\frac{1}{2}}.$$

□

Corollary 6.3 *Every isotropy subgroup of the action of $Z_2^\tau \times_s (SO(3)\times SO(3))$ on $T^*GL^+(3)$ is conjugate to an isotropy subgroup of the action of $Z_2^\tau \times SO(3)^D$ on $gl(3)^*$ given by (38) with $SO(3) = SO(3)^D$. Conversely, every isotropy subgroup of the action of $Z_2^\tau \times SO(3)^D$ on $gl(3)^*$ is an isotropy subgroup of $Z_2^\tau \times_s (SO(3) \times SO(3))$ on $T^*GL^+(3)$.*

Proof: If (σ, g) fixes (A, Π) then by part (i) of Lemma 6.2, g commutes with A and so must lie in:

1. $SO(3)$ if $A = diag(a, a, a)$,
2. $O(2)$ if $A = diag(a, a, b)$,
3. D_2 if $A = diag(a, b, c)$,

or a group conjugate to one of these for the other possibilities for A. From this and part (ii) of the lemma it follows that the isotropy subgroup Σ of (A, Π) under the action (40) must be the intersection of the isotropy subgroup of $\pi = A^{-\frac{1}{2}} \Pi A^{\frac{1}{2}}$ under the action (38) and a group conjugate to one of the groups $Z_2^\tau \times SO(3)$, $Z_2^\tau \times O(2)$ and $Z_2^\tau \times D_2$. But all these intersections are themselves isotropy subgroups of the action (38). Hence Σ is an isotropy subgroup of the $Z_2^\tau \times SO(3)^D$ action on $gl(3)^*$.

Conversely, if Σ is the isotropy subgroup of π under the action (38) then it is the isotropy subgroup of $(I, A^{\frac{1}{2}} \pi A^{-\frac{1}{2}})$ under the action (40).

□

Thus the isotropy subgroups for the action of $Z_2^\tau \times_s (SO(3) \times SO(3))$ on $T^*GL(3)^+$ are all conjugate to those shown in Figure 5, with $SO(3)$ identified with $SO(3)^D$. However two subgroups of $Z_2^\tau \times SO(3)^D$ which are not conjugate in $Z_2^\tau \times SO(3)^D$ can be conjugate in $Z_2^\tau \times_s (SO(3) \times SO(3))$. In fact straightforward calculations give the following conjugacies.

Proposition 6.4 *In $Z_2^\tau \times_s (SO(3) \times SO(3))$:*

1. *$Z_2^\tau \times Z_2^D$ is conjugate to $\widetilde{D}_2^D$;*
2. *Z_2^τ is conjugate to $\widetilde{Z}_2^D$.*

□

We can also compute representative elements (Q, P) for each symmetry type. More precisely, the second and third columns of Table 2 show the elements of the form (Q, P) with Q diagonal which are fixed by the symmetry groups in the first column.

Isotropy subgroup	Q	P	Description
$\mathbf{Z}_2^\tau \times SO(3)^D$	diag (a,a,a)	diag (α,α,α)	spherical equilibrium
$\mathbf{Z}_2^\tau \times O(2)^D$	diag (a,a,b)	diag (α,α,β)	spheroidal equilibrium
$\mathbf{Z}_2^\tau \times D_2^D$	diag (a,b,c)	diag (α,β,γ)	ellipsoidal equilibrium
$\widetilde{O(2)}^D$	diag (a,a,b)	$\begin{pmatrix} \alpha & -\delta & 0 \\ \delta & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$	Maclaurin spheroid
$\mathbf{Z}_2^\tau \times \mathbf{Z}_2^D$	diag (a,b,c)	$\begin{pmatrix} \alpha & \delta & 0 \\ \delta & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$	self-adjoint, type S ellipsoid
$\widetilde{D_2}^D$	diag (a,b,c)	$\begin{pmatrix} \alpha & -\delta & 0 \\ \delta & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$	skew-adjoint, type S ellipsoid
$\mathbf{Z}_2^D$	diag (a,b,c)	$\begin{pmatrix} \alpha & \delta & 0 \\ \epsilon & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$	general type S ellipsoid
$\mathbf{Z}_2^\tau$	diag (a,b,c)	$\begin{pmatrix} \alpha & \delta & \epsilon \\ \delta & \beta & \phi \\ \epsilon & \phi & \gamma \end{pmatrix}$	self-adjoint type I ellipsoid
$(\widetilde{\mathbf{Z}_2})^D$	diag (a,b,c)	$\begin{pmatrix} \alpha & -\delta & -\epsilon \\ \delta & \beta & \phi \\ \epsilon & \phi & \gamma \end{pmatrix}$	skew-adjoint type II or III ellipsoid
$\mathbf{1}$	diag (a,b,c)	general matrix	general ellipsoid of type I, II or III.

