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Integrability and Algebraic Solutions for Planar Polynomial Differential Systems with Emphasis on the Quadratic Systems

Laurent Cairó, Marc R. Feix and Jaume Llibre

Abstract: The paper is divided into two parts. In the first one we present a survey about the theory of Darboux for the integrability of polynomial differential equations. In the second part we apply all mentioned results on Darboux theory to study the integrability of real quadratic systems having an invariant conic. The fact that two intersecting straight lines or two parallel straight lines are particular cases of conics allows us to study simultaneously the integrability of quadratic systems having at least two invariant straight lines.

Key words: Integrability, quadratic differential systems, algebraic solutions.

1. Introduction

By definition a *polynomial system* is a differential system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y), \tag{1}$$

where P and Q are polynomials with coefficients in $\mathbf{F}$, where $\mathbf{F}$ will denote either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system. The polynomial systems of degree 2 will be called *quadratic systems*. In this paper we only consider polynomial systems (1) such that P and Q are relatively prime. In other words, we only consider polynomial systems (1) having finitely many singular points. This work contributes to show the link between the theories of polynomial systems and algebraic curves. Indeed, already in 1878, Darboux [7] showed how the first integrals of polynomial systems possessing sufficient algebraic solutions are constructed (see Darboux Theorem in Section 2). In particular, he proved that if a polynomial system of degree m has at least $m(m+1)/2$ algebraic solutions, then it has a first integral. On the other hand such links were also suggested in 1900 by the way that Hilbert [12] stated his 16th problem, in two parts: the first one about the topology of real algebraic curves and the second one about the maximum number of limit cycles of polynomial systems having a given degree. Recently, this link appeared in the theory of the center for quadratic systems. See the work of Schlomiuk [27, 28, 29], and in particular the Theorem of Schlomiuk–Guckenheimer–Rand, in Section 3, and our Theorem 4. But this link is not restricted to quadratic systems and there is a wide literature on this question in recent years; a general reference is the paper of Pearson, Lloyd and Christopher [22].

In the first part of this paper (Sections 2 and 3) we present a short survey about the Darboux theory of integrability; in the second part (Sections 4 and 5) we apply all mentioned results on Darboux theory to the study of the integrability of real quadratic systems.

· A renewal of interest in Darboux theory, with several improvements, began in recent decades. As a mather of fact the best two improvements to Darboux Theorem are due to Jouanoulou [13], in 1979, and to Prelle and Singer [24], in 1983. The first showing that if the number of algebraic solutions of a polynomial system of degree m is at least $2+[m(m+1)/2]$, then the system has a rational first integral. The second proving that if a polynomial system has an elementary first integral, then the integral can be computed by using the algebraic solutions of the system. Some recent improvements to Darboux theory were made by Chavarriga, Llibre and Sotomayor [4]. Essentially these improvements are based on the fact that suitable singular points reduce the number $m(m+1)/2$ of algebraic solutions necessary for the integrability of the polynomial system. Other recent interesting related results have been published by Christopher [6, 17], Gasull [10], Kooij [17], Zholadek [35], among other authors (see Section 3).

Quadratic systems have been studied intensively during this century, specially in the last thirty years. At this moment more than 1000 papers have been published on this subject (see for instance the bibliographical survey of Reyn [26]). However, few results are known about integrable quadratic systems except, of course, those for the class of quadratic systems having a center (see Kapteyn–Bautin Theorem in Section 2), the class of quadratic Hamiltonian systems (see Artés and Llibre [1]), and few others (see [34]).

Starting with Section 4, we apply the Darboux theory to study the integrability of quadratic systems having an invariant conic, i.e. an invariant algebraic curve of degree 2. The results appear in Section 5, where we obtain the normal forms. These normal forms concern the nine different types of conics, namely: ellipse, complex ellipse, hyperbola, two complex straight lines intersecting in a real point, parabola, two parallel straight lines, two complex straight lines and one double straight line. The fact that two intersecting straight lines or two parallel straight lines are particular cases of conics, allows us to study the integrability of the quadratic systems having at least two invariant algebraic curves of degree 1, i.e. having invariant straight lines. Although our results about the integrability of quadratic system summarized in Theorems 13, 15, 17, 19, 20, 23, 26, 28 and 29, and in Propositions 25 and 27 are not complete, they contribute to partially fill the big gap on this subject.

Roughly speaking we have studied the "generic" Darboux integrability of real quadratic systems having an invariant conic. More explicitly, we characterize the real quadratic systems which have an invariant conic and their integrability is forced by the existence of invariant straight lines. If the integrability of such systems is forced by the existence of algebraic solutions of degree larger than 1, then more restrictions on the coefficients of the quadratic systems are necessary for the existence of the algebraic solution, and consequently the integrability in general is less generic. See Section 2 for a precise definition of Darboux integrability of a polynomial system.

Since the Lotka–Volterra systems have a special interest for their applications to biology and physics, for them we also study the integrability forced by the existence of a second invariant conic (see Theorem 21).

2. The Darboux theory of integrability

In this section we start to study the link between the integrability of a polynomial system and its algebraic solutions. We restrict our attention to polynomial systems in $\mathbf{R}^2$ or $\mathbf{C}^2$, althought the results of this section and many of those of the other sections can be extended easily to polynomial systems in $\mathbf{R}^n$ or $\mathbf{C}^n$. The main contribution in this direction are Darboux results [7].

First we need some preliminary definitions. We denote by

$$D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \tag{2}$$

the differential operator associated to system (1). System (1) is *integrable* on an open subset U of $\mathbf{F}^2$ if there exists a nonconstant analytic function $H : U \to \mathbf{F}$, called a *first integral* of the system on U, which is constant on all solution curves $(x(t), y(t))$ of (1) on U; i.e. $H(x(t), y(t)) =$ constant, for all values of t for which the solution $(x(t), y(t))$ is defined on U. Clearly H is a first integral of system (1) on U if and only if $DH \equiv 0$ on U.

An *invariant algebraic curve* of system (1) is an algebraic curve $f(x, y) = 0$ with $f \in \mathbf{F}[x, y]$, such that for some polynomial $K \in \mathbf{F}[x, y]$ we have

$$Df = Kf, \tag{3}$$

where K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that if (x_0, y_0) is a singular point of system (1), then either $K(x_0, y_0) = 0$, or $f(x_0, y_0) = 0$, and if the polynomial system has degree m, then any cofactor has at most $m - 1$ as degree. We say that the curve $f = 0$ with $f \in \mathbf{F}[x, y]$ is an *algebraic solution* of system (1) if and only if it is an invariant algebraic curve and f is an irreducible polynomial over $\mathbf{F}[x, y]$. The following proposition is easy to prove.

Proposition 1. *Suppose that $f \in \mathbf{F}[x, y]$ and let $f = f_1^{n_1} \cdots f_r^{n_r}$ be the factorization of f in irreducible factors over $\mathbf{F}[x, y]$. If $Df = Kf$ then f_i is a divisor of Df_i in $\mathbf{F}[x, y]$ for all $i = 1, \ldots, r$.*

¿¿From Proposition 1 it follows that if $f = f_1^{n_1} \cdots f_r^{n_r} = 0$ is an invariant algebraic curve of system (1), then $f_i = 0$ is also an invariant algebraic curve of system (1) for every $i = 1, \cdots, r$.

Let U be an open subset of $\mathbf{F}^2$ and let $R : U \to \mathbf{F}$ be an analytic function which is not identically zero on U. The function R is an *integrating factor* of system (1) on U if we have

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \tag{4}$$

or equivalently $\operatorname{div}(RP, RQ) = 0$, or $DR = -R\operatorname{div}(P,Q)$. The *first integral* H *associated to the integrating factor* R is given by $H(x,y) = \int R(x,y)\; P(x,y)dy + f(x)$, satisfying $\frac{\partial H}{\partial x} = -RQ$.

For a proof of the next result see [29] page 438.

Proposition 2. *If system (1) has an integrating factor of the form* $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ *where each* f_i *is a nonconstant polynomial of* $\mathbf{F}[x,y]$ *and* $\lambda_i \in \mathbf{F}\backslash\{0\}$*, then the curves* $f_i = 0$ *are invariant algebraic curves of the system.*

As far as we know, the problem of integrating a polynomial system by using its algebraic solutions was considered for the first time by Darboux in [7]. His main results are summarized in the following theorem.

Darboux Theorem. *Suppose that a polynomial system (1) of degree* m *admits* q *invariant algebraic curves* $f_i = 0$ *with cofactors* K_i*, for* $i = 1, \ldots, q$.

(a) If $q \geq 1 + m(m+1)/2$*, then the function* $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ *for suitable* $\lambda_i \in \mathbf{F}$ *not all zero is a first integral and* $\sum_{i=1}^{q} \lambda_i K_i = 0$.

(b) If $q = m(m+1)/2$*, then the function* $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$*, for suitable* $\lambda_i \in \mathbf{F}$ *not all zero, is a first integral and* $\sum_{i=1}^{q} \lambda_i K_i = 0$*, or an integrating factor and* $\sum_{i=1}^{q} \lambda_i K_i = -\operatorname{div}(P,Q)$.

(c) *If* $q < m(m+1)/2$ *and there exist* $\lambda_i \in \mathbf{F}$ *not all zero such that* $\sum_{i=1}^{q} \lambda_i K_i = 0$*, then* $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ *is a first integral.*

(d) If $q < m(m+1)/2$ *and there exist* $\lambda_i \in \mathbf{F}$ *not all zero such that* $\sum_{i=1}^{q} \lambda_i K_i = -\operatorname{div}(P,Q)$*, then* $R = f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ *is an integrating factor.*

The proof of Darboux Theorem becomes clear looking to the proof of the theorem of Chavarriga–Llibre–Sotomayor given in Section 3.

A first integral of system (1) of the form $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ with $\lambda_i \in \mathbf{F}$ and $f_i \in \mathbf{F}[x,y]$ is called a *Darboux first integral.* An integrating factor of system (1) of the form $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ with $\lambda_i \in \mathbf{F}$ and $f_i \in \mathbf{F}[x,y]$ is called a *Darboux integrating factor.* We say that system (1) is *Darboux integrable* if it has a Darboux first integral or a Darboux integrating factor.

If among the invariant algebraic curves a complex conjugate pair $f = 0$ and $\bar{f} = 0$ occurs, then the first integral will have a factor of the form $f^{\lambda} \bar{f}^{\bar{\lambda}}$, which is just the real valued function $[(\mathrm{Re} f)^2 + (\mathrm{Im} f)^2]^{\mathrm{Re}\,\lambda} \exp(-2\,\mathrm{Im}\,\lambda \arctan(\mathrm{Im}\, f/\mathrm{Re}\, f))$. We note that if the polynomial system (1) is real, and we have a complex invariant curve $f = 0$, then we also have its conjugate $\bar{f} = 0$ as an invariant curve.

If a polynomial system has sufficient algebraic solutions, the theory of Darboux not only allows to obtain first integrals, but it also works for obtaining invariants. Roughly speaking, with a first integral we can describe completely the phase portrait of the polynomial system, while with an invariant we only can describe its asymptotic behaviour. A nonconstant analytic function I in the variables

x, y and t such that $I(x(t), y(t), t)$ is constant for all solution curves $(x(t), y(t))$ of system (1) in an open subset U of $\mathbf{F}^2$, is a *Darboux invariant* or simply an *invariant* of the system on U. The next proposition shows us how to obtain an invariant of system (1) knowing sufficient algebraic solutions of the system.

Proposition 3. *Suppose that a polynomial system (1) of degree m admits q algebraic solutions $f_i = 0$ with cofactors K_i for $i = 1, \ldots, q$. If there exist $\mu_i \in \mathbf{F}$ not all zero such that $\sum_{i=1}^{q} \mu_i K_i = -s$ with $s \in \mathbf{F}$, then the function $f_i^{\mu_1} \cdots f_q^{\mu_q} e^{st}$ is an invariant of system (1).*

Proof. In order to show that $f_i^{\mu_1} \cdots f_q^{\mu_q} e^{st}$ is an invariant, it is sufficient to see that $D(f_i^{\mu_1} \cdots f_q^{\mu_q} e^{st}) \equiv 0$, where D is the differential operator defined in (2). Thus, we have

$$D(f_i^{\mu_1} \cdots f_q^{\mu_q} e^{st}) = (f_i^{\mu_1} \cdots f_q^{\mu_q} e^{st})(\sum_{i=1}^{q} \mu_i \frac{Df_i}{f_i} + s)$$

$$= (f_i^{\mu_1} \cdots f_q^{\mu_q} e^{st})(\sum_{i=1}^{q} \mu_i K_i + s) \equiv 0.$$

■

In the rest of this section we present applications to quadratic system of each statement of Darboux Theorem.

