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Simply Connected Algebras

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Abstract: The main aim of this survey is to discuss the class of simply connected algebras, their characterisations, construction techniques and examples. It is an expanded version of a series of lectures given by the author at the "Workshop em Representacoes de Algebras", held at IME-USP.

Key words: Simply connected, strongly simply connected, Hochschild cohomology, tilting theory.

The present survey is an expanded, but generally faithful, version of a series of three lectures given by the author at the "Workshop em Representações de Álgebras" held at the IME from the 14th to the 16th of July 1999, prior to the Conference on Representations of Algebras in São Paulo which took place from the 19th to the 24th. The objective of the lectures was to introduce the participants to an important class of finite dimensional algebras over an algebraically closed field, namely that of the simply connected algebras. Their importance became apparent since the study of representation-finite simply connected algebras, first introduced by Bongartz and Gabriel, see [BG]. Indeed, for an arbitrary representation-finite algebra A, the indecomposable $A-$modules can be lifted to indecomposable modules over a simply connected algebra $\tilde{A}$, contained inside a certain Galois covering of the standard form of A, see [BrG]. Thus, covering techniques allow to reduce many problems of the study of representation-finite algebras to problems about representation-finite simply connected algebras. The latter are by now considered to be well-understood (see [BG, BLS, BrG]) but little is known about covering techniques or simply connected algebras in the representation-infinite case. Certain classes of simply connected algebras have been extensively investigated, for instance, the strongly simply connected algebras, introduced by Skowroński in [S2], but, so far, an effective criterion allowing to decide whether a given algebra is simply connected or not does not exist. However, several partial results are known (see, for instance, [AL, AP, S2]) and the purpose of these notes is precisely to present the existing characterizations, construction techniques, and examples of classes of simply connected algebras.

This paper consists of the following sections:

1. The fundamental group and simple connectedness.
2. The separation condition.
3. The fundamental groups of a one-point extension.
4. The Hochschild cohomology spaces and simple connectedness.
5. Strongly simply connected algebras.
6. Tilting and simple connectedness.
7. Simply connected mesh algebras.

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1. The fundamental group and simple connectedness.

1.1. Notation. Throughout, k will denote an algebraically closed field. By algebra is always meant an associative, finite dimensional $k-$algebra with an identity, which we assume moreover to be basic, but not necessarily connected.

A *quiver* Q is defined by its set of points Q_0, and its set of arrows Q_1. A *relation* from a point x to a point y is a linear combination $p = \sum_{i=1}^{m} \lambda_i w_i$ where, for each i (such that $1 \leq i \leq m$), λ_i is a non-zero scalar, and w_i is a path in Q with source x and target y. Such a relation is called a *monomial relation* if $m = 1$, and a *commutativity relation* if it is of the form $\rho = w_1 - w_2$. A set of relations generates an ideal I in the path algebra kQ of a finite quiver Q. Such an ideal is called *admissible* if, for any cyclic path w in Q, there exists $s > 0$ such that $w^s \in I$. If I is an admissible ideal of kQ, then the pair (Q, I) is called a *bound quiver*.

To each algebra A corresponds a quiver Q_A. Indeed, let $e_1, \ldots, e_n$ be a complete set of primitive orthogonal idempotents of A. Then Q_A is defined as follows: the points $\{1, \ldots, n\}$ of Q_A are in bijective correspondence with the e_i, and the arrows from i to j are in bijective correspondence with a basis of the $k-$vector space $e_i(rad\, A/rad^2 A)e_j$. While it is easily shown that this construction does not depend on the choice of a particular complete set of primitive orthogonal idempotents, or of basis of the above vector space, it induces a surjective algebra morphism $\nu : kQ_A \to A$ which heavily depends on these basis: indeed, ν is defined by mapping the stationary path at $i \in (Q_A)_0$ to the corresponding idempotent e_i, and the arrow $\alpha \in (Q_A)_1$ to the corresponding basis vector x_α of $rad\, A/rad^2 A$, and is extended in the obvious way. Then the kernel I_ν of ν is an admissible ideal, and we have $A \cong kQ_A/I_\nu$. The bound quiver (Q_A, I_ν) is called a *presentation* of A. If I_ν is generated by paths (thus, by monomial relations), then (Q_A, I_ν) is called a *monomial presentation*.