Table 2: Isotropy subgroups for the action of $\mathbf{Z}_2^\tau \times_s (SO(3) \times SO(3))$ on $T^*GL^+(3)$, their fixed points (Q,P) with Q diagonal, and their corresponding relative equilibria.

6.3 Symmetry Types of Riemann Ellipsoids

Finally we match the isotropy subgroups computed in the previous sections to Riemann's classification of relative equilibria. Every relative equilibrium has symmetry group conjugate to that of a motion of the form:

$$Q(t) = e^{t\Omega_L} A e^{-t\Omega_R}$$

with $A = diag(a,b,c)$. In this case the momentum at $t=0$ is given by:

$$P(0) = \Omega_L A - A\Omega_R$$

$$= \begin{pmatrix} 0 & -b\omega_{L,3} + a\omega_{R,3} & c\omega_{L,2} - a\omega_{R,2} \\ a\omega_{L,3} - b\omega_{R,3} & 0 & -c\omega_{L,1} + b\omega_{R,1} \\ -a\omega_{L,2} + c\omega_{R,2} & b\omega_{L,1} - c\omega_{R,1} & 0 \end{pmatrix} \quad (41)$$

where $\Omega_L = \hat{\omega}_L$ with $\omega_L = (\omega_{L,1}, \omega_{L,2}, \omega_{L,3})$ and similarly for Ω_R. If the relative equilibrium has symmetry group equal to one of those listed in Table 2, then $P(0)$ must have the corresponding form given in the table. We now apply this observation to the different symmetry groups in turn.

$\mathbf{Z_2^\tau \times SO(3)^D}$, $\mathbf{Z_2^\tau \times O(2)^D}$, $\mathbf{Z_2^\tau \times D_2^D}$

In each of these cases, comparing the form given in (41) with that in Table 2 shows that the relative equilibria must be equilibria. In the case of $Z_2^\tau \times D_2^D$, where a, b and c are distinct, this follows immediately from the fact that ω_L and ω_R must be 0. For $Z_2^\tau \times O(2)^D$, with $a = b \neq c$, the comparison shows that $\omega_L = \omega_R = (0, 0, \omega_3)$, say, where ω_3 may be non-zero. However $e^{t\Omega_L} = e^{t\Omega_R}$ commutes with A, and so $Q(t) \equiv A$. Similarly, for $Z_2^\tau \times SO(3)^D$, where $a = b = c$, we also have that $e^{t\Omega_L} = e^{t\Omega_R}$ commutes with A. Thus the relative equilibria with symmetry group conjugate to $Z_2^\tau \times SO(3)^D, Z_2^\tau \times O(2)^D$ and $Z_2^\tau \times D_2^D$, must be, respectively, spherical, spheroidal and ellipsoidal equilibria. Conversely equilibrium solutions will always have symmetry groups conjugate to one of these groups.