Example 1. If $abc \neq 0$ then the real quadratic system

$$\dot{x} = x(ax + c), \qquad \dot{y} = y(2ax + by + c), \tag{5}$$

has exactly the following five invariant straight lines (i.e. algebraic solutions of degree 1): $f_1 = x = 0$, $f_2 = ax + c = 0$, $f_3 = y = 0$, $f_4 = ax + by = 0$, $f_5 = ax + by + c = 0$. Then, by Darboux Theorem (a) we know that system (4) must have a first integral of the form $H = f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} f_4^{\lambda_4} f_5^{\lambda_5}$ with $\lambda_i \in \mathbf{F}$ satisfying $\sum_{i=1}^{5} \lambda_i K_i = 0$, where K_i is the cofactor of f_i. An easy computation shows that $K_1 = ax + c, K_2 = ax, K_3 = 2ax + by + c, K_4 = ax + by + c$ and $K_5 = ax + by$. Then, a solution of $\sum_{i=1}^{5} \lambda_i K_i = 0$ is $\lambda_1 = \lambda_5 = -1$, $\lambda_2 = \lambda_4 = 1$ and $\lambda_3 = 0$. Therefore a first integral of system (5) is

$$H = \frac{(ax + c)(ax + by)}{x(ax + by + c)}.$$

■

Example 2. If $r_{12} = cB(A - a) + aC(b - B) = 0$, and $a^2C^2 + c^2B^2 \neq 0$, then the real quadratic system

$$\dot{x} = x(ax + by + c), \qquad \dot{y} = y(Ax + By + C), \tag{6}$$

has the following three invariant straight lines: $f_1 = x = 0$, $f_2 = y = 0$, $f_3 = aCx + cBy + cC = 0$. Then, by Darboux Theorem (b) we know that the function $H = f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a first integral if there exists $\lambda_i \in \mathbf{F}$ not all zero such that $\sum_{i=1}^{3} \lambda_i K_i = 0$, where K_i is the cofactor of f_i. An easy computation shows that $K_1 = ax + by + c$, $K_2 = Ax + By + C$ and $K_3 = ax + By$. Then a solution of $\sum_{i=1}^{3} \lambda_i K_i = 0$ is $\lambda_1 = (A-a)B$, $\lambda_2 = a(b-B)$ and $\lambda_3 = aB - bA$. Hence a first integral of system (6) is

$$H = x^{(A-a)B} y^{a(b-B)} \left(aCx + cBy + cC\right)^{aB-bA}.$$

■

Example 3. If $a_{02} \neq 0$ then the real quadratic system

$$\dot{x} = x^2 - 1 = P, \quad \dot{y} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 = Q, \tag{7}$$

with $a_{00} = (2a_{11} + a_{01}^2 - 1)/(4a_{02})$, $a_{10} = a_{01}a_{11}/(2a_{02})$ and $a_{20} = a_{11}(a_{11} - 2)/(4a_{02})$, has the following three algebraic solutions: two straight lines $f_1 = x+1$, $f_2 = x - 1$, and one hyperbola

$$f_3 = \frac{a_{11}(a_{11}-2)}{4a_{02}}x^2 + (a_{11}-1)xy + a_{02}y^2 + \frac{a_{01}(a_{11}-1)}{2a_{02}}x + a_{01}y + \frac{a_{01}^2+1}{4a_{02}} = 0.$$

Their cofactors are $K_1 = x - 1$, $K_2 = x + 1$ and $K_3 = (a_{11}+1)x + 2a_{02}y + a_{01}$, respectively. Since $\sum_{i=1}^{3} \lambda_i K_i = -\mathrm{div}(P,Q)$ for $\lambda_1 = \lambda_2 = -1/2$, and $\lambda_3 = -1$, from Darboux Theorem (b) it follows that $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor. By computing its associated first integral we obtain

$$H = -2\mathrm{arctanh}\left[\frac{(a_{11}-1)x + 2a_{02}y + a_{01}}{(x^2-1)^{1/2}}\right] - \ln\left|x + (x^2-1)^{1/2}\right|.$$

■

Example 4. If $a \neq 0$ the quadratic system

$$\dot{x} = -y(ay+b) - (x^2+y^2-1), \qquad \dot{y} = x(ay+b), \tag{8}$$

has the algebraic solutions $f_1 = ay + b = 0$ with cofactor $K_1 = ax$, and $f_2 = x^2 + y^2 - 1 = 0$ with cofactor $K_2 = -2x$. Since $2K_1 + aK_2 = 0$, by Darboux Theorem (c) we have that $H = (ay+b)^2(x^2+y^2-1)^a$ is a first integral of system (8). ■

Example 5. The real quadratic system

$$\dot{x} = -y - b(x^2+y^2) = P, \qquad \dot{y} = x = Q, \tag{9}$$

has the invariant algebraic curve $f_1 = x^2 + y^2$ with cofactor $K_1 = -2bx$. Since $K_1 = \mathrm{div}(P,Q)$, from Darboux Theorem (d), it follows that f_1^{-1} is an integrating factor. Then an easy computation shows that $H = \exp(2by)(x^2 + y^2)$ is a first integral of system (9). ∎

An interesting application of Darboux Theorem (d) allows us to present a new and shorter proof of the sufficient conditions for the classification theorem of centers of quadratic systems due to Kapteyn [14, 15] and Bautin [2]. The first proof of this fact was due to Kapteyn in the 1910s. Recently Schlomiuk [28] gave a generic proof based also on the algebraic solutions (see later). A first step in order to get a different generic proof based on the algebraic solutions and on integrating factors appears in the Ph. D. Thesis of Giné [11].

Kapteyn–Bautin Theorem. *Any quadratic system candidate to have a center can be written in the form*

$$\dot{x} = -y - bx^2 - Cxy - dy^2, \qquad \dot{y} = x + ax^2 + Axy - ay^2. \tag{10}$$

This system has a center at the origin if and only if one of the following conditions holds

(KB_I) $A - 2b = C + 2a = 0$,

(KB_{II}) $C = a = 0$,

(KB_{III}) $b + d = 0$,

(KB_{IV}) $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$.

The following result gives a very short proof of the sufficient conditions of Kapteyn–Bautin Theorem.

Theorem 4. *If system (10) satisfies one of the four conditions of the Kapteyn–Bautin Theorem, then it has a center at the origin.*

Proof. Since system (10) has a linear center at the origin, to prove that system (10) satisfying one of the four conditions of the Kapteyn–Bautin Theorem has a center at the origin, it is sufficient to show that it has a first integral in a neighbourhood of the origin.

Assume that system (10) *satisfies condition* (KB_I). Then it is easy to check that the system is Hamiltonian, i.e. $\dot{x} = -\partial H/\partial y$, $\dot{y} = \partial H/\partial x$ with $H = \frac{1}{2}\left(x^2 + y^2\right) + \frac{a}{3}x^3 + bx^2y - axy^2 + \frac{d}{3}y^3$. Therefore H is a first integral defined in a neighbourhood of the origin.

Suppose that system (10) *satisfies condition* (KB_{II}). Then the system can be written in the form

$$\dot{x} = -y - bx^2 - dy^2, \qquad \dot{y} = x + Axy.$$

If $A \neq 0$ this system has the invariant straight line $f_1 = 1 + Ay = 0$ with cofactor $K_1 = Ax$. The divergence of the system is $(A - 2b)x$. Then, if $A(A - 2b) \neq 0$ we have the divergence of the system is equal to $\left(1 - \frac{2b}{A}\right) K_1$. Hence, by Darboux Theorem (d) we obtain that $(1 + Ay)^{2b/A-1}$ is an integrating factor of system (10). Since this integrating factor is not zero at the origin, the first integral is defined in a neighbourhood of the origin, and consequently the origin is a center.

We can assume that $A - 2b \neq 0$, otherwise we would be under the assumptions of condition (KB_I). So, it remains only to study the case $A = 0$ and $b \neq 0$. Then, the system becomes $\dot{x} = -y - bx^2 - dy^2$, $\dot{y} = x$. This system has the algebraic solution $f_1 = 2b^2(bx^2 + dy^2) + (b - d)(2by - 1) = 0$ with cofactor $K_1 = -2bx$, which is equal to the divergence of the system. Therefore, by Darboux Theorem (d) we obtain that f_1^{-1} is an integrating factor. Hence the first integral associated to this integrating factor is defined at the origin if $b - d \neq 0$, and consequently the origin is a center.

Now we suppose that in addition $b - d = 0$. Then the system goes over to $\dot{x} = -y - b(x^2 + y^2)$, $\dot{y} = x$. By Example 5 we know that $H = \exp(2by)(x^2 + y^2)$ is a first integral, which is defined at the origin, and therefore the origin must be a center.

Assume that system (10) *satisfies condition* (KB_{III}). As Frommer observed in [9] (see also [28]) the form of system (10) with $b + d = 0$ is preserved under a rotation of axes. After performing a rotation of axes of an angle θ, the new coefficient a' of x^2 in the second equation of system (8) becomes of the form $a' = a\cos^3\theta + \alpha\cos^2\theta\sin\theta + \beta\cos\theta\sin^2\theta + d\sin^3\theta$. Therefore, if $a \neq 0$ we can find θ such that $a' = 0$. So we can assume that $a = 0$, and consequently $C \neq 0$; otherwise we would be under the assumptions of condition (KB_{II}).

The system $\dot{x} = -y - bx^2 - Cxy + by^2$, $\dot{y} = x + Axy$, has the algebraic solutions $f_1 = 1 + Ay = 0$ if $A \neq 0$ with cofactor $K_1 = Ax$, and $f_2 = (1 - by)^2 + C(1 - by)x - b(A + b)x^2 = 0$ with cofactor $K_2 = -2bx - Cy$. Since the divergence of the system is equal to $K_1 + K_2$, by Darboux Theorem (d) we obtain that $f_1^{-1}f_2^{-1}$ is an integrating factor. Hence, again the first integral associated to the integrating factor is defined at the origin, and consequently the origin is a center.

We remark that if $A = 0$ then f_1 is not an algebraic solution of the system, but then the divergence of the system is equal to K_2 and the integrating factor of the system is f_2^{-1}, and using the same arguments we obtain that the origin is a center.

Suppose that system (10) *satisfies condition* (K_{IV}). Then, if $d \neq 0$ the system becomes

$$\dot{x} = -y + \frac{a^2 + 2d^2}{d}x^2 + 2axy - dy^2, \qquad \dot{y} = x + ax^2 + \frac{3a^2 + d^2}{d}xy - ay^2.$$

We note that if $d = 0$ then we are under the assumptions of condition K_{II}. This system has the algebraic solution $f_1 = (a^2 + d^2)\left[(dy - ax)^2 + 2dy\right] + d^2 = 0$ with cofactor $K_1 = 2(a^2 + d^2)x/d$. Therefore the divergence of the system is equal to $\frac{5}{2}K_1$. Hence, by Darboux Theorem (d) the function $f_1^{-5/2}$ is an integrating factor of the system. Since $d \neq 0$, its associated first integral is defined in a neighbourhood of the origin, and consequently the origin is a center. ∎

3. Improvements and topics related with Darboux theory

Let us present first some results on the cases not covered by Darboux Theorem due to Chavarriga, Llibre and Sotomayor [4]. In order to be more precise we need some notation and definitions.

If $S(x,y) = \sum_{i+j=0}^{m-1} a_{ij}x^i y^j$ is a polynomial of degree $m-1$ with $m(m+1)/2$ coefficients in $\mathbf{F}$, then we write $S \in \mathbf{F}_{m-1}[x,y]$. We identify the linear vector space $\mathbf{F}_{m-1}[x,y]$ with $\mathbf{F}^{m(m+1)/2}$ through the isomorphism

$$S \to (a_{00}, a_{10}, a_{01}, \ldots, a_{m-1,0}, a_{m-2,1}, \ldots, a_{0,m-1}).$$

We say that p points $(x_k, y_k), k = 1, \ldots, p$, are *independent* with respect to $\mathbf{F}_{m-1}[x,y]$ if the intersection of the p hyperplanes

$$\sum_{i+j=0}^{m-1} x_k^i y_k^j a_{ij} = 0, \qquad k = 1, \ldots, p,$$

in $\mathbf{F}^{m(m+1)/2}$ is a linear subspace of dimension $[m(m+1)/2] - p$.

We remark that the maximum number of isolated singular points of system (1) is m^2 (by Bezout Theorem), and that the maximum number of independent isolated singular points of system (1) is $m(m+1)/2$ and that $m(m+1)/2 < m^2$ for $m \geq 2$.

A singular point (x_0, y_0) of system (1) is called *weak* if the divergence, $\operatorname{div}(P,Q)$, of system (1) at (x_0, y_0) is zero.

The next result [4] improves statements (a) and (b) of Darboux Theorem when the polynomial system has independent singular points, or weak independent singular points. Since the statements of this theorem are slightly different from the corresponding ones of [4] we shall prove it.

Chavarriga–Llibre–Sotomayor Theorem (CLS Theorem). *Suppose that a polynomial system (1) of degree m admits q invariant algebraic curves $f_i = 0$ with cofactors K_i, for $i = 1, \ldots, q$, and p independent singular points (x_k, y_k), for $k = 1, \ldots, p$ such that $f_i(x_k, y_k) \neq 0$.*

(a) If $q = [m(m+1)/2] + 1 - p > 0$, then system (1) has a first integral of the form $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ where $\lambda_i \in \mathbf{F}$ are not all zero and $\sum_{i=1}^{q} \lambda_i K_i = 0$.

(b) If $q = [m(m+1)/2] - p > 0$ and the p independent singular points are weak, then the function $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ for suitable $\lambda_i \in \mathbf{F}$ not all zero is a first integral and $\sum_{i=1}^{q} \lambda_i K_i = 0$, or it is an integrating factor and $\sum_{i=1}^{q} \lambda_i K_i = -\operatorname{div}(P,Q)$.

Proof. By hypothesis we have q algebraic solutions $f_i = 0, i = 1, \ldots, q$, of system (1). That is, the f_i's are polynomials such that $Df_i = K_i f_i$ with K_i the cofactor polynomial of degree $m-1$, or equivalently $K_i \in \mathbf{F}_{m-1}[x,y]$. We note that the dimension of $\mathbf{F}_{m-1}[x,y]$ as a vector space over $\mathbf{F}$ is $1+2+\ldots+m = m(m+1)/2$.

Since (x_k, y_k) is a singular point, $P(x_k, y_k) = Q(x_k, y_k) = 0$. Then, from $Df_i = P[\partial f_i/\partial x] + Q[\partial f_i/\partial y]$, it follows that $K_i(x_k, y_k) f_i(x_k, y_k) = 0$. By assumption $f_i(x_k, y_k) \neq 0$, therefore $K_i(x_k, y_k) = 0$ for $i = 1, \ldots, p$. Consequently, since the p singular points are independent, all the polynomials K_i belong to a linear subspace S of $\mathbf{F}_{m-1}[x,y]$ of dimension $[m(m+1)/2] - p$.

Let us prove statement (a). Since $q = [m(m+1)/2]+1-p > [m(m+1)/2]-p$, we obtain that the q polynomials K_i must be linearly dependent on S. So, there are $\lambda_i \in \mathbf{F}$ not all zero such that $\sum\limits_{i=1}^{q} \lambda_i K_i = 0$. Therefore, we obtain

$$D(f_1^{\lambda_1} \cdots f_q^{\lambda_q}) = (f_1^{\lambda_1} \cdots f_q^{\lambda_q}) \left(\sum_{i=1}^{q} \lambda_i \frac{Df_i}{f_i} \right) = (f_1^{\lambda_1} \cdots f_q^{\lambda_q}) \left(\sum_{i=1}^{q} \lambda_i K_i \right) \equiv 0,$$

i.e. $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ is a first integral of system (1), proving the statement (a).

To prove statement (b), let $K = \operatorname{div}(P, Q)$, clearly $K \in \mathbf{F}_{m-1}[x, y]$. Since the singular points (x_k, y_k) are weak, $K(x_k, y_k) = 0$ for $k = 1, \ldots, p$. So K belongs to the linear subspace S.