It is sometimes convenient to consider an algebra $A \cong kQ/I$ as a $k-$category as in [BG]. The class A_0 of objects of this category is the set Q_0 of points in Q, and the set $A(x, y)$ of morphisms from x to y is the $k-$vector space $kQ(x, y)$ of linear combinations of paths in Q with source x and target y, modulo the subspace $I(x, y) = I \cap kQ(x, y)$. A full subcategory B of A is called *convex* if, for any path $x_0 \to x_1 \to \ldots \to x_t$ in (the quiver of) A, with $x_0, x_t \in B_0$, we have $x_i \in B_0$ for all i with $0 < i < t$. The algebra A is called *triangular* if its quiver Q_A is *acyclic* (that is, has no oriented cycles). Except where otherwise (explicitly) specified, we shall be concerned exclusively with triangular algebras.

Let A be an algebra. By an $A-$module is meant a finitely generated right $A-$module. We denote by $modA$ the category of $A-$modules and $A-$linear maps.

It is well-known that, if $A = kQ/I$, then $modA$ is equivalent to the category of all bound representations of (Q, I), we may thus identify an $A-$module M with the corresponding representation $(M_x, M_\alpha)_{x\in Q_0, \alpha\in Q_1}$. For each $x \in Q_0$, we denote by $S(x)$ the corresponding simple $A-$module, and by $P(x)$ (or $I(x)$) the projective cover (or the injective envelope, respectively) of $S(x)$. The algebra A is called *schurian* if, for all $x, y \in A_0$, we have $dim_k\ Hom_A(P(x), P(y)) \leq 1$. For general facts about the representation theory of an algebra A, we refer the reader to [ARS, R1].

1.2. Simple connectedness. We start by defining the fundamental group of a bound quiver. Let (Q, I) be a connected bound quiver. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is called *minimal* if $m \geq 2$ and, for every non-empty proper subset J of $\{1, \ldots, m\}$, we have $\sum_{j\in J} \lambda_j w_j \notin I(x, y)$.

For an arrow $\alpha : x \to y$, we denote by $\alpha^{-1} : y \to x$ its formal inverse. A *walk* in Q from x to y is a formal composition $\alpha_1^{\epsilon_1}\alpha_2^{\epsilon_2} \ldots \alpha_t^{\epsilon_t}$ (where $\alpha_i \in Q_1, \epsilon_i \in \{-1, +1\}$ for all i such that $1 \leq i \leq t$) with source x and target y. We denote by e_x the stationary path at x. If w is a walk from x to y, and w' is a walk from y to z, then their composition ww' is a walk from x to z. We thus have a (partially defined) product of walks. Clearly, if w is a walk from x to y, then $w = e_x w = w e_y$.

We now define the homotopy relation $\sim$ on (Q, I) (see [G, MP]) to be the smallest equivalence relation on the set of all walks in Q satisfying the following conditions:

(a) For each arrow $\alpha : x \to y$ in Q, we have $\alpha\alpha^{-1} \sim e_x$ and $\alpha^{-1}\alpha \sim e_y$.

(b) For each minimal relation $\sum_{i=1}^{m} \lambda_i w_i$ in I, we have $w_i \sim w_j$ for all i, j such that $1 \leq i, j \leq m$.

(c) If u, v, w and w' are walks, and $u \sim v$, then $wuw' \sim wvw'$, whenever these products are defined.

We denote by $\overline{w}$ the homotopy class of a walk w. Clearly, the product of walks induces a product of homotopy classes: given two walks u and v, the products $\overline{uv}$ is defined whenever the product uv is defined and, moreover, $\overline{uv} = \overline{u}\,\overline{v}$.

Let $x \in Q_0$ be a fixed point, which we call the base point, and consider the set of all homotopy classes of walks of source x and target x. On this set, the product of homotopy classes is everywhere defined, and is easily seen to induce a group structure. This group $\pi_1(Q, I, x)$ is called the *fundamental group* of the bound quiver (Q, I) with base point x. Now, for any two points $x, y \in Q_0$, the fundamental groups $\pi_1(Q, I, x)$ and $\pi_1(Q, I, y)$ are isomorphic: indeed since Q is connected, there exists a walk u, say, from x to y, thus an isomorphism $\pi_1(Q, I, x) \to \pi_1(Q, I, y)$ given by $\overline{w} \to \overline{u}^{-1}\overline{w}\,\overline{u}$. Thus the fundamental group does not depend on the choice of a base point, we denote it simply by $\pi_1(Q, I)$ and call it the *fundamental group* of (Q, I).

On the other hand, it follows from the definition of the homotopy relation that the fundamental group depends essentially on the minimal relations, thus on the ideal I. We are then led to the following definition.