$\widetilde{\mathbf{O(2)}}^{\mathbf{D}}$

Comparing (41) with the entry for $\widetilde{O(2)}^D$ in Table 2 we see that relative equilibria with this symmetry group are spheroids, with axis of symmetry parallel to the z-axis, and $\omega_{L,i} = 0 = \omega_{R,i}$ for $i = 2$ and 3. These are spheroids rotating about their axes of symmetry, ie the Maclaurin spheroids described in section 3.4.

$\mathbf{Z_2^\tau \times Z_2^D}$, $\mathbf{\tilde{D}_2^D}$, $\mathbf{Z_2^\tau}$, $\mathbf{Z_2^D}$, $\mathbf{\tilde{Z}_2^D}$, $\mathbf{I}$

If the symmetry group of a relative equilibrium contains Z_2^D then the form of $P(0)$ given in Table 2 implies that $\omega_{L,i} = 0 = \omega_{R,i}$ for $i = 2$ and 3, and so ω_L and ω_R are parallel to a principal axis. Thus a relative equilibrium with symmetry group containing a group conjugate to Z_2^D must be of type S. The converse is also true.

The symmetry group of a trajectory contains Z_2^τ if and only if it is self-adjoint. If it contains the group $\widetilde{Z}_2^D$ then it is skew-adjoint. Every symmetric-adjoint trajectory must have symmetry group conjugate to one of those listed in Table 2 containing an element of the form $(\tau, (g, h))$.

The identification of relative equilibria with the remaining symmetry types now follows easily. Relative equilibria with symmetry group $Z_2^\tau \times Z_2^D$ are precisely the self-adjoint relative equilibria of type S. Since $\widetilde{D}_2^D$ contains both Z_2^D and $\widetilde{Z}_2^D$, this corresponds to the skew-adjoint relative equilibria of type S which are not

Maclaurin spheroids. The isotropy subgroups which are conjugate to $Z_2^\tau \times Z_2^D$ and $\widetilde{D}_2^D$ are symmetry groups of symmetric-adjoint ellipsoids.

Relative equilibria with symmetry groups conjugate to Z_2^D are precisely the S-type ellipsoids which are not symmetric-adjoint. Those with symmetry groups equal to Z_2^τ and $\widetilde{Z}_2^D$ are respectively self-adjoint and skew-adjoint ellipsoids of types *I, II* and *III*, while those with symmetry groups conjugate to these are the corresponding symmetric-adjoint ellipsoids. Finally the relative equilibria with trivial symmetry groups are the ellipsoids of types *I, II* and *III* which are not symmetric-adjoint. These identifications are summarized in the final column of Table 2.

The partial ordering of the set of symmetry groups of trajectories of affine rigid bodies given by inclusion has implications for the possible symmetry changes that can occur in families of relative equilibria. Consider a family of relative equilibria, all of which have the same symmetry group, but which bifurcates from another relative equilibrium with a different symmetry group as a parameter is varied. Elementary considerations show that the symmetry group of the bifurcating relative equilibria must be a subgroup of the symmetry group of the relative equilibrium they are bifurcating from. Thus the inclusions of symmetry groups shown in figure 5 can be used to exclude the possibility of certain bifurcations occurring.

For example the Maclaurin spheroids can bifurcate to skew-adjoint S-type ellipsoids, but not to self-adjoint S-type ellipsoids. From figure 2 and the accompanying discussion it is apparent that the skew-adjoint S-type ellipsoids can indeed be regarded as bifurcating from the Maclaurin spheroids as the parameter e is varied. At first sight it may also appear that the self-adjoint S-type ellipsoids bifurcate in a similar way. However closer inspection shows that they join the family of Maclaurin spheroids at the point where the angular velocities satisfy $\omega_L = \omega_R$ and the Maclaurin spheroid is stationary. These stationary axi-symmetric ellipsoids have symmetry groups conjugate to either $\mathbf{Z_2^\tau \times SO(3)^D}$ or $\mathbf{Z_2^\tau \times O(2)^D}$, from which transitions to the symmetry type $\mathbf{Z_2^\tau \times Z_2^D}$ are allowed.

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