On the other hand, since $\dim S = q = [m(m+1)/2] - p$ and we have $q+1$ polynomials $K_1, \ldots, K_q$, K in S, it follows that $K_1, \ldots, K_q$, K are linearly dependent on S. Therefore, we obtain $\lambda_i \in \mathbf{F}$ and $\lambda \in \mathbf{F}$ not all zero such that $\left(\sum\limits_{i=1}^{q} \lambda_i K_i \right) + \lambda K = 0$.

If $\lambda = 0$ then, as in the proof of statement (a), we obtain that $f_1^{\lambda_1} \cdots f_q^{\lambda_q}$ is a first integral of system (1).

If we assume now that $\lambda \neq 0$, and if moreover $\mu_i = \lambda_i / \lambda$, then we have $K = -\sum\limits_{i=1}^{q} \mu_i K_i$. Therefore, we obtain

$$D(f_1^{\mu_1} \cdots f_q^{\mu_q}) = (f_1^{\mu_1} \cdots f_q^{\mu_q}) \left(\sum_{i=1}^{q} \mu_i K_i \right) =$$

$$= -(f_1^{\mu_1} \cdots f_q^{\mu_q}) K = -(f_1^{\mu_1} \cdots f_q^{\mu_q}) \operatorname{div}(P, Q),$$

i.e. $f_1^{\mu_1} \cdots f_q^{\mu_q}$ is an integrating factor of system (1). ∎

The next three propositions are different applications of the CLS Theorem to quadratic systems. The space $\mathbf{F}^{m(m+1)/2}$ is now $\mathbf{F}^3$ and its elements are (a_{00}, a_{10}, a_{01}). The plane associated to a point (x_i, y_i) is now $a_{00} + x_i a_{10} + y_i a_{01} = 0$. Therefore, since the vector $(1, x_i, y_i) \neq (0, 0, 0)$ we have that a unique point is always independent. Since the vectors $(1, x_i, y_i), (1, x_j, y_j)$ are independent if the points $(x_i, y_i) \neq (x_j, y_j)$, two points are always independent. Finally, since the three points $(1, x_i, y_i), (1, x_j, y_j), (1, x_k, y_k)$ are never collinear if the points $(x_i, y_i), (x_j, y_j), (x_k, y_k)$ are isolated singular points of the quadratic system (see the proof of Lemma 11.1 of [34]), it follows that three isolated singular points of any quadratic system are always independent. Of course, since the dimension of $\mathbf{F}^3$ is 3, four points are always dependent. In short, we have proved the following result.

Proposition 5. *One, two or three isolated singular points of a quadratic system are always independent with respect to* $\mathbf{F}_1[x, y]$. For the quadratic systems (1) the divergence vanishes on a straight line. Since no more than two isolated singular points of a quadratic system can be collinear (see again the proof of Lemma 11.1 of [34]), we have the following well–known result that will be used later on.

Proposition 6. *A quadratic system has at most two weak isolated singular points.*

The next proposition presents an application of statement (a) of CLS Theorem. Any real quadratic system which has as invariant algebraic curves two complex straight lines intersecting in a real point can be written, after an affine change of coordinates in the form (11) (see Proposition 12 for more details). In the next proposition we will use the fact that the invariant algebraic curve $x^2+y^2=0$ gives two algebraic solutions in $\mathbf{C}^2$.

Proposition 7. *Consider the quadratic system*

$$\begin{array}{rcl} \dot{x} & = & \frac{a}{2}\left(x^2+y^2\right)+\frac{c}{2}x+2y\left(px+qy+r\right)=P(x,y), \\ \dot{y} & = & \frac{b}{2}\left(x^2+y^2\right)+\frac{c}{2}y-2x\left(px+qy+r\right)=Q(x,y). \end{array} \tag{11}$$

If $cp+4rq=0$ and $(c^2+r^2)(p^2+q^2)\neq 0$, then the system has 3 singular points and a Darboux first integral

$$H=(x^2+y^2)^{2r}\exp\left(c\arctan\left(\frac{y}{x}\right)\right).$$

Otherwise it has at most 2 singular points or a straight line of singular points.

Proof. To study the singular points of system (11) we compute the resultant of the polynomials P and Q with respect to the variable y, which turns out to be a polynomial of the form $xT(x)=x(A+Bx+Cx^2+Dx^3)$. By the properties of the resultant, we know that if (x_0,y_0) is a singular point of system (11), then x_0 is a root of the resultant. We define

$$\omega=27A^2D^2+2(2B^2-9AC)BD+(4AC-B^2)C^2.$$

Then ω is equal to

$$64(cp+4rq)^2\left[b^2+(a+4q)^2\right]^2[16(cq-br)^2+$$

$$+16(cp-ar)^2+c^2(a^2+b^2+8aq-8bp)]^2.$$

It is well known for a polynomial $T(x)$ of degree 3 ($D\neq 0$) that $T(x)$ has a unique simple real root if $\omega>0$, one simple real root and one double root, or a triple real root if $\omega=0$, and three simple real roots if $\omega<0$.

If $cp+4rq=0$ and $(c^2+r^2)(p^2+q^2)\neq 0$, then $\omega=0$ and working with the roots of $T(x)$, the resultant has 3 real roots, and one of them is double. Otherwise $\omega>0$ and the resultant has at most 2 real roots, or it is identically zero. Then the statements of the proposition with respect to singular points are proved.

It is easy to check that $x^2+y^2=0$ is an invariant algebraic solution of system (11). By Proposition 1, $f_1=x+yi=0$ and $\bar{f}_2=x-yi=0$ are two algebraic solutions of system (11) in $\mathbf{C}^2$. If $cp+4rq=0$ and $(c^2+r^2)(p^2+q^2)\neq 0$, then system (11) has three real singular points, one of them being the origin. The other two are not contained in the algebraic curves $x + yi = 0$ and $x - yi = 0$. So, by applying CLS Theorem (a) we obtain that $H = f_1^{\lambda_1} \bar{f}_2^{\bar{\lambda}_1}$ is a Darboux first integral of system (11) with $\lambda_1 K_1 + \lambda_2 K_2 = 0$, being K_1 the cofactor of f_1. Hence the proposition is proved. ■

Before presenting another application of CLS Theorem, let us recall the theorem due to Schlomiuk, Guckenheimer and Rand [27] concerning the Kapteyn–Bautin Theorem. We denote the points of $\mathbf{R}^5$ as (a, b, d, A, C) and we define the following submanifolds of $\mathbf{R}^5$

$$\begin{aligned} M_I &= \{(a,b,d,A,C) \in \mathbf{R}^5 : A - 2b = C + 2a = 0\}, \\ M_{II} &= \{(a,b,d,A,C) \in \mathbf{R}^5 : C = a = 0\}, \\ M_{III} &= \{(a,b,d,A,C) \in \mathbf{R}^5 : b + d = 0\}, \\ M_{IV} &= \{(a,b,d,A,C) \in \mathbf{R}^5 : C + 2a = A + 3b + 5d = a^2 + bd + 2d^2\}. \end{aligned}$$

According to the Kapteyn-Bautin Theorem the quadratic systems (10) have a center at the origin if and only if $(a, b, d, A, C) \in M_I \cup M_{II} \cup M_{III} \cup M_{IV}$.

We recall (see Section 2) that systems (10) in M_I are Hamiltonian with H equal to a cubic polynomial. Therefore all solutions of these systems are algebraic.

We say that a system (10) in M_{II} satisfies property P_{II} if it has as invariant algebraic curve a real straight line and a real conic solution irreducible over $\mathbf{C}$. Both invariant algebraic curves do not pass through the origin.

We say that a system (10) in M_{III} satisfies property P_{III} if it has as invariant algebraic curve a real straight line L and a real conic which is not the line L with multiplicity 2. Both invariant algebraic curves do not pass through the origin.

We say that a system (10) in M_{IV} satisfies property P_{IV} if it has as invariant algebraic curve a parabola and an irreducible cubic curve. Both invariant algebraic curves do not pass through the origin.

Schlomiuk–Guckenheimer–Rand Theorem. *The property P_i is generic for the real quadratic systems of M_i for $i = II, III, IV$.*

For a proof of this theorem see [27] or [28].

As an application of CLS Theorem (b) we have the following corollary of the Schlomiuk–Guckenheimer–Rand Theorem, which is a particular case of Theorem 3.

Corollary 8. *For $i = II, III, IV$ the real quadratic systems of M_i generically have a center at the origin.*

Proof. From the Schlomiuk–Guckenheimer–Rand Theorem we have that, generically, the systems of M_i have two algebraic solutions which do not pass through the origin. Since the origin is a weak focus, from CLS Theorem (b) with $q = 2$ and $p = 1$, it follows that such systems have a first integral which is defined in a neighborhood of the origin. Hence, the origin is a center. ■

Jouanolou [13] proved that if the number of algebraic solutions is $q \geq 2 + [m(m+1)/2]$, then the exponents λ_i in the statement of Darboux Theorem may be chosen to be integers and hence we have a rational first integral in this case. Following [29] we state Jouanolou Theorem in Singer's formulation [30].

Jouanoulou Theorem. *If a polynomial system (1) of degree m admits q algebraic solutions $f_i = 0, i = 1, \ldots, q$, being $f_1, \ldots, f_q$ relatively prime in $\mathbf{C}[x, y]$, then one and only one of the following statements hold.*
(a) $q < 2 + [m(m+1)/2]$.
(b) There exist integers n_i not all zero such that $f_1^{n_1} \cdots f_q^{n_q}$ is a rational first integral. In this case, if $f = 0$ is any algebraic solution, then either there exist $c_1, c_2 \in \mathbf{C}$ not both zero such that f divides $c_1 \prod_{i \in I} f_i^{n_i} - c_2 \prod_{j \in J} f_j^{|n_j|}$ where $I = \{i : n_i \geq 0\}$ and $J = \{j : n_j < 0\}$, or f divides the greatest common divisor of P and Q.

We note that Example 1 of Section 2 has a rational first integral because for $m = 2$ we have $2 + [m(m+1)/2] = 5$, and the system of this example has 5 algebraic solutions.

From the Theorems of Darboux and Jouanolou it follows easily the next result (see also [29]).

Corollary 9. *For a polynomial system (1) of degree m, one and only one of the following statements hold.*
(a) System (1) has a finite number $q < 2 + [m(m+1)/2]$ of algebraic solutions.
(b) System (1) has an infinite number of algebraic solutions and admits a rational first integral of the form $f_1^{n_1} \cdots f_q^{n_q}$ where $f_i = 0$ for $i = 1, \ldots,, q$ are algebraic solutions with f_i and f_j relatively prime if $i \neq j, q \geq 2 + [m(m+1)/2]$, and the n_i are integers not all zero. Moreover, the degree of every algebraic solution of system (1) is bounded by

$$N = \max \left\{ \sum_{n_i > 0} n_i \deg f_i, \sum_{n_i < 0} |n_i| \deg f_i \right\}.$$

Note that in both cases of Corollary 9 we can obtain a natural number N which bounds the degrees of all algebraic solutions of a given polynomial system (1). We remark that the degrees of all invariant algebraic curves of a given polynomial system (1) is not bounded. This is due to the fact that if a polynomial system (1) has an invariant algebraic curve $f = 0$, then also $f^n = 0$ is an invariant algebraic curve for all $n \in \mathbf{N}$.

One natural open question due to Prelle and Singer [24] (see also [29] and [4]) is the following.

Open Question 10. *Give an effective procedure to find an upper bound N for the degrees of the algebraic solutions of a given polynomial system (1) with degree $m \geq 2$.*

For a fixed degree $m \geq 2$ does not exist an upper bound $N(m)$ for the degrees of the algebraic solutions of all polynomial systems (1) of degree m as the following example shows for $m = 2$. We consider the system

$$\dot{x} = y - \frac{2}{n} xy, \qquad \dot{y} = -x + \frac{n+2}{n}(x^2 - y^2), \tag{12}$$

which has the first integral

$$H = \left[\left(1 - \frac{n+2}{n} x \right)^2 - \frac{n+2}{n} y^2 \right] \left(1 - \frac{2}{n} x \right)^{-n-2}.$$

(See for more details expressions (29) and (31) of [21]). Therefore, system (12) has algebraic solutions of degree $n+2$.

We do not know examples of polynomial systems with algebraic solutions solutions of arbitrary degree which are not Darboux integrable. So another natural question is:

Open Question 11. *For a fixed degree $m \geq 2$ prove the existence of an upper bound $N(m)$ for the degrees of the algebraic solutions of all polynomial systems (1) of degree m which are not Darboux integrable.*

In the case $m = 2$ we know that $N(2) \geq 4$, see [32]. This last open question is essentially due to Poincaré [23].

The next theorem due to Prelle-Singer [24] concerns the *elementary first integrals.* Roughly speaking an elementary first integral is a first integral expressible in terms of exponentials, logarithms and algebraic functions. The notion of elementary function of one variable is due to Liouville who, between 1833 and 1841, used it in the theory of integration. Elementary functions of two variables are defined by starting with the field of rational functions in two variables $\mathbf{C}[x,y]$ and using extension fields but with two commuting derivations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. For more details see [24]. Of course, Darboux first integrals are elementary functions.

Prelle-Singer Theorem. *If the system (1) has an elementary first integral, then it has an integrating factor of the form $f_1^{n_1} \cdots f_q^{n_q}$ with $f_i \in \mathbf{C}\,[x,y]$ and $n_i \in \mathbf{Z}$ and each $f_i = 0$ is an algebraic solution.*

We remark that this theorem says that if a polynomial system (1) has an elementary first integral, then this integral can be computed by using the algebraic solutions of the system.

The next theorem is due to Christopher, see [6] or [17]. Recently Zholadek [35] rediscovered it. The theorem shows that for the integrability of a polynomial system (1) of degree m we do not need many algebraic solutions; when these solutions are in generic position, it is enough that the sum of their degrees be $m+1$.

Christopher Theorem. *Let $f_i = 0$ for $i = 1, \ldots, q$ be q irreducible algebraic curves in $\mathbf{C}^2$, and let $k = \sum_{i=1}^{q} \deg f_i$. We assume*

(i) there are no points at which f_i and its first derivatives all vanish,

(ii) the highest order terms of f_i have no repeated factors,

(iii) no more than two curves meet at any point in the finite plane and are not tangent at these points,

(iv) no two curves have a common factor in their highest order terms,

then any polynomial vector field X of degree m tangent to all $f_i = 0$ is of the form described below.