DEFINITION [AS]. A connected triangular algebra A is called *simply connected* if, for every presentation (Q_A, I_ν) of A, the fundamental group $\pi_1(Q_A, I_\nu)$ is trivial.

Before giving examples, some comments are in order. First, in the case of representation-finite algebras, the above definition coincides with that in [BG]: this indeed follows from [BrG](1.2) and [MP](4.3). On the other hand, the following proposition explains the reason for the name of simply connected algebras.

PROPOSITION [S1](4.2). Let A be a connected triangular algebra. Then A is simply connected if and only if it admits no proper Galois covering. ∎

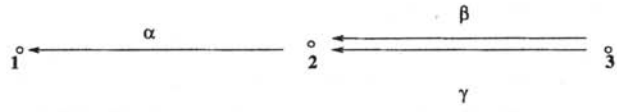
Clearly, neither the definition, nor the above proposition, allow to recognise easily whether a given algebra is simply connected or not. We thus need criteria.

1.3 Examples.

(a) A monomial relation is never a minimal relation. Consequently, if I is an ideal in the path algebra kQ of a quiver Q which is generated by monomial relations, then $\pi_1(Q, I)$ equals the fundamental group $\pi_1(Q)$ of Q considered as a graph. In particular, $\pi_1(Q, I)$ is trivial if and only if Q is a tree. As a consequence, if A is an algebra whose quiver Q_A is a tree, then A is simply connected.

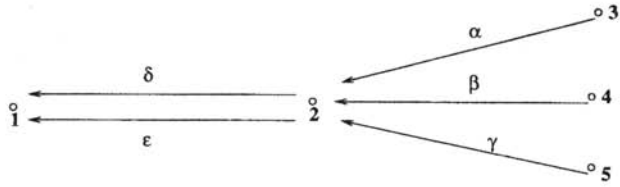
(b) For the same reason as in (a), a hereditary algebra is simply connected if and only if its quiver is a tree.

(c) Let $A = kQ/I$, where Q is the quiver



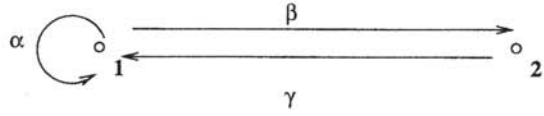
and I is generated by $\beta\alpha$. Since I is generated by a monomial relation, we have $\pi_1(Q, I) = \pi_1(Q) \cong \mathbb{Z}$. Indeed, taking 3 as base point, we see that $\pi_1(Q, I)$ is generated by the homotopy class $\overline{\beta}\ \overline{\gamma}^{-1}$. On the other hand, let $A' = kQ/I'$, where Q is the same quiver as above, and I' is generated by $\beta\alpha - \gamma\alpha$. Then $A \cong A'$. Indeed, let $\{x'_\alpha, x'_\beta, x'_\gamma\}$ be a basis of $rad\ A'/rad^2 A'$ where to each arrow in Q is associated a basis vector (so that $x'_\beta x'_\alpha = x'_\gamma x'_\alpha$). We obtain a basis $\{x_\alpha, x_\beta, x_\gamma\}$ of $rad\ A/rad^2 A$ by setting $x_\alpha = x'_\alpha, x_\beta = x'_\beta - x'_\gamma$ and $x_\gamma = x'_\gamma$. Clearly, $x_\alpha x_\beta = 0$. This shows that $A \cong A'$. On the other hand, $\beta\alpha - \gamma\alpha$ is a minimal relation. Hence $\overline{\beta}\ \overline{\alpha} = \overline{\gamma}\ \overline{\alpha}$ in $\pi_1(Q, I')$ and $\overline{\beta}\ \overline{\gamma}^{-1} = 1$. This shows that $\pi_1(Q, I') = 1$. In particular, A is not simply connected.

(d) Let $A = kQ/I$, where Q is the quiver.



and I is generated by $\alpha\delta, \beta\epsilon$ and $\gamma\delta - \gamma\epsilon$. Then clearly $\pi_1(Q, I) = 1$. Moreover, A is simply connected. Indeed, any choice of a basis of $rad\ A/rad^2 A$ will lead to at least one minimal relation with target 1 and source $i \in \{3, 4, 5\}$.