(a) If $m > k-1$ *then*

$$X = Y\left(\prod_{i=1}^{q} f_i\right) + \sum_{i=1}^{q} h_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{q} f_j\right) X_{f_i}, \tag{13}$$

where $X_{f_i} = (-\partial f_i/\partial y, \partial f_i/\partial x)$ *is a Hamiltonian vector field, the* h_i *are polynomials of degree* $\leq m-k+1$ *and* Y *is a polynomial vector field of degree* $\leq m-k$.
(b) If $m = k-1$ *then*

$$X = \sum_{i=1}^{q} \alpha_i \left(\prod_{\substack{j=1 \\ j\neq i}}^{q} f_j\right) X_{f_i},$$

with $\alpha_i \in \mathbf{C}$. *In this case a Darboux first integral exists.*
(c) If $m < k-1$ *then* $X \equiv 0$.

We note that if $f = 0$ is an irreducible algebraic curve of degree $m+1$ in $\mathbf{C}^2$ satisfying assumptions (i) and (ii) of the previous theorem, then there is a unique vector field of degree m (modulus a multiplicative constant) tangent to $f = 0$, namely $X_f = (-\partial f/\partial y, \partial f/\partial x)$.

Kooij and Christopher in [17] improved statement (b) of the previous theorem.
Kooij–Christopher Proposition. *Under the assumptions of statement (b) of Christopher Theorem it follows that the polynomial system (1) has an integrating factor of the form* $(f_1 \cdots f_q)^{-1}$ *and a first integral of the form* $f_1^{\alpha_1} \cdots f_n^{\alpha_n}$, *where the constants* α_i *are as in statement (b) of Christopher Theorem.*

The next theorem due to Zholadek [35] extends Christopher Theorem to the case where the algebraic curves $f_i = 0$ have singular points, i.e. points (x_0, y_0) where $f_i(x_0,y_0) = (\partial f_i/\partial x)(x_0,y_0) = (\partial f_i/\partial y)(x_0,y_0) = 0$.
Zholadek Theorem. *Let* $f_i = 0$, *for* $i = 1,\ldots,q$, *be* q *irreducible algebraic curves in* $\mathbf{C}^2$ *satisfying the conditions (ii)-(iv) from the previous theorem and*
(i') all the singular points of $f_i = 0$ *are double points (transversal self–intersections), then any polynomial vector field* X *of degree* m *tangent to all* $f_i = 0$ *is of one of the forms described below.*
Let $D(x,y)$ *be a polynomial of degree d such that the curve* $D = 0$ *goes through the singular points of all* $f_i = 0$ *and that* D *and* f_i *have no common factors in their highest order terms and let* $k = \left(\sum_{i=1}^{q} \deg f_i\right)$, *then any polynomial vector field* X *of degree* m *tangent to all* $f_i = 0$ *is of the one of the forms described below*
(a) If $m > k-d-1$, *then formula (13) holds but with* Y/D *instead of* Y, *and* h_i/D *instead of* h_i.
(b) If $m = k-d-1$ *then statement (b) of the previous theorem divided by* D *holds.*
(c) If $m < k-d-1$ *then* $X \equiv 0$.

We note that the denominator D of the previous theorem is not fixed.

The previous two theorems give us the expression of the polynomial systems which have a given generic set of irreducible algebraic curves as solutions. The next result due to Gasull [10] gives a characterization of all polynomial systems which have a given set of irreducible invariant algebraic curves.

Gasull Lemma. *Suppose that a polynomial system (1) admits q algebraic solutions $f_i = 0$ for $i = 1, \ldots, q$. We denote by $F = \prod_{i=1}^{q} f_i, F_x = \partial F/\partial x$, $F_y = \partial F/\partial y, d = \gcd(F_x, F_y)$, $H_1, H_2, H_3, K, P_K, Q_K \in \mathbf{F}[x, y]$ satisfying $F_x P_K + F_y Q_K = KF$. Then the system can be written in the form*

$$\dot{x} = \frac{1}{H_3}\left(H_1 P_K - H_2 \frac{F_y}{d}\right), \qquad \dot{y} = \frac{1}{H_3}\left(H_1 Q_K + H_2 \frac{F_x}{d}\right),$$

with suitable H_1, H_2, H_3, K, P_K and Q_K, such that $H_1 P_K - H_2 \frac{F_y}{d}$ and $H_1 Q_K + H_2 \frac{F_x}{d}$ are polynomials divisible by H_3.

The following lemma due to Christopher [6] tells us how must be the higher degree terms of an invariant algebraic curve $f = 0$ of a polynomial system (1). We find preliminary versions of this lemma in other authors, see for instance Theorem 1 of Yablonskii [33].

Christopher Lemma. *Suppose that a polynomial system (1) of degree m has the invariant algebraic curve $f = 0$ of degree n. Let P_m, Q_m and f_n be the homogeneous components of P, Q and f of degree m and n respectively. Then the irreducible factors of f_n must be factors of $yP_m - xQ_m$.*

We note that the irreducible factors of f_n in $\mathbf{R}[x, y]$ are either linear or quadratic because f_n is a homogeneous polynomial. While the irreducible factors of f_n in $\mathbf{C}[x, y]$ are always linear. We also remark that $yP_m - xQ_m$ is the maximum degree of $\dot{\theta}$ if we write system (1) in polar coordinates $x = r\cos\theta, y = r\sin\theta$.

4. Darboux integrability for real quadratic systems

In this section we apply the previous results about the integrability of a real polynomial system (1) of degree m to the particular case $m = 2$, i.e. to real quadratic systems.

We remark that a real quadratic system can have real algebraic solutions and complex algebraic solutions, and that both types of algebraic solutions must be taken into account when we study its integrability. Moreover, if $f_i = 0$ is a complex (non real) algebraic solution, then its conjugate $\bar{f}_i = 0$ is also an algebraic solution, and its cofactor is the conjugate cofactor of f_i. On the other hand, if we have an equality of the form

$$\sum_{i=1}^{q_1} \left(\lambda_i f_i + \mu_i \bar{f}_i\right) + \sum_{j=1}^{q_2} \eta_j g_j = 0,$$

with $\lambda_i, \mu_i \in \mathbf{C}$, $\eta_j \in \mathbf{R}$, $f_i, \bar{f}_i \in \mathbf{C}[x, y]$ and $g_j \in \mathbf{R}[x, y]$, then $\mu_i = \bar{\lambda}_i$.

In the rest of this section we assume that system (1) is a real quadratic system having q real or complex algebraic solutions $f_i = 0$ with cofactors $K_i \neq 0$ for $i = 1, \ldots, q$. Also we suppose that $\operatorname{div}(P, Q) \neq 0$, otherwise the system should be Hamiltonian. Then the following statements hold.

(i) If $q \geq 5$ then the quadratic system has a real rational first integral of the form $f_1^{n_1} \cdots f_q^{n_q}$ with the n_i's integers such that $\sum_{i=1}^{q} n_i K_i = 0$, and consequently all the solutions of the quadratic system are algebraic (Jouanolou Theorem).
(ii) If the quadratic system has exactly $q = 4$ algebraic solutions, then it has a real first integral of the form $f_1^{\lambda_1} \cdots f_4^{\lambda_4}$ with $\lambda_i \in \mathbf{R}$ if $f_i \in \mathbf{R}[x,y]$, $\lambda_i \in \mathbf{C}$ if $f_i \in \mathbf{C}[x,y]$ and $\sum_{i=1}^{4} \lambda_i K_i = 0$ (Darboux Theorem (a)).
(iii) If the quadratic system has exactly $q = 3$ algebraic solutions, then the real function $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ for suitable $\lambda_i \in \mathbf{R}$ if $f_i \in \mathbf{R}[x,y]$, $\lambda_i \in \mathbf{C}$ if $f_i \in \mathbf{C}[x,y]$ is either a first integral and $\sum_{i=1}^{3} \lambda_i K_i = 0$, or it is an integrating factor and $\sum_{i=1}^{3} \lambda_i K_i = -\mathrm{div}(P,Q)$ (Darboux Theorem (b)).
(iv) Suppose that the quadratic system has exactly $q = 2$ algebraic solutions.

(a) If there exist $\lambda_i \in \mathbf{R}$ if $f_i \in \mathbf{R}[x,y]$, $\lambda_i \in \mathbf{C}$ if $f_i \in \mathbf{C}[x,y]$ such that $\lambda_1 K_1 + \lambda_2 K_2 = 0$, then $f_1^{\lambda_1} f_2^{\lambda_2}$ is a real first integral (Darboux Theorem (c)).

(b) If there exist $\lambda_i \in \mathbf{R}$ if $f_i \in \mathbf{R}[x,y]$, $\lambda_i \in \mathbf{C}$ if $f_i \in \mathbf{C}[x,y]$ such that $\lambda_1 K_1 + \lambda_2 K_2 = -\mathrm{div}(P,Q)$, then $f_1^{\lambda_1} f_2^{\lambda_2}$ is a real integrating factor (Darboux Theorem (d)).

(c) If the quadratic system has 2 singular points not contained in $f_i = 0$ for $i = 1,2$, then it has a real first integral of the form $f_1^{\lambda_1} f_2^{\lambda_2}$ with $\lambda_i \in \mathbf{R}$ if $f_i \in \mathbf{R}[x,y]$, $\lambda_i \in \mathbf{C}$ if $f_i \in \mathbf{C}[x,y]$ and $\lambda_1 K_1 + \lambda_2 K_2 = 0$ (CLS Theorem (a)).

(d) If the quadratic system has 1 week singular point not contained in $f_i = 0$ for $i = 1,2$, then the real function $f_1^{\lambda_1} f_2^{\lambda_2}$ for suitable $\lambda_i \in \mathbf{R}$ if $f_i \in \mathbf{R}[x,y]$, $\lambda_i \in \mathbf{C}$ if $f_i \in \mathbf{C}[x,y]$, is either a first integral and $\lambda_1 K_1 + \lambda_2 K_2 = 0$, or an integrating factor and $\lambda_1 K_1 + \lambda_2 K_2 = -\mathrm{div}(P,Q)$ (CLS Theorem (b)).

(v) Suppose that the quadratic system has exactly $q = 1$ algebraic solutions. Then $f_1 \in \mathbf{R}[x,y]$.

(a) If there exist $\lambda_i \in \mathbf{R}$ such that $\lambda_1 K_1 = -\mathrm{div}(P,Q)$, then $f_1^{\lambda_1}$ is an integrating factor (Darboux Theorem (d)).

(b) If the quadratic system has 2 weak singular points not contained in $f_1 = 0$, then the function $f_1^{\lambda_1}$ for suitable $\lambda_1 \in \mathbf{R}$ is an integrating factor and $\lambda_1 K_1 = -\mathrm{div}(P,Q)$ (CLS Theorem (b)).

We remark that CLS Theorem (a) cannot be applied to quadratic systems having a unique algebraic curve $f_1 = 0$, because its application needs 3 singular points outside $f_1 = 0$, that is in $K_1 = 0$, and we know that 3 singular points cannot be on an straight line (otherwise the straight line would be formed by singular points and this is not allowed in this paper, see Section 1).

5. Real quadratic systems having an invariant conic

Quadratic systems with an algebraic solution of degree 2 have been studied by many authors. For instance, Qin Yuan-xum [25] studied the quadratic systems having an ellipse as limit cycle. Druzhkova [8] formulated in terms of the coefficients of the quadratic system necessary and sufficient conditions for the existence and uniqueness of an algebraic solution of second degree. A nice result is the following: a quadratic system having an algebraic solution of degree 2 has at most one limit cycle. This result is essentially due to Qin Yuan-xun and Kooij and Zegeling [18] (see also Christopher [5] and Gasull [10]).

We want to study the integrability of quadratic systems which have an invariant conic, i.e. an invariant algebraic curve $f(x,y)=0$, with $f(x,y)$ a real polynomial of degree 2.

The conics in $\mathbf{R}^2$ are classified as ellipses (E), complex ellipses (CE), hyperbolas (H), two complex straight lines intersecting in a real point (p), two intersecting straight lines (LV), parabolas (P), two parallel straight lines (PL), two complex straight lines (CL) and one double straight line (DL). After an affine change of coordinates, we can assume that the above conics have equations $x^2+y^2-1=0$, $x^2+y^2+1=0$, $x^2-y^2-1=0$, $x^2+y^2=0$, $xy=0$, $y-x^2=0$, $x^2-1=0$, $x^2+1=0$ and $x^2=0$, respectively.

In order to apply the theory of Darboux to study the integrability of a real quadratic system having as invariant algebraic curve a conic, and simplify the computations, we start obtaining the normal forms for such systems.

We say that a quadratic system is of type (E) if it has an invariant ellipse. In a similar way we define the quadratic systems of type (CE), (H), (p), (LV), (P), (PL), (CL) and (DL).

Proposition 12. *A real quadratic system having an invariant conic after an affine change of coordinates writes in one of the following nine forms*

$$\begin{array}{ll}
\dot{x}=\frac{a}{2}(x^2+y^2-1)+2y(px+qy+r), & \dot{y}=\frac{b}{2}(x^2+y^2-1)-2x(px+qy+r),\\
\dot{x}=\frac{a}{2}(x^2+y^2+1)+2y(px+qy+r), & \dot{y}=\frac{b}{2}(x^2+y^2+1)-2x(px+qy+r),\\
\dot{x}=\frac{a}{2}(x^2-y^2-1)-2y(px+qy+r), & \dot{y}=-\frac{b}{2}(x^2-y^2-1)-2x(px+qy+r),\\
\dot{x}=\frac{a}{2}(x^2+y^2)+\frac{c}{2}x+2y(px+qy+r), & \dot{y}=\frac{b}{2}(x^2+y^2)+\frac{c}{2}y-2x(px+qy+r),\\
\dot{x}=x(ax+by+c), & \dot{y}=y(Ax+By+C),\\
\dot{x}=\frac{b}{2}xy-\frac{a}{2}(y-x^2)+px+qy+r, & \dot{y}=by^2+c(y-x^2)+2x(px+qy+r),\\
\dot{x}=\frac{a}{2}(x^2-1), & \dot{y}=Q(x,y),\\
\dot{x}=\frac{a}{2}\left(x^2+1\right), & \dot{y}=Q(x,y),\\
\dot{x}=x(ax+by+c), & \dot{y}=Q(x,y),
\end{array}$$

((E), (CE), (H), (p), (LV), (P), (PL), (CL) *and* (DL) *respectively), if the invariant conic is an ellipse, a complex ellipse, a hyperbola, two complex straight lines intersecting in a real point, two intersecting straight lines, a parabola, two parallel*

straight lines, two complex straight lines, and one double straight line, respectively. Here $Q(x,y)$ denotes an arbitrary polynomial of degree 2. Moreover, except for the system (LV), the cofactor of the invariant conic is $ax+by+c$, where the constants b and c are zero if they do not appear in the system.

Proof. The proposition is easy to prove using Christopher Theorem for quadratic systems of type (E), (CE), (H), (P), and (LV), or using Gasull Lemma for quadratic systems of any type. Let us show how to proceed in the case (E). Assume that one algebraic solution for the quadratic system ($m=2$) is the ellipse $f_1 = x^2+y^2-1 = 0$. Then $q=1$, $k=2$ and as $m > k-1$, we must apply the statement (a) of Christopher Theorem. We see that the degree of h_1 is $m-k+1=1$, the degree of the polynomial vector field Y is $m-k=0$ and finally $X_{f_1} = (-2y, 2x)$, from where the expression for (E) follows. Let us show how the Gasull Lema applies in the case (p). We have $F = f_1 = x^2+y^2$, i.e. $F_x = 2x$, $F_y = 2y$, $d=2$ and from the equation $2xP_K + 2yQ_K = K(x^2+y^2)$ and because our system must be quadratic, it follows that we must take as P_K and Q_K two quadratic polynomials, say $P_K = [cx + a(x^2+y^2)]/2$, $Q_K = [cy + b(x^2+y^2)]/2$ and $K = ax+by+c$, then the choice $H_1 = 1$, $H_2 = -2(px+qy+r)$ and $H_3 = 1$ gives the right answer. Finally, let us show how the Gasull Lema applies in the case (PL). Here $F = f_1 = x$, $F_x = 1$, $F_y = 0$ and $d = 1$. Now the equation $P_K + 0Q_K = Kx$ gives $P_K = x$, $K = 1$ and Q_K an arbitrary quadratic polynomial, from where the expression for (PL) follows. ■

We note that six of these nine normal forms appeared in [10], and that system (DL) of Proposition 12 is also the normal form for a quadratic system having an invariant straight line. We remark that real quadratic system having an invariant conic of type (LV), (PL), (p) or (CL) have at least two algebraic solutions. Real quadratic system having an invariant conic of type (E), (CE), (H), (P) or (DL) have in general only one algebraic solution. In what follows we study the integrability of each type of quadratic system given above.

Theorem 13 (Ellipse Theorem). *Let $f_1 = x^2+y^2-1$. Then for a system of type (E) the following statements hold.*

(a) If $ap+bq=0$ and $(b^2+p^2)(p^2+q^2) \neq 0$, then the straight line $f_2 = px+qy+r = 0$ is an algebraic solution of (E) and $f_1^{\lambda_1} f_2^{\lambda_2}$ is a Darboux integrating factor for $\lambda_1 = \lambda_2 = -1$, which gives the first integral

$$H = \frac{(px+qy+r)^b}{(x^2+y^2-1)^{2p}}$$

(b) If $p=q=0$ and $r(a^2+b^2) \neq 0$, then $f_1^{\lambda_1}$ is a Darboux integrating factor for $\lambda_1 = -1$, which gives the first integral

$$H = ay - bx + \ln\left|x^2+y^2-1\right|^{2r}.$$

(c) If $b=p=0$ and $a(q^2+r^2) \neq 0$, then $f_1^{\lambda_1}$ is a Darboux integrating factor for $\lambda_1 = (2q-a)/a$, which gives the first integral

$$H = (qy+r)^a \left(x^2+y^2-1\right)^{2q}.$$

Moreover, if $q \neq 0$ this system has the invariant straight line $qy + r = 0$.

Proof. From Proposition 12 the cofactor of f_1 is $K_1 = ax + by$. Since $ap + bq = 0$, we obtain that $f_2 = 0$ is an algebraic solution of system (E) with cofactor $K_2 = -2qx + 2py$. The divergence of system (E) is div $= (a-2q)x + (b+2p)y$. Therefore, we have $K_1 + K_2 =$ div. Then, by Darboux Theorem (d), $f_1^{-1} f_2^{-1}$ is a Darboux integrating factor of system (E). Hence, if $(b^2 + p^2)(p^2 + q^2) \neq 0$ statement (a) follows.

If $p = q = 0$, then $K_1 =$ div. Therefore, by Darboux Theorem (d), f_1^{-1} is a Darboux integrating factor of system (E). Hence, if $r(a^2 + b^2) \neq 0$ statement (b) follows.

If $b = p = 0$ and $a \neq 0$, then $(a - 2q)K_1 = a\,\mathrm{div}$. So, by Darboux Theorem (d), $f_1^{(2q-a)/a}$ is a Darboux integrating factor. Hence, if $a(q^2 + r^2) \neq 0$ we obtain statement (c). ■

Proposition 14. *Statements (a) and (c) of Theorem 13 contain all real quadratic systems which have an invariant ellipse, an invariant straight line, and verify that these two algebraic solutions force the integrability.*

Proof. From Christopher Lemma we know that all real invariant straight lines of the systems of type (E) must be of the form $f_2 = (4p - b)x + (4q + a)y +$ constant. On the other hand, since the cofactor of $f_1 = x^2 + y^2 - 1$ is $K_1 = ax + by$, and the divergence of systems (E) is div $= (a - 2q)x + (b + 2p)y$, we have that the independent term of K_2 must be zero if we want for the systems $\lambda_1 K_1 + \lambda_2 K_2 = -$div, or $\lambda_1 K_1 + \lambda_2 K_2 = 0$ to have a solution. In other terms, for obtaining an integrating factor or a first integral for a system (E) only using the algebraic solutions $f_1 = 0$ and $f_2 = 0$ we need that the independent term of the cofactor of f_2 be zero. In short, using these two restrictions, the first in the form of f_2 and the second in the form of K_2, the proposition follows easily. ■

Theorem 15 (Complex ellipse Theorem). *For a system of type (CE) the three statements of Theorem 13 hold interchanging $x^2 + y^2 - 1$ with $x^2 + y^2 + 1$.*

Proof. The same proof as in Theorem 13. ■

Proposition 16. *Statements (a) and (c) of Theorem 15 contain all real quadratic systems which have an invariant complex ellipse, an invariant straight line, and verify that these two algebraic solutions force the integrability.*

Proof. The same proof as in Proposition 14. ■

Theorem 17 (Hyperbola Theorem). *Let $f_1 = x^2 - y^2 - 1$. Then for a system of type (H) the following statements hold.*

(a) If $ap - bq = 0$ and $(b^2 + p^2)(p^2 + q^2) \neq 0$, then the straight line $f_2 = px + qy + r = 0$ is an algebraic solution of (H) and $f_1^{\lambda_1} f_2^{\lambda_2}$ is a Darboux integrating factor with $\lambda_1 = \lambda_2 = -1$, which gives the first integral

$$H = (px + qy + r)^b \left(x^2 - y^2 - 1\right)^{2p}.$$

(b) If $p = q = 0$ and $r\left(a^2 + b^2\right) \neq 0$, then $f_1^{\lambda_1}$ is a Darboux integrating factor for $\lambda_1 = -1$, which gives the first integral

$$H = ay + bx + \ln\left|x^2 - y^2 - 1\right|^{2r}.$$

(c) If $b=p=0$ and $a\left(q^2+r^2\right)\neq 0$, then $f_1^{\lambda_1}$ is a Darboux integrating factor for $\lambda_1=(2q-a)/a$, which gives the first integral

$$H=(qy+r)^a\left(x^2-y^2-1\right)^{2q}.$$

Moreover, if $q\neq 0$ this system has the invariant straight line $qy+r=0$.
(d) If $a+b=r=0$ and $a(p^2+q^2)\neq 0$, then the straight line $f_2=x-y=0$ is an algebraic solution of (H) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux integrating factor with $\lambda_1=(2q-2p-a)/a$, $\lambda_2=1$, which gives the first integral H equal to

$$(x^2-y^2-1)^{2(q-p)}((p-q)[(a-4p)x^2+$$

$$+2(2p-2q-a)xy+(a+4q)y^2]+a(p+q))^a.$$

(e) If $a-b=r=0$ and $a(p^2+q^2)\neq 0$, then the straight line $f_2=x+y=0$ is an algebra ic solution of (H) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux integrating factor with $\lambda_1=(2p+2q-a)/a, \lambda_2=1$, which gives the first integral H equal to

$$\left(x^2-y^2-1\right)^{2(q+p)}((p+q)\left[(a+4p)x^2+\right.$$

$$\left.+2(2p+2q+a)xy+(a+4q)y^2\right]+a(p-q))^a.$$

Proof. The first three statements can be proved as in Theorem 13. We prove statement (d) and in a similar way statement (e) can be shown.

We assume that $a+b=r=0$ and $a\left(p^2+q^2\right)\neq 0$. Then it is easy to check that $f_2=0$ is an algebraic solution of system (H) with cofactor $K_2=2(px+qy)$. The divergence of (H) is div $=(a-2q)x+(b-2p)y$. Then the equation $\lambda_1K_1+\lambda_2K_2=-\text{div}$, has the solution $\lambda_1=(2q-2p-a)/a, \lambda_2=1$. Therefore, by Darboux Theorem (d), statement (d) follows. ∎

Proposition 18. *Theorem 17 contains all real quadratic systems which have an invariant hyperbola, an invariant straight line, and verify that these two algebraic solutions force the integrability.*

Proof. The same as in Proposition 14. ∎

We recall that Proposition 6 tells us that systems (p) satisfying $cp+4rq=0$ and $\left(c^2+r^2\right)\left(p^2+q^2\right)\neq 0$, have a Darboux first integral. The next theorem characterizes all real quadratic

systems of type (p) which are integrable having at least three algebraic

solutions of degree 1. Of course, systems (p) always have two complex straight lines $x+yi=0$ and $x-yi=0$.

Theorem 19 (Two complex straight lines intersecting in a real point Theorem). *Let $f_1=x+yi$ and $f_2=\bar{f}_1$. If a system (p) has a third algebraic solution $f_3=0$ of degree 1 then it verifies one of the following two statements.*
(a) If $p[4ar-c(4p-b)]+q[4br-c(4q+a)]=0$ and $(4p-b)^2+(4q+a)^2\neq 0$, then the straight line $f_3=(4p-b)x+(4q+a)y+4r-[4br-c(4q+a)]/(4p)=0$ is an algebraic

solution of (p) and $f_1^{\lambda_1}\bar{f}_1^{\bar{\lambda}_1}f_3^{\lambda_3}$ *is a Darboux integral for* $\lambda_1=(cp+4rq)(8r-2ci)$, $\lambda_2=\bar{\lambda}_1$, $\lambda_3=a\left(c^2+16r^2\right)+4c(cq-4rp)$. *A real expression of it is*

$$
\begin{aligned}
H &= \left(x^2+y^2\right)^{8r(cp+4rq)}\exp\left(4c(cp+4rq)\arctan\left(\frac{y}{x}\right)\right)\cdot\\
&\cdot\quad ((4p-b)x+(4q+a)y+4r-[4br-c(4q+a)]/(4p))^{a\left(c^2+16r^2\right)+4c(cq-4rp)}.
\end{aligned}
$$

(b) If $r=0$ *and* $c\left[(4p-b)^2+(4q+a)^2\right]\neq 0$, *then the straight line* $f_3=(4p-b)x+(4q+a)y=0$ *is an algebraic solution of (p) and* $f_1^{\lambda_1}\bar{f}_1^{\bar{\lambda}_1}f_3^{\lambda_3}$ *is a Darboux integrating factor for* $\lambda_1=-\left(\left[a^2+b^2+6(aq-bp)+8\left(p^2+q^2\right)\right]+\ 2(bq+ap)i\right)/\left[(4p-b)^2+(4q+a)^2\right]$, $\lambda_2=\bar{\lambda}_1$, $\lambda_3=-4\left[p(4p-b)+q(4q+a)\right]/\left[(4p-b)^2+(4q+a)^2\right]$. *Moreover,* $f_1^{\mu_1}\bar{f}_1^{\bar{\mu}_1}f_3^{\mu_3}e^{st}$ *is a Darboux invariant of (p) for* $\mu_2=\bar{\mu}_1$, $\mu_1=-4([p(4p-b)+q(4q+a)]-(bq+ap)i)/\left(c[(4p-b)^2+(4q+a)^2]\right)$, $\mu_3=2[b(4p-b)-a(4q+a)]s/\left(c[(4p-b)^2+(4q+a)^2]\right)$.

Proof. An easy computation shows that the cofactor of f_1 is

$$
K_1=\left[\frac{a}{2}+\left(\frac{b}{2}-2p\right)i\right]x+
$$

$$
+\left[\frac{b}{2}-\left(\frac{a}{2}+2q\right)i\right]y+\frac{c}{2}-2ri.
$$

Of course, the cofactor of f_2 is $K_2=\bar{K}_1$. With the assumptions of statement (a), the cofactor of f_3 is $K_3=2(-qx+py)$. Since $p[4ar-c(4p-b)]+q[4br-c(4q+a)]=0$, it follows that $\lambda_1K_1+\bar{\lambda}_1\bar{K}_1+\lambda_3K_3=0$ for $\lambda_1=(cp+4rq)(8r-2ci)$, and $\lambda_3=a\left(c^2+16r^2\right)+4c(cq-4rp)$. Therefore, by Darboux Theorem (b) we obtain that $f_1^{\lambda_1}\bar{f}_1^{\bar{\lambda}_1}f_3^{\lambda_3}$ is a first integral, and consequently statement (a) is proved.

Now we suppose the hypothesis of statement (b). Then the cofactor of f_3 is $K_3=-2qx+2py+c/2$. Since $c\left[(4p-b)^2+(4q+a)^2\right]\neq 0$, the system $\lambda_1K_1+\bar{\lambda}_1\bar{K}_1+\lambda_3K_3=(2q-a)x-(2p+b)y-c$ has a unique solution for λ_1 and λ_3, the one described in the theorem. Then, by Darboux Theorem (b) $f_1^{\lambda_1}\bar{f}_1^{\bar{\lambda}_1}f_3^{\lambda_3}$ is an integrating factor of (p). Since the system $\mu_1K_1+\bar{\mu}_1\bar{K}_1+\mu_3K_3=-s$ has a unique solution for μ_1 and μ_3, the one described in the theorem, by Proposition 3, $f_1^{\mu_1}\bar{f}_1^{\bar{\mu}_1}f_3^{\mu_3}e^{st}$ is an invariant of (p). Hence, statement (b) is proved.