(e) The following example, due to Riedtmann, explains the triangularity assumption in the definition. Let Q be the quiver.



and I be the ideal generated by $\alpha^2 - \beta\gamma, \gamma\beta - \gamma\alpha\beta, \alpha^4$. We claim that $\pi_1(Q, I) = 1$. Choose 1 as base point. It suffices to show that each of the cyclic walks α and $\beta\gamma$ has a trivial homotopy class. Now, $\gamma\beta - \gamma\alpha\beta$ is a minimal relation. Hence $\overline{\gamma}\overline{\beta} = \overline{\gamma}\ \overline{\alpha}\overline{\beta}$ which yields $\overline{\alpha} = 1$. Similarly, $\alpha^2 - \beta\gamma$ yields $\overline{\beta}\overline{\gamma} = 1$.

2. The separation condition.

2.1. We now look at a combinatorial condition which is easy to verify and is closely related to simple connectedness (in fact, is equivalent to it in the representation-finite case). This is the so-called separation condition of Bautista, Larrión and Salmerón [BLS]. In order to define it, we let A be a triangular algebra (not necessarily connected), an $A-$module M is called *separated* if, for each connected component C of A, the restriction $M|_C$ of M to C is zero or is indecomposable. This can be expressed in terms of supports: the *support* of an $A-$module M is the full subcategory *supp* M of A generated by all $x \in A_0$ such that $M_x \neq 0$. Thus, an $A-$module M is separated if and only if the supports of its distinct indecomposable summands lie in distinct connected components of A. For instance, any indecomposable module over a connected algebra is (trivially) separated.

DEFINITION. Let A be a triangular algebra, and $x \in A_0$.

(a) Let A^x denote the full subcategory of A generated by the non-predecessors of x in Q_A. Then x is a *separating point* if the restrictions to A^x of $rad\ P(x)_A$ is separated as an A^x-module. The algebra A is said to satisfy the *separation condition* if each $x \in A_0$ is a separating point.

(b) Let ${}^{x}A$ denote the full subcategory of A generated by the non-successors of x in Q_A. Then x is a *coseparating point* if the restriction to ${}^{x}A$ of $I(x)/S(x)$ is separated as an ${}^{x}A-$module. The algebra A is said to satisfy the *coseparation condition* if each $x \in A_0$ is a coseparating point.

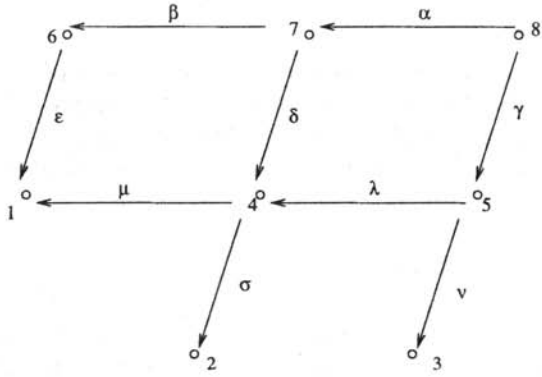
2.2. Examples.

(a) An algebra whose quiver is a tree always satisfies the separation (and the coseparation) condition.

(b) A hereditary algebra satisfies the separation (or the coseparation) condition if and only if its quiver is a tree.

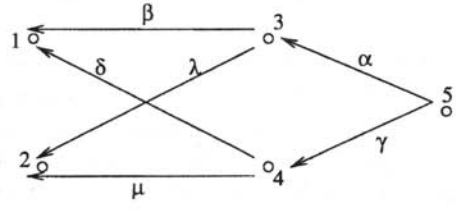
(c) The algebra of (1.3)(d) does not satisfy the separation condition: the point 2 is not separating, since $rad\ P(3) \cong \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus S(2)$ has two distinct indecomposable summands lying in the same connected component of A^3.

(d) The algebra given by the quiver



bound by $\alpha\delta = \gamma\lambda$, $\beta\epsilon = \delta\mu$, $\alpha\beta = 0$, $\lambda\mu = 0$ satisfies the separation condition.

(e) Clearly, if A satisfies the coseparation condition, then its opposite algebra A^{op} satisfies the separation condition, and conversely. There exist however examples of algebras satisfying one of these conditions, but not the other. Let A be given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\alpha\lambda = \gamma\mu$. Each indecomposable injective has an indecomposable (or zero) quotient by its socle, hence A satisfies the coseparation condition. On the other hand, A does not satisfy the separation condition (neither 3 nor 4 is separating).

2.3. We now investigate the relation between the separation condition and simple connectedness. For this purpose, we recall that the *one-point extension* of an algebra B by a $B-$module M is the matrix algebra

$$A = B[M] = \begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$$

where the operations are induced from the matrix operations and the module structure of M. The quiver Q_A of A then contains the quiver Q_B of B as a full convex subquiver, and there is an additional (extension) point which is a source.