In order to check that the statements of the theorem give all algebraic solutions of systems (p) of degree 1, we must find all polynomials f_3 of degree 1 satisfying (3). This tedious task can be made easier with the help of an algebraic manipulator and Chirstopher Lemma, which fixes the form of f_3 to be $f_3=(4p-b)x+(4q+a)y+\text{constant}$. ∎

The systems (LV) are called *Lotka–Volterra systems*; their study was started to be studied by Kotka and Volterra in [19, 20, 31]. Later on Kolmogorov studied them in [16] and some authors call the systems (LV) *Kolmogorov systems*.

The next theorem characterizes all Lotka–Volterra systems which are integrable having at least three algebraic solutions of degree 1. Of course, Lotka–

Volterra systems in the normal form (LV) have always two invariant straight lines $x = 0$ and $y = 0$.

Theorem 20 (Two intersecting straight lines Theorem). *Let $f_1 = x$ and $f_2 = y$ be the two trivial algebraic solutions of a Lotka–Volterra system (LV). Then the following statement holds*

(a) If $r_{12} = cB(A-a) + aC(b-B) = 0$ the expression $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor of (LV) with $\lambda_1 = [(C-c)B+Cb]/(cB-Cb)$ and $\lambda_2 = (2cB-Cb)/(cB-Cb)$. If the condition is $aB - bA = 0$, then the expression $f_1^{\mu_1} f_2^{\mu_2} e^{st}$ is a Darboux invariant of the (LV) provided that μ_1 and μ_2 be a solution of the system

$$\begin{aligned} c\mu_1 + C\mu_2 &= -s, \\ a\mu_1 + b\mu_2 &= 0, \\ A\mu_1 + B\mu_2 &= 0. \end{aligned}$$

If the (LV) has a third algebraic solution $f_3 = 0$ of degree 1, then modulus the symmetry $(x, y, a, b, c, A, B, C) \to (y, x, B, A, C, b, a, c)$ it verifies one of the following statements.

(b) If $b = 0$ and $c \neq 0$, then the straight line $f_3 = ax + c = 0$ is an algebraic solution of (LV) and $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = (C-c)/c$, $\lambda_2 = -2$, $\lambda_3 = (cA - ac - aC)/(ac)$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant of (LV) for $\mu_1 = -s/c$, $\mu_2 = 0$, $\mu_3 = s/c$.

(c) If $c = C$ and $(A-a)(B-b) \neq 0$, then the straight line $f_3 = (A-a)x + (B-b)y = 0$ is an algebraic solution of (LV) and $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = b/(B-b)$, $\lambda_2 = A/(a-A)$, $\lambda_3 = (ab - 2aB + AB)/[(a-A)(B-b)]$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant of (LV) for $\mu_1 = sB/[C(b-B)]$, $\mu_2 = as/[C(A-a)]$, $\mu_3 = (aB - bA)s/[C(a-A)(B-b)]$.

(d) If $r_{12} = 0$ and $acBC(a-A)(b-B) \neq 0$, then the straight line $f_3 = aCx + cBy + cC = 0$ is an algebraic solution of (LV) and $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux first integral for $\lambda_1 = (A-a)B$, $\lambda_2 = a(b-B)$, $\lambda_3 = aB - bA$.

Proof. As the cofactors of f_1 and f_2 are respectively $K_1 = ax + by + c$ and $K_2 = C + Ax + By + C$, under the assumptions of statement (a) the equations $\lambda_1 K_1 + \lambda_2 K_2 + \operatorname{div}(P, Q)$ and $\mu_1 K_1 + \mu_2 K_2 + s$ have, as solution, the stated values for the λ_i and μ_i.

Under the assumptions of statement (b) the cofactor of f_3 is $K_3 = ax$. Since $acB \neq 0$, the system $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 = (2a + A)x + 2By + c + C$ has a unique solution $\lambda_1 = (C-c)/c, \lambda_2 = -2, \lambda_3 = (cA - ac - aC)/(ac)$. Therefore, by Darboux Theorem (b) we obtain that $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is an integrating factor of system (LV). The system $\mu_1 K_1 + \mu_2 K_2 + \mu_3 K_3 = -s$ has a unique solution $\mu_1 = -s/c, \mu_2 = 0, \mu_3 = s/c$. Hence, by Proposition 3, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is an invariant of (LV). So, statement (b) is proved.

Now we assume the hypotheses of statement (c). Then the cofactor of f_3 is $K_3 = ax + By + c$. Since $c(A-a)(B-b) \neq 0$, the system $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 = (2a+A)x + (b+2B)y + 2c$ has a unique solution $\lambda_1 = b/(B-b), \lambda_2 = A/(a-A), \lambda_3 = (ab - 2aB + AB)/[(a-A)(B-b)]$. Then, by Darboux Theorem (b) we obtain that $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is an integrating factor of (LV). The system $\mu_1 K_1 + \mu_2 K_2 + \mu_3 K_3 = -s$ has a unique solution $\mu_1 = sB/[C(b-B)]$, $\mu_2 = as/[C(A-a)]$, $\mu_3 = (aB - bA)s/[C(a-A)(B-b)]$. Therefore, from Proposition 3 it follows that $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is an invariant of (LV).

Statement (d) is proved in Example 2 of Section 2.

Finally, in order to check that modulus the symmetry $(x, y, a, b, c, A, B.C) \to (y, x, B, A, C, b, a, c)$, the statements of the theorem give all algebraic solutions of the Lotka–Volterra system (LV) of degree 1, we must find all polynomials f_3 of degree 1 satisfying equation (3). This tedious task is made with the help of the algebraic manipulator Maple, and Christopher Lemma. We note that Christopher Lemma says that f_3 must be of the form $f_3 = (a-A)x + (b-B)y +$ constant, $f_3 = x+$constant, or $f_3 = y+$constant. Moreover, due to the mentioned symmetry it is sufficient to consider the first two possibilities. ∎

We remark that the integrability of the Lotka–Volterra systems described in statement (d) of Theorem 20 can also be detected by using either Darboux Theorem (d), Chavarriga–Llibre–Sotomayor Theorem (b), or Christopher Theorem and Kooij–Christopher Proposition.

The next theorem characterizes all Lotka–Volterra systems which are integrable having an invariant conic.

Theorem 21. *Let $f_1 = x$ and $f_2 = y$. If a Lotka–Volterra system (LV) has a third algebraic solution $f_3 = 0$ of degree 2, then modulus the symmetry $(x, y, a, b, c, A, B, C) \to (y, x, B, A, C, b, a, c)$ it verifies one of the following statements*

(a) If $B = 2b$, $a(2c + C) - cA = 0$ and $acC \neq 0$, then the parabola $f_3 = C(ax + c)^2 + c^2 By = 0$ is an algebraic solution of (LV) and $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = -2$, $\lambda_2 = (c - C)/C$, $\lambda_3 = -(2c + C)/(2C)$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant of (LV) for $\mu_1 = -s/c$, $\mu_2 = 0$, $\mu_3 = s/(2c)$.

(b) If $b + B = 0$, $a(2c + C) - cA = 0$ and $ac(c + C) \neq 0$, then the hyperbola $f_3 = (c+C)(ax+c)^2 + 2acBxy = 0$ is an algebraic solution of (LV) and $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = -c/(c + C)$, $\lambda_2 = -(2c + C)/(c + C)$, $\lambda_3 = (c - C)/[2(c + C)]$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant of (LV) for $\mu_1 = \mu_2 = -s/(c + C), \mu_3 = (3c + C)s/[2c(c + C)]$.

(c) If $(c + 2C)B - bC = 0$, $a(2c + C) - cA = 0$ and $acC \neq 0$, then the parabola $f_3 = (aCx + cBy + cC)^2 - 4acBCxy = 0$ is an algebraic solution of (LV) and $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = -1/2$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant of (LV) for $\mu_1 = -s/(2c)$, $\mu_2 = -s/(2C)$, $\mu_3 = (c + C)s/(2cC)$.

(d) If $c = 2C$, $A(2B - b) - a(3B - 2b) = 0$ and $ac(2B - b) \neq 0$, then the parabola $f_3 = acx + [ax + (2B - b)y]^2 = 0$ is an algebraic solution of (LV) and $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = (B - b)/(b - 2B)$, $\lambda_2 = -2$, $\lambda_3 = b/[2(b - 2B)]$. Moreover, $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant of (LV) for $\mu_1 = 2Bs/[c(b - 2B)]$, $\mu_2 = 2s/c$, $\mu_3 = 2(B - b)s/[c(b - 2B)]$.

(e) If $c = C$, $aB - bA + 2(A - a)(b - B) = 0$ and $caB(b - B)(A - a) \neq 0$, then the parabola $f_3 = c[(A - 2a)^2 x + aBy] + a[(A - 2a)x - By]^2 = 0$ is an algebraic solution of (LV) and $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a first integral for $\lambda_1 = (A - 2a)/(a - A), \lambda_2 =$

$a/(a-A), \lambda_3 = 1$.

Proof. The cofactors of f_1 and f_2 are $K_1 = ax + by + c$ and $K_2 = Ax + By + C$ respectively. Under the suitable assumptions for each statement it is easy to check that $f_3 = 0$ is an algebraic solution of system (LV).

For statements (a)–(d) we must check that $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3$ is equal to minus the divergence of system (LV), i.e. equal to $-(2a+A)x-(b+2B)y-(c+C)$. Then, according to Darboux Theorem (b) $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor of system (LV). Similarly for statement (e) we see that $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 = 0$ accepts a non trivial solution provided that $aB-bA+2(A-a)(b-B)=0$, which is precisely the second relation needed for the existence of f_3.

Again, to see for statements (a)–(d) that $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is an invariant of system (LV), from Proposition 3 it is sufficient to verify that $\mu_1 K_1 + \mu_2 K_2 + \mu_3 K_3 = -s$.

Finally, in order to check that modulus the symmetry $(x, y, a, b, c, A, B.C) \to (y, x, B, A, C, b, a, c)$, the statements of the theorem give all algebraic solutions of the Lotka–Volterra system (LV) of degree 2, we must find all polynomials f_3 of degree 2 satisfying equation (3). We note that Christopher Lemma and the mentioned symmetry imply that f_3 must be of the form $f_3 = f_{00} + f_{10}x + f_{01}y + x^2$, $f_3 = f_{00} + f_{10}x + f_{01}y + xy$, $f_3 = f_{00} + f_{10}x + f_{01}y + x[(a-A)x + (b-B)y]$, $f_3 = f_{00} + f_{10}x + f_{01}y + [(a-A)x + (b-B)y]^2$. ■

We must mention that statement (d) of Theorem 20 and the invariants of statements (a) and (c) of Theorem 20, and (b), (c) and (d) of Theorem 21 were already found by Cairó and Feix [3] using the Carleman method.

Proposition 22. *Let $f_1 = x$ and $f_2 = y$. If $aB - bA = 0$, then $f_1^{\mu_1} f_2^{\mu_2} e^{st}$ is a Darboux invariant of system (LV) for $\mu_1 = Bs(cB - bC)$, $\mu_2 = bs(cB - bC)$.*

Proof. The cofactors of f_1 and f_2 are $K_1 = ax + by + c$ and $K_2 = Ax + By + C$ respectively. Then the system $\lambda_1 K_1 + \lambda_2 K_2 = -s$ has solutions if and only if $aB - bA = 0$. It is easy to check that $\mu_1 = Bs(cB - bC)$, $\mu_2 = bs(cB - bC)$ is one of the solutions of the above system. Therefore, from Proposition 3, the proposition follows. ■

Theorem 23 (Parabola Theorem). *Let $f_1 = y - x^2$. Then for a system of type (P) the following statements hold.*

(a) If $2b^2r + (a-2q)(a^2 + 2bp - 2aq) = 0$, $cb - (a^2 + 2bp - 2aq) = 0$ and $b \neq 0$, then the straight line $f_2 = bx + 2q - a = 0$ is an algebraic solution of (P) and $H = f_1^{\lambda_1} f_2^{\lambda_2}$ is a Darboux first integral for $\lambda_1 = 1, \lambda_2 = -2$; i.e.

$$H = \frac{y - x^2}{(bx + 2q - a)^2}.$$

(b) If $b = 0$, $a = 2q \neq 0$, and $A^2 = p^2 - 4qr > 0$, then the two parallel straight lines $f_2 = 2qx + p + |A| = 0$ and $f_3 = 2qx + p - |A| = 0$ are algebraic solutions of (P) and $H = f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux first integral for $\lambda_1 = |A|$, $\lambda_2 = c - p - |A|$, $\lambda_3 = p - c - |A|$; i.e.

$$H = (y - x^2)^{|A|} (2qx + p + |A|)^{c-p-|A|} (2qx + p - |A|)^{p-c-|A|}.$$

(c) If $b=0, a=2q$, $p^2-4qr=0$ and $q(p-c)\neq 0$, then the straight line $f_2=2qx+p=0$ is an algebraic solution of (P) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux integrating factor for $\lambda_1=-1$, $\lambda_2=-2$, which gives the first integral

$$H=\frac{2(c-p)}{2qx+p}+\ln\left|\frac{4q(y-x^2)}{(2qx+p)^2}\right|.$$

(d) If $b=0$, $ap-2cq=0$ and $aq\neq 0$, then the straight line $f_2=2cqx+aqy+ra=0$ is an algebraic solution of (P) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux integrating factor for $\lambda_1=-(a+2q)/a, \lambda_2=0$, which gives the first integral

$$H=\frac{(2cqx+aqy+ra)^a}{(y-x^2)^{2q}}.$$

(e) If $a=b=q=0$ and $p\neq 0$, then the straight line $f_2=px+r=0$ is an algebraic solution of (P) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux integrating factor for $\lambda_1=\lambda_2=-1$, which gives the first integral

$$H=\frac{(y-x^2)^p}{(px+r)^c}.$$

(f) If $16b^2r-(a-2q)(7a^2-24aq+16q^2)=0$, $16bp+16q^2-40aq+15a^2=0$, $8bc+7a^2-24aq+16q^2=0$ and $b\neq 0$, then the straight line $f_2=2(2q-a)x+by+(a-2q)^2/b=0$ is an algebraic solution of (P) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux first integral for $\lambda_1=-1, \lambda_2=1$, which gives the first integral

$$H=\frac{2(2q-a)x+by+(a-2q)^2/b}{y-x^2}.$$

(g) If $64b^2r-a(4q-a)^2=0$, $32bp+(a-4q)(3a+4q)=0$, $16bc+(a-4q)(5a-4q)=0$ and $b(3a-4q)\neq 0$, then the straight line $f_2=(4q-a)x+2by+(4q-a)^2/(8b)=0$ is an algebraic solution of (P) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux integrating factor for $\lambda_1=-1/2, \lambda_2=-2$, which gives the first integral

$$H=\arctan\left(\frac{4b\sqrt{y-x^2}}{4bx+4q-a}\right)+\frac{(3a-4q)\sqrt{y-x^2}}{2\left[(4q-a)x+2by+(4q-a)^2/(8b)\right]}.$$