Dually, the *one-point coextension* of B by M is the matrix algebra

$$[M]B = \begin{bmatrix} k & 0 \\ DM & B \end{bmatrix}$$

(where $D = Hom_k(\_, k) : mod\ B \to mod\ B^{op}$ is the standard duality), where the operations are induced from the matrix operations and the module structure of M. The quiver Q_A of A then contains the quiver Q_B of B as a full convex subquiver, and there is an additional (coextension) point which is a sink.

It is important to observe that triangular algebras can be constructed as repeated one-point extensions (or as repeated one-point coextensions). We have the following lemma (which follows from the proof of [S2] (2.3)).

LEMMA. Let $A = B[M]$, where $M = rad\ P(x)_A$. If B is simply connected and x is separating, then A is simply connected.

Proof. Let (Q_A, I) be an arbitrary presentation of A, and let $B = B_1 \times \ldots \times B_t$, and $M = M_1 \oplus \ldots \oplus M_t$, where each B_i is connected, and M_i is a B_i-module (for $1 \leq i \leq t$). For each i with $1 \leq i \leq t$, let C_i denote the full convex subcategory generated by x and all objects in B_i. In order to prove that $\pi_1(Q_A, I) = 1$, it suffices to prove that each of the groups $\pi_1(Q_{C_i}, I \cap kQ_{C_i})$ is trivial. Therefore, we may assume that B is connected and M is indecomposable.

Let $\alpha_i : x \to y_i$ (with $1 \leq i \leq m$) be all arrows of Q_A with source x. We show that, for any i, j with $1 \leq i, j \leq m$, the walk $\alpha_i^{-1}\alpha_j$ is homotopic to a walk lying entirely within Q_B. This will imply that any cyclic walk in Q_A passing through x is homotopic to a cyclic walk lying entirely within Q_B, and hence its homotopy class is trivial.

Suppose that this is not the case. Then we may assume that there exists t with $1 \leq t \leq m$ such that, for every i, j with $1 \leq i \leq t < j \leq m$, the walk $\alpha_i^{-1}\alpha_j$ is not homotopic to any walk lying entirely within Q_B. Let $M_1 = \sum_{i=1}^{t} \alpha_i A$ and $M_2 = \sum_{j=t+1}^{m} \alpha_j A$. Then $M_1 \cap M_2 \neq 0$, because M is indecomposable. Thus

there exists $z \in A_0$ such that $(M_1 \cap M_2)_z \neq 0$ and, for every $y \in A_0$ distinct from x and z, and lying on a path $x \to \ldots \to y \to \ldots \to z$, we have $(M_1 \cap M_2)_y = 0$. Then there exist paths $p_{i_s} : y_i \to \ldots \to z$ and non-zero scalars $\lambda_{i_s} \in k$ with $1 \leq s \leq r_i$, $1 \leq i \leq m$, such that:

(1) $\rho = \sum_{i=1}^{m} \sum_{s=1}^{r_i} \lambda_{i_s} \alpha_i p_{i_s}$ is a minimal relation.

(2) There exist i_0, s_0 such that $1 \leq i_0 \leq t$, $1 \leq s_0 \leq r_{i_0}$ and $\lambda_{i_0 s_{i_0}} \alpha_{i_0} p_{i_0 s_0} \notin I$.

(3) There exist j_0, q_0 such that $t+1 \leq j_0 \leq m$, $1 \leq q_0 \leq r_{j_0}$ and $\lambda_{j_0 q_0} \alpha_{j_0} p_{j_0 q_0} \notin I$.

The minimality of ρ implies that $\alpha_{i_0} p_{i_0 s_0} \sim \alpha_{j_0} p_{j_0 q_0}$. Hence $\alpha_{i_0}^{-1} \alpha_{j_0} \sim p_{i_0 s_0} p_{j_0 q_0}^{-1}$, and $p_{i_0 s_0} p_{j_0 q_0}^{-1}$ lies entirely within Q_B, a contradiction. Therefore $\pi_1(Q_A, I)$ is trivial and so A is simply connected. ∎

2.4. As we shall see, a direct consequence of (2.3) is that any algebra satisfying the separation condition is simply connected. This provides a large class of examples of simply connected algebras. In order to prove it, however, we need the following easy lemma.