(h) If $3a-4q=0$, $3c-2p=0$, $54b^2r+16q^3-36bpq+9\sqrt{2}|A|^{3/2}$ $-\sqrt{2}(10q^2-15bp)|A|=0$ and $b|A|\neq 0$ where $A^2=2q^2-3bp>0$, then the straight line $f_2=2(2q+\sqrt{2}|A|)x+3by+[8\sqrt{2}q^3-12\sqrt{2}bpq+(2q^2+3bp)|A|+3|A|^3]/(3b|A|)=0$ is an invariant algebraic solution of (P) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux integrating factor for $\lambda_1=-5/2$, $\lambda_2=0$, which gives the first integral

$$H=\frac{54b^3x^3-81b^3xy-54b^2px-54b^2qy-36bpq+16q^3+4\sqrt{2}|A|^3}{(y-x^2)^{3/2}}.$$

(i) If $3a-4q=0$, $3bp-2q^2=0$, $27b^2r-4q^3=0$, $9bc-4q^2=0$ and $b\neq 0$, then the straight line $f_2=12qx+9by+4q^2/b=0$ is an algebraic solution of system (P) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is a Darboux first integral for $\lambda_1=-1,\lambda_2=1$, which gives the first integral

$$H=\frac{12qx+9by+4q^2/b}{y-x^2}.$$

(j) If $3a-4q=0$, $27b^2r+8q^3-18bpq=0$, $9bc+8q^2-18bp=0$ and $(4q-3a)(3bp-2q^2)\neq 0$, then the parallel straight lines $f_2=12qx+9by+4q^2/b=0$ and $f_3=12qx+9by+2(9bp-4q^2)/b=0$ are algebraic solutions of (P) and $f_1^{\lambda_1}f_2^{\lambda_2}f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1=\lambda_3=-1$, $\lambda_2=-1/2$, which gives the first integral

$$H=\frac{9b^2x^2+24bqx+9b^2y+8q^2+2(3bx+2q)(12bqx+9b^2y+4q^2)^{1/2}}{9b^2\,(y-x^2)}.$$

(k) If $3bp-2q^2=0$, $27b^2r-4q^3=0$, $9bc+4q^2-6aq=0$ and $bq(3a-4q)\neq 0$, then the straight line $f_2=12qx+9by+4q^2/b=0$ is an algebraic solution of (P) and $f_1^{\lambda_1}f_2^{\lambda_2}$ is an integrating factor of (P) for $\lambda_1=-1$, $\lambda_2=-3/2$, which gives the first integral

$$H=\frac{6a-8q}{\sqrt{bf_2}}-\ln\left|\frac{9b^2x^2+\left(24bq+6b\sqrt{bf_2}\right)x+9b^2y+8q^2+4q\sqrt{bf_2}}{9b^2\,(y-x^2)}\right|.$$

Proof. From Proposition 12 we know that the cofactor of f_1 is $K_1=ax+by+c$. We denote the divergence of system (P) by div. Under the assumptions of statement (a), the cofactor of f_2 is $K_2=[ax+by+2p+(a-2q)a/b]/2$. Since $K_1-2K_2=0$, from Darboux Theorem (c), statement (a) follows. Similarly for statement (b), the cofactors of f_2 and f_3 are $K_2=qx+(p-|A|)/2$ and $K_3=qx+(p+|A|)/2$. Since $\sum_{i=1}^{3}\lambda_iK_i=0$ for the values of λ_i's given in (b), by Darboux Theorem (b), statement (b) is proved. For statement (c), the cofactor of f_2 is $K_2=qx+p/2$. Since $K_1+2K_2=\mathrm{div}$, from Darboux Theorem (d), statement (c) follows. In statement (d), it is easy to check that $f_2=0$ is an algebraic solution of (P). Since $(a+2q)K_1=-a\,\mathrm{div}$, by Darboux Theorem (d), we obtain statement (d). For statement (e), the cofactor of f_2 is $K_2=p$. Since $K_1+K_2=\mathrm{div}$, by Darboux Theorem (d), we get statement (e). For statement (f). So the cofactor of f_2 is $K_2=ax+by+c$. Consequently $K_1=K_2$, and by Darboux Theorem (c), statement (f) follows. For statement (g), the cofactor of f_2 is $K_2=(q+a/4)x+by-a(a-4q)/(8b)$. Since $\frac{1}{2}K_1+2K_2=\mathrm{div}$, by Darboux Theorem (d), statement (g) is proved. For statement (h), the cofactor of f_2 is $K_2=(4q-\sqrt{2}|A|)x/3+by+[(6bp+4q^2)|A|+\sqrt{2}(12bpq-8q^3)-6|A|^3]/(18b|A|)$. Consequently $f_2=0$ is an algebraic solution. Since $5K_1=2\mathrm{div}$, by Darboux Theorem (d), statement (h) is proved. For statement (i), the cofactor of f_2 is $K_2=4qx/3+by+4q^2/(9b)$. Since $K_1=K_2$, by Darboux Theorem (c), we obtain statement (i). For statement (j), the cofactors of f_2 and f_3 are $K_2 = 4qx/3 + by + 2(9bp - 4q^2)/(9b)$ and $K_3 = 4qx/3 + by + 4q^2/(9b)$. Consequently, $K_1 + \frac{1}{2}K_2 + K_3 = \text{div}$, by Darboux Theorem (b), statement (j) follows. Finally, for statement (l), the cofactor of f_2 is $K_2 = 4qx/3 + by + 4q^2/(9b)$. Since $K_1 + \frac{3}{2}K_2 = \text{div}$, by Darboux Theorem (d), statement (k) follows. ∎

Other cases of integrability related to Theorem 23 have been found. They have been omitted here due to the lenght of the expressions involved.

Proposition 24. *Theorem 23 contains all real quadratic systems which have an invariant parabola, an invariant straight line, and verify that these two algebraic solutions force the integrability.*

Proof. From Christopher Lemma we know that all real straight lines of systems of type (P) must be of the form

$$f_2 = \left[2q - \frac{a}{2} + \sqrt{\left(\frac{a}{2} - 2q\right)^2 + 2b(c-2p)}\right] x + by + \text{constant},$$
$$f_2 = \left[2q - \frac{a}{2} - \sqrt{\left(\frac{a}{2} - 2q\right)^2 + 2b(c-2p)}\right] x + by + \text{constant},$$
$$f_2 = x + \text{constant}.$$

The cofactor of $f_1 = y - x^2$ is $K_1 = ax + by + c$. We denote the cofactor of f_2 by $K_2 = k_{00} + k_{10}x + k_{01}y$. Since the divergence of systems (P) is $\text{div} = (a+2q)x + 5by/2 + c + p$, a necessary condition in order that the systems $\lambda_1 K_1 + \lambda_2 K_2 = -\text{div}$ or $\lambda_1 K_1 + \lambda_2 K_2 = 0$, can have solutions is that

$$\begin{vmatrix} a & k_{10} & a+2q \\ b & k_{01} & 5b/2 \\ c & k_{00} & c+p \end{vmatrix} = 0.$$

Then, by using these two restrictions, the first in the form of f_2 and the second given by the determinant, it is a straightforward but tedious work to prove the proposition. ∎

We write system (PL) in the form

$$\dot{x} = x^2 - 1, \qquad \dot{y} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2. \tag{14}$$

Proposition 25. *Let $f_1 = x + 1$ and $f_2 = x - 1$. If $a_{02} = 0$ in the system (14), then $f_1^{\lambda_1} f_2^{\lambda_2}$ is a Darboux integrating factor for $\lambda_1 = (a_{01} - a_{11} - 2)/2$ and $\lambda_2 = -(a_{01} + a_{11} + 2)/2$.*

Proof. Let K_1 and K_2 be the cofactors of f_1 and f_2 respectively. Then $K_1 = x - 1$ and $K_2 = x + 1$. From Darboux Theorem (d) and since $\lambda_1 K_1 + \lambda_2 K_2$ is equal to minus the divergence of system (14), the lemma follows. ∎

Since we are interested in systems (14) which are integrable, by Proposition 15 we can assume that $a_{02} \neq 0$. The next theorem characterizes all systems (14) with $a_{02} \neq 0$ which are integrable having at least three algebraic solutions of degree

1. Of course, systems (14) always have the two algebraic solutions $x+1=0$ and $x-1=0$.

Theorem 26 (Two parallel straight lines Theorem). *Let $f_1 = x+1$, $f_2 = x-1$ and $A^2 = (a_{11}-1)^2 - 4a_{20}a_{02} \geq 0$. If a system (14) with $a_{02} \neq 0$ has a third algebraic solution $f_3 = 0$ of degree 1, then it verifies one of the following statements.*

(a) If $4-12\,a_{20}\,a_{02}+5\,a_{11}^2+4\,a_{02}\,a_{00}-8\,a_{11}-8\,a_{02}^2a_{00}\,a_{20}-4\,a_{02}\,a_{00}\,a_{11}+a_{11}\,a_{01}^2-a_{01}^2+2\,a_{02}^2a_{10}^2-2\,a_{02}\,a_{10}\,a_{11}\,a_{01}+4\,a_{11}\,a_{20}\,a_{02}+2\,a_{01}^2a_{20}\,a_{02}+2\,a_{02}\,a_{00}\,a_{11}^2-a_{11}^3+\left(2\,a_{11}+a_{01}^2-4\,a_{02}\,a_{00}-3\right)\,|A|-|A|^3=0$ and $|A| \neq 1$, then the straight line $f_3 = (a_{11}-1+|A|)x+2a_{02}y-(2a_{02}a_{10}-a_{11}a_{01}+a_{01}-a_{01}|A|)/(|A|-1)=0$ is an algebraic solution of (14), and $f_1^{\lambda_1}f_2^{\lambda_2}f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = (A^2-1+2a_{02}a_{10}-a_{11}a_{01})/[2(1-|A|)]$, $\lambda_2 = (A^2-1-2a_{02}a_{10}+a_{11}a_{01})/[2(1-|A|)]$, $\lambda_3=-2$. Moreover, $f_1^{\mu_1}f_2^{\mu_2}f_3^{\mu_3}e^{st}$ is a Darboux invariant of (14) for $\mu_1 = s/2, \mu_2 = -s/2, \mu_3 = 0$.

(b) If $a_{10}a_{11}-2a_{02}a_{10}=0, A^2=1$ and $B^2 = a_{01}^2+2a_{11}-4a_{02}a_{00} > 0$, then the two parallel straight lines $f_3 = a_{11}x+2a_{02}y+a_{01}+|B|=0$ and $f_4 = a_{11}x+2a_{02}y+a_{01}-|B|=0$ are algebraic solutions of (14), and $f_1^{\lambda_1}f_2^{\lambda_2}f_3^{\lambda_3}f_4^{\lambda_4}$ is a Darboux first integral for $\lambda_1 = |B|/2, \lambda_2 = -\lambda_1, \lambda_3 = -1, \lambda_4 = 1$.

(c) If $a_{01}a_{11}-2a_{02}a_{10}=0, A^2=1$ and $B=0$, then the straight line $f_3 = a_{11}x+2a_{02}y+a_{01}=0$ is an algebraic solution of (14), and $f_1^{\lambda_1}f_2^{\lambda_2}f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1=\lambda_2=-1$ and $\lambda_3=-2$, which gives the first integral

$$H = \frac{4}{a_{11}x+2a_{02}y+a_{01}} + \ln\left|\frac{x-1}{x+1}\right|.$$

Moreover, $f_1^{\mu_1}f_2^{\mu_2}f_3^{\mu_3}e^{st}$ is a Darboux invariant of (14) for $\mu_1 = s/2, \mu_2 = -s/2$, $\mu_3 = 0$.

(d) If $4-12a_{20}a_{02}+5a_{11}^2-8a_{11}-8a_{02}^2a_{00}a_{20}-4a_{02}a_{00}a_{11}+a_{11}a_{01}^2-a_{01}^2+2a_{02}^2a_{10}^2-2a_{02}a_{10}a_{11}a_{01}+4a_{11}a_{20}a_{02}+2a_{01}^2a_{20}a_{02}+2a_{02}a_{00}a_{11}^2-a_{11}^3-(2a_{11}+a_{01}^2-4a_{02}a_{00}-3)|A|+|A|^3=0$ and $|A| \neq -1$, then the straight line $f_3 = (a_{11}-1-|A|)x+2a_{02}y+(2a_{02}a_{10}-a_{11}a_{01}-a_{01}-a_{01}|A|)/(|A|+1)+2a_{01}=0$ is an algebraic solution of (14), and $f_1^{\lambda_1}f_2^{\lambda_2}f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = \left(A^2-1+2a_{02}a_{10}-a_{11}a_{01}\right)/[2(1+|A|)]$, $\lambda_2 = \left(A^2-1-2a_{02}a_{10}+a_{11}a_{01}\right)$, $\lambda_3 = -2$. Moreover, $f_1^{\mu_1}f_2^{\mu_2}f_3^{\mu_3}e^{st}$ is a Darboux invariant of (14) for $\mu_1 = s/2$, $\mu_2 = -s/2$, $\mu_3 = 0$.