LEMMA [AL1] (3.1). Let $A = B[M]$, where $M = rad\ P(x)_A$. If A satisfies the separation condition, then so does B.

Proof. We must show that any $y \in B_0$ is separating. Since A is triangular, there is no path from y to x. Hence, the indecomposable projective $B-$module $P(y)$, when considered as an $A-$module, equals $P(y)_A$. If x precedes y, then $B^y = A^y$, and y is separating in B because it is so in A. If x does not precede y, then A^y is generated by B^y and x. Assume that $rad\ P(y)_B$ is not a separated $B-$module. Then there exist two distinct indecomposable summands M_1 and M_2 of $rad\ P(y)_B$ whose supports lie in the same connected component of B^y. But M_1 and M_2 lie in distinct connected components of A^y, a contradiction. ∎

2.5. We proceed to prove that the separation condition implies simple connectedness. It is reasonable to ask whether the converse of this statement is true. This is the case for representation-finite algebras: a representation-finite algebra is simply connected if and only if it satisfies separation condition (see [BLS] or [BrG](2.9)). This is not true for representation-infinite algebras: the algebras of examples (1.3)(d) and (2.2)(e) are simply connected, but do not satisfy the separation condition. We have however a partial converse, for which we need a definition. A schurian algebra A is called $\tilde{\boldsymbol{A}}$ *−free* if it contains no full subcategory $B \cong kQ$, where the underlying graph of Q is the euclidean diagram $\tilde{\boldsymbol{A}}_m$ (for some $m \geq 1$). We then have the following theorem, of which the first statement is [S2] (2.3) and the second is [AS1] (1.2) Remark 1.

THEOREM. Let A be a triangular algebra.

(a) If A satisfies the separation condition, then A is simply connected.

(b) If A is simply connected, schurian and $\tilde{\boldsymbol{A}}$ −free, then it satisfies the separation condition.

Proof.

(a) This follows from (2.3) and induction. Since A is triangular, we may choose a source x in its quiver, and let B be the full convex subcategory of A generated by all objects of A except x. Then $A = B[M]$, where $M = rad\ P(x)_A$. Since A satisfies the separation condition, then x is a separating point and, by (2.4), each of the connected components of B satisfies the separation condition. By the inductions hypothesis, this implies that (each of the connected components of) B is simply connected. Applying (2.3) completes the proof.

(b) Since A is schurian and simply connected, it follows from [BrG](2.3) that its first homology group vanishes. But then, since A is $\tilde{\boldsymbol{A}}$ −free, it follows from [Bo](2.3), [BrG](2.9), that it satisfies the separation condition. ∎

2.6. The following theorem (which, in view of (2.5)(a), generalises (2.3)) gives a necessary and sufficient condition so that a one-point extension (or coextension) satisfies the separation condition (or the coseparation condition, respectively).

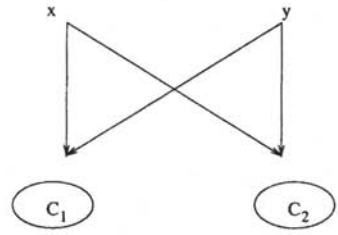
THEOREM [AL1](3.1).

(a) Let $A = B[M]$, where $M = rad\ P(x)$, then A satisfies the separation condition if and only if B satisfies the separation condition, and x is a separating point.

(b) Let $A = [M]B$, where $M = I(x)/S(x)$, then A satisfies the coseparation condition if and only if B satisfies the coseparation condition, and x is a coseparating point.

Proof. We only prove (a), since the proof of (b) is similar. Assume that A satisfies the separation condition. Then, clearly, x is separating and, by (2.4), B satisfies the separation condition.

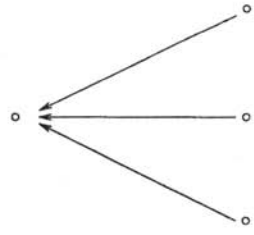
Conversely, assume that B satisfies the separation condition and that x is separating. We must show that any $y \in B_0$ is separating in A. If x precedes y, then clearly y is separating in A, since $(A^y)_0 \cup \{y\} = (B^y)_0 \cup \{y\}$ in this case. If x does not precede y, then A^y is generated by B^y and x. Assume that $rad\ P(y)_A$ is not a separated A^y−module. Then there exist two distinct indecomposable summands M_1, M_2 of $rad\ P(y)_A$ whose supports are connected in A^y. Since they are not connected in B^y (because B satisfies the separation condition), there exist two distinct connected components of B^y, say C_1 and C_2, containing respectively the supports of M_1 and M_2, and connected in A^y via the extension point x. In fact, each of C_1, C_2 is connected to x by a single arrow: let $x \to x_1 - x_2 - \ldots - x_t$ with $x_t \in (C_i)_0$ be a walk of least length from x to C_i, this implies $x_j \neq x$ for all j such that $1 \leq j \leq t$, and thus $x_j \in (B^y)_0$ for all j, so that $x_1 \in (C_i)_0$. Thus the restriction of M to each C_i is non-zero, and we have the following picture



We observe that C_1 and C_2 belong to the same component of B, since they are connected through y. Moreover, the restriction of M to this component is indecomposable, since M is a separated $B-$module. In particular, there exist $s > 1$ and a walk $c_1 = z_0 - z_1 - \ldots - z_{s-1} - z_s = c_2$ in $supp\, M$, with $c_1 \in (C_1)_0, c_2 \in (C_2)_0$ and $z_i \notin (B^y)_0$ for all i such that $1 \leq i < s$, because C_1 and C_2 are disconnected in B^y. Thus, each z_i precedes y. On the other hand, since $z_i \in (supp\, M)_0$, there exists a path from x to z_i. Consequently, x precedes y, a contradiction. ■

We deduce from this theorem an easy inductive procedure for constructing all (triangular) algebras satisfying the separation (or the coseparation) condition: an algebra A satisfies the separation (or the coseparation) condition if and only if there exists a sequence of algebras $A_0, A_1, \ldots A_n = A$, with A_0 semisimple, and, for each i such that $0 \leq i < n$, a separated A_i-module M_i such that $A_{i+1} = A_i[M_i]$ (or $A_{i+1} = [M_i]A_i$, respectively).

We notice that if B is an algebra satisfying the separation condition, and M is a separated $B-$module, then $[M]B$ usually does not satisfy the separation condition (even if B also satisfies the coseparation condition). Indeed, let B be the (representation-finite) hereditary algebra given by the quiver



and M be the unique indecomposable $B-$module with dimension-vector $2\begin{smallmatrix}1\\1\end{smallmatrix}$. Then $[M]B$ is the algebra of example (1.3)(d): it does not satisfy the separation condition.

3. The fundamental groups of a one-point extension.

3.1. Let $A = B[M]$, where $M = rad\, P(x)_A$. Any presentation (Q_A, I_ν) of A

induces a presentation (Q_B, I'_ν) of B by setting $I'_\nu = I_\nu \cap kQ_B$. Our objective is to relate the fundamental groups of (Q_A, I_ν) and (Q_B, I'_ν). We start by computing a direct sum decomposition of M.

We let $\approx$ denote the smallest equivalence relation on the set of all arrows of source x such that $\alpha \approx \beta$ whenever there exist $y \in A_0$ and a minimal relation $\sum_{i=1}^{m} \lambda_i w_i \in I_\nu(x,y)$ such that $w_1 = \alpha w'_1$ and $w_2 = \beta w'_2$. We denote by $[\alpha]_\nu$ the equivalence class of α, and define a module $M_{[\alpha]_\nu}$ as follows. Let $A_{[\alpha]_\nu}$ be the quotient algebra of A obtained by deleting all arrows β of source x such that $\beta \notin [\alpha]_\nu$, let $P(x)_{[\alpha]_\nu}$ be the indecomposable projective $A_{[\alpha]_\nu}$–module corresponding to x and, finally, let $M_{[\alpha]_\nu} = rad\ P(x)_{[\alpha]_\nu}$. Since $A_{[\alpha]_\nu}$ is a quotient of A, the $A_{[\alpha]_\nu}$–module $M_{[\alpha]_\nu}$ has a canonical A–module structure. We then have the following result.

PROPOSITION [AP](2.1). Let A be a triangular algebra, x be a source in A, and $M = rad\ P(x)$.

(a) For any presentation (Q_A, I_ν) of A, we have $M = \oplus_{[\alpha]_\nu} M_{[\alpha]_\nu}$.

(b) There exists a presentation (Q_A, I_μ) of A such that, for each arrow α of source x, the module $M_{[\alpha]_\mu}$ is indecomposable. Thus, $M = \oplus_{[\alpha]_\mu} M_{[\alpha]_\mu}$ is an indecomposable decomposition. ∎

For instance, in the example of a non-simply connected algebra in (1.3) (c), we have two presentations (Q, I) and (Q, I'). For the first, we have $M_{[\beta]} = S(2)$ and $M_{[\gamma]} = P(2)$, while, for the second, $rad\ P(3) = M_{[\beta]} = M_{[\gamma]}$.