(e) If $16a_{00}a_{02}-8a_{11}+16+4a_{02}^2a_{10}^2-a_{02}a_{10}a_{01}a_{11}+a_{01}^2a_{11}^2-4a_{01}^2=0$ and $A^2=1$, then the straight line $f_3 = (a_{11}-2)x+2a_{02}y+a_{02}a_{10}+a_{01}-a_{01}a_{11}/2$ is an algebraic solution of (14), and $f_1^{\lambda_1}f_2^{\lambda_2}f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 = (2a_{02}a_{10}-a_{01}a_{11})/4$, $\lambda_2 = -\lambda_1$, $\lambda_3 = -2$. Moreover, $f_1^{\mu_1}f_2^{\mu_2}f_3^{\mu_3}e^{st}$ is a Darboux invariant of (14) for $\mu_1 = s/2$, $\mu_2 = -s/2$, $\mu_3 = 0$.

Proof. Taking into account that the cofactors of f_1 and f_2 are $K_1 = x-1$ and $K_2 = x+1$ respectively, and under suitable assumptions for each statement it is easy to check that $f_3 = 0$ is an algebraic solution of system (14), i.e. f_3 verifies equation (3) where K_3, the cofactor of f_3 is of the form $px+qy+r$, and where we have written the system (14) in the form $\dot{x} = P(x,y), \dot{y} = Q(x,y)$.

For all statements except statement (b) we must check that $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3$ is equal to minus the divergence of system (14), i.e. equal to $(2 + a_{11})x + 2a_{02}y + a_{01}$. Then, from Darboux Theorem (b) $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a Darboux integrating factor of system (14).

For statement (b) we must verify that $\sum_{i=1}^{4} \lambda_i K_i = 0$, where the cofactors of f_3 and f_4 are $K_3 = a_{11}x/2 + a_{02}y + a_{01}/2 - |B|/2$ and $K_4 = a_{11}x/2 + a_{02}y + a_{01}/2 + |B|/2$. Hence, by Darboux Theorem (a) $f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} f_4^{\lambda_4}$ is a first integral of system (14).

To verify for all statements except statement (b) that $f_1^{\mu_1} f_2^{\mu_2} f_3^{\mu_3} e^{st}$ is a Darboux invariant of system (14), from Proposition 3 it is sufficient to check that $\mu_1 K_1 + \mu_2 K_2 + \mu_3 K_3 = -s$.

Finally, in order to show that the statements of the theorem provide all algebraic solutions of (14) of degree 1, we must find all polynomials f_3 of degree 1 satisfying equation (3). By Christopher Lemma, f_3 must be of the form $f_3 = (a_{11} - 1 + |A|)x + 2a_{02}y + \text{constant}$, $f_3 = (a_{11} - 1 - |A|)x + 2a_{02}y + \text{constant}$, $f_3 = x + \text{constant}$. It is easy to verify that the unique invariant straight lines of system (14) of the last form are f_1 and f_2. Now knowing these two possible expressions for f_3, it is tedious but easy to verify that the theorem give all algebraic solutions of (14) of degree 1. ∎

We assume that $a \neq 0$ for the systems (CL), otherwise $x = \text{constant}$ is a first integral. Therefore, doing a rescaling of the time variable (if necessary) we can assume that $a = 2$, and consequently systems (CL) can be written in the form

$$\dot{x} = x^2 + 1, \qquad \dot{y} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2. \tag{15}$$

Proposition 27. *Let $f_1 = x + i$. If $a_{02} = 0$ for a system (15) of type (CL), then $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1}$ is a Darboux integrating factor for $\lambda_1 = -\frac{1}{2}(2 + a_{11} + a_{01}i)$.*

Proof. The cofactor of f_1 is $K_1 = x - i$. From Darboux Theorem (d) since $\lambda_1 K_1 + \bar{\lambda}_1 \bar{K}_1$ is equal to minus the divergence of system (15), the lemma follows. ∎

Due to Proposition 27 in the rest of the section we assume that $a_{02} \neq 0$. The next theorem classifies all real quadratic systems of type (CL) with $a_{02} \neq 0$ which are integrable having at least three algebraic solutions of degree 1. Of course, systems (CL) always have two complex straight lines $x + i = 0$ and $x - i = 0$.

Theorem 28 (Two complex straight lines Theorem). *Let $f_1 = x + i$ and $A^2 = (a_{11} - 1)^2 - 4a_{20}a_{02} \geq 0$. If a system (CL) with $a_{02} \neq 0$ has a third algebraic solution $f_3 = 0$ of degree 1, then it verifies one of the following statements.*

(a) If $-4 + 12a_{20}a_{02} + 4a_{02}a_{00} + 8a_{11} - 5a_{11}^2 - a_{01}^2 + 2a_{02}^2a_{10}^2 - 2a_{02}a_{10}a_{01}a_{11} + a_{01}^2a_{11} + 2a_{02}a_{00}a_{11}^2 - 8a_{02}^2a_{00}a_{20} - 4a_{02}a_{00}a_{11} - 4a_{11}a_{20}a_{02} + 2a_{01}^2a_{20}a_{02} + a_{11}^3 + (3 - 2a_{11} + a_{01}^2 - 4a_{02}a_{00})|A| + |A|^3 = 0$ and $|A| \neq 1$, then the straight line $f_3 = (a_{11} - 1 + |A|)x + 2a_{02}y + 2a_{01} - (2a_{02}a_{10} - a_{01}a_{11} - a_{01} + a_{01}|A|)/(|A| - 1) = 0$ is an algebraic solution of (15) and $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3}$ is a Darboux integrating factor for $\lambda_1 =$

$\left[1 - A^2 - (a_{01}a_{11} - 2a_{02}a_{10})\, i\right] / (2(|A| - 1))$, $\lambda_3 = -2$. *Moreover,* $f_1^{\mu_1} \bar{f}_1^{\bar{\mu}_1} f_3^{\mu_3} e^{st}$ *is a Darboux invariant of (15) for* $\mu_1 = -(si)/2, \mu_3 = 0$.

(b) If $a_{01}a_{11} - 2a_{02}a_{10} = 0$, $A^2 = 1$, *and* $B^2 = a_{01}^2 - 2a_{11} - 4a_{02}a_{00} > 0$, *then the two parallel straight lines* $f_3 = a_{11}x + 2a_{02}y + a_{01} - |B| = 0$ *and* $f_4 = a_{11}x + 2a_{02}y + a_{01} + |B| = 0$ *are algebraic solutions of (15) and* $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3} f_4^{\lambda_4}$ *is a Darboux first integral for* $\lambda_1 = |B|i/2$, $\lambda_3 = -1$, $\lambda_4 = 1$; *i.e.*

$$H = \exp\left(-|B| \arctan\left(\frac{1}{x}\right)\right) \frac{a_{11}x + 2a_{02}y + a_{01} + |B|}{a_{11}x + 2a_{02}y + a_{01} - |B|}.$$

(c) If $a_{01}a_{11} - 2a_{02}a_{10} = 0$, $A^2 = 1$ *and* $B = 0$, *then the straight line* $f_3 = a_{11}x + 2a_{02}y + a_{01} = 0$ *is an algebraic solution of (15) and* $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3}$ *is a Darboux integrating factor for* $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = -2$, *which gives the first integral*

$$H = \frac{2}{a_{11}x + 2a_{02}y + a_{01}} + \arctan(x).$$

Moreover, $f_1^{\mu_1} \bar{f}_1^{\bar{\mu}_1} f_3^{\mu_3} e^{st}$ *is a Darboux invariant of (15) for* $\mu_1 = -(si)/2, \mu_3 = 0$.

(d) If $-4 + 12a_{20}a_{02} + 4a_{02}a_{00} + 8a_{11} - 5a_{11}^2 - a_{01}^2 + 2a_{02}^2 a_{10}^2 - 2a_{02}a_{10}a_{01}a_{11} + a_{01}^2 a_{11} + 2a_{02}a_{00}a_{11}^2 - 8a_{02}^2 a_{00}a_{20} - 4a_{02}a_{00}a_{11} - 4a_{11}a_{20}a_{02} + 2a_{01}^2 a_{20}a_{02} + a_{11}^3 - \left(3 - 2a_{11} + a_{01}^2 - 4a_{02}a_{00}\right)|A| - |A|^3 = 0$ *and* $|A| \neq -1$, *then the straight line* $f_3 = (a_{11} - 1 - |A|)\, x + 2a_{02}y + 2a_{01} + (2a_{02}a_{10} - a_{01}a_{11} \ - a_{01} - a_{01}|A|) / (|A| + 1) = 0$ *is an algebraic solution of (15) and* $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} \ f_3^{\lambda_3}$ *is a Darboux integrating factor for* $\lambda_1 = \left[A^2 - 1 + (a_{01}a_{11} - 2a_{02}a_{10})\, i\right] / \ (2(|A| + 1))$, $\lambda_3 = -2$. *Moreover,* $f_1^{\mu_1} \bar{f}_1^{\bar{\mu}_1} f_3^{\mu_3} e^{st}$ *is a Darboux invariant of (15) for* $\mu_1 = -(si)/2, \mu_3 = 0$.

(e) If $16a_{00}a_{02} + 8a_{11} - 16 + 4a_{02}^2 a_{10}^2 - 4a_{02}a_{10}a_{01}a_{11} + a_{01}^2 a_{11}^2 - 4a_{01}^2 = 0$ *and* $A^2 = 1$, *then the straight line* $f_3 = (a_{11} - 2)\, x + 2a_{02}y + a_{01} + a_{02}a_{10} - a_{01}a_{11}/2 = 0$ *is an algebraic solution of (15) and* $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3}$ *is a Darboux integrating factor for* $\lambda_1 = (a_{01}a_{11} - 2a_{02}a_{10})\, i/4$, $\lambda_3 = -2$. *Moreover,* $f_1^{\mu_1} \bar{f}_1^{\bar{\mu}_1} f_3^{\mu_3} e^{st}$ *is a Darboux invariant of (15) for* $\mu_1 = -(si)/2, \mu_3 = 0$.

(f) If $4a_{00}a_{02} + 2a_{11} - 2 + 4a_{02}^2 a_{10}^2 - 4a_{02}a_{10}a_{01}a_{11} + a_{01}^2 a_{11}^2 - a_{01}^2 = 0$ *and* $A^2 = 0$, *then the straight line* $f_3 = (a_{11} - 1)\, x + 2a_{02}y + a_{01} + 2a_{02}a_{10} - a_{01}a_{11} = 0$ *is an algebraic solution of (15) and* $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3}$ *is a Darboux integrating factor for* $\lambda_1 = -\left[1 - (a_{01}a_{11} - 2a_{02}a_{10})\, i\right]/2$, $\lambda_3 = -2$. *Moreover,* $f_1^{\mu_1} \bar{f}_1^{\bar{\mu}_1} f_3^{\mu_3} e^{st}$ *is a Darboux invariant of (15) for* $\mu_1 = -(si)/2, \mu_3 = 0$.

Proof. It is easy to verify that under the suitable assumptions each statement has the corresponding invariant straight line, the corresponding integrating factor or first integral, and the corresponding invariant.

In order to check that the statements of the theorem provide all algebraic solutions of systems (CL) of degree 1, we must find all polynomials f_3 of degree 1 satisfying equation (3) with a suitable cofactor, and where we write the system (CL) in the form $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$. By using Christopher Lemma we know that f_3 must have one of the following three forms $f_3 = (a_{11} - 1 + |A|)\, x + 2a_{02}y + \text{constant}$, $f_3 = (a_{11} - 1 - |A|)\, x + 2a_{02}y + \text{constant}$, $f_3 = x + \text{constant}$. It is easy to check that the unique invariant straight lines of the last form are $f_1 = 0$ and $\bar{f}_1 = 0$. Now, working with the first two expressions for f_3, we can prove after some tedious computations that all invariant straight lines of system (15) with $a_{02} \neq 0$ are given in the statements of the theorem. ■

We write system (DL) in the form

$$\dot{x} = x(ax + by + c), \qquad \dot{y} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2. \quad (16)$$

If this system, which has the invariant straight line $x = 0$, has another invariant straight line or an invariant conic, then after an affine change of coordinates (if necessary) it can be written as one of the first eight normal forms of Proposition 12. Hence we do not need to study the integrability of systems (DL) if they have some invariant algebraic curve of degree 1 or 2 different from $x = 0$.

In the next proposition we study the integrability of systems (DL) when a priori we do not know any algebraic solution of them except the invariant straight line $x = 0$.

Theorem 29 (One double straight line Theorem). *Let $f_1 = x$ and assume $b(2a + a_{11}) = a(b + 2a_{02})$, $c(b + 2a_{02}) = b(c + a_{01})$ and $a \neq 0$, then $f_1^{\lambda_1}$ is a Darboux integrating factor for $\lambda_1 = -(2a + a_{11})/a$ which gives the first integral*

$$H = \left(\frac{2\,a a_{00}}{a + a_{11}} + \frac{2\,a a_{10}\,x}{a_{11}} + 2\,cy + 2\,axy - \frac{2\,a a_{20}\,x^2}{a - a_{11}} + by^2\right) x^{-\frac{a + a_{11}}{a}}$$

Proof. The cofactor of f_1 is $K_1 = ax + by + c$. Then, under the assumptions it follows that the critical points which are on the cofactor namely $(-(by + c)/a$, $(a_{10}\,ab - ca^2 + 2\,a_{20}\,bc + \sqrt{\Delta})/b/(2\,a_{20}\,b - a_{11}a + a^2)$ and $(-(by + c)/a$, $(a_{10}\,ab - ca^2 + 2\,a_{20}\,bc - \sqrt{\Delta})/b/(2\,a_{20}\,b - a_{11}a + a^2)$ with $\Delta = a^2(a_{10}{}^2b^2 + a^2c^2 + 2\,c^2 a_{20}\,b) + 2\,bca(-a_{10}\,a + ca_{20})a_{11} - 2\,ba^2(a^2 + 2\,a_{20}\,b)a_{00} + 2\,a_{11}\,a^3ba_{00}$ are weak. Therefore, $\lambda_1 K_1 = -(2a + a_{11})x - (b + 2a_{02})y - (c + a_{01})$ is minus the divergence of system (16). This allows to compute λ_1 and following the CLS Theorem $f_1^{\lambda_1}$ is an integrating factor from which one can deduce the above mentioned first integral straightforwardly, proving the theorem. ■

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Laurent Cairó, Marc R. Feix
Département de Mathématiques.
MAPMO, UMR 6628 -
Université d'Orléans,
BP 6759, 45067.
Orléans, Cédex 2,
lcairo@labomath.univ-orleans.fr
France.

Jaume Llibre
Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
08193 – Bellaterra, Barcelona
jllibre@mat.uab.es
Spain