3.2. We now reformulate (3.1) in terms of some numerical invariants. Let $c(x)$ denote the number of connected components of B, and $t(x)$ denote the number of summands in an indecomposable decomposition of $M = rad\ P(x)$. Thus, by definition, the source x is separating if and only if $c(x) = t(x)$. Also, given a presentation (Q_A, I_ν) of A, let $t(\nu)$ denote the number of equivalence classes under $\approx$ in the set of all arrows of source x. With this notation, (3.1) becomes the following corollary.

COROLLARY [AP](2.2). Let A be a triangular algebra, and x be a source in A.

(a) For any presentations (Q_A, I_ν) of A, we have $c(x) \leq t(\nu) \leq t(x)$.

(b) There exists a presentation (Q_A, I_μ) of A such that $t(\mu) = t(x)$. ∎

3.3. We now state our main result of this section. Let A be a connected triangular algebra, and x be a source in A. Then $A = B[M]$, where B is generated by all objects of A except x, and $M = rad\ P(x)$. Let $B = B_1 \times \ldots \times B_{c(x)}$, where each B_i is connected, then write $Q^{(i)} = Q_{B_i}$ and $I_\nu^{(i)} = I_\nu \cap kQ^{(i)}$ for each i such that $1 \leq i \leq c(x)$. The embedding of $Q^{(i)}$ inside Q_A induces a canonical group morphism $f_i : \pi_1(Q^{(i)}, I_\nu^{(i)}) \to \pi_1(Q_A, I_\nu)$. Thus, for any abelian group G, and

any i, there exists an induced morphism of abelian groups

$$f_i^* = Hom(f_i, G) : Hom(\pi_1(Q_A, I_\nu), G) \to Hom(\pi_1(Q^{(i)}, I_\nu^{(i)}), G)$$

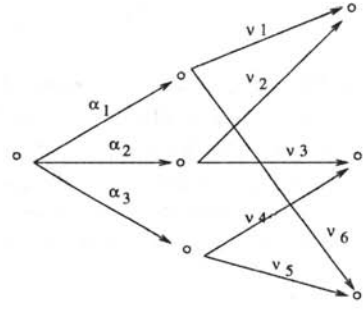
and hence a morphism of abelian groups

$$f^* = (f_i^*)_i : Hom(\pi_1(Q_A, I_\nu), G) \to \prod_{i=1}^{c(x)} Hom(\pi_1(Q^{(i)}, I_\nu^{(i)}), G).$$

We now exhibit an exact sequence of abelian groups allowing to compute the kernel and the cokernel of f^*. We need some additional notation. Let $\beta_1, \ldots, \beta_{t(\nu)}$ be a complete set of representatives of the classes $[\alpha]_\nu$ of the arrows α of source x (under the equivalence $\approx$). For each i, such that $1 \leq i \leq t(\nu)$, let $l(i)$ denote the number of those tuples of paths $(v_1, v_2, \ldots, v_{2s-1}, v_{2s})$ such that there exist minimal relations

$$\lambda_{i1}\alpha_i v_{2i-1} + \lambda_{i2}\alpha_{i+1}v_{2i} + \sum_{j\geq 3} \lambda_{ij}u_{ij} \in I_\nu(x, y_i) \text{ for } i \in \{1, \ldots, s\}$$

where $\alpha_1, \ldots, \alpha_s, \alpha_{s+1} = \alpha_1$ are pairwise distinct arrows in the class $[\beta_i]_\nu$, and the λ_{ij} are non-zero scalars. We illustrate here the case $s = 3$.



THEOREM [AP](2.4). Let A be a connected triangular algebra, x be a source in A, and G be any abelian group. There exists an exact sequence of abelian groups

$$0 \to G^{t(\nu)-c(x)} \to Hom(\pi_1(Q_A, I_\nu), G) \xrightarrow{f^*} \prod_{i=1}^{c(x)} Hom(\pi_1(Q^{(i)}, I_\nu^{(i)}), G) \to \prod_{i=1}^{t(\nu)} G^{l(i)}.$$

■

For instance, if the point x is separating, then $c(x) = t(x)$, hence (3.2)(a) implies that $c(x) = t(\nu)$, so that f^* is injective.

An example of this situation is provided by the algebra A given by the bound quiver of (2.2)(e). The algebra A is simply connected, but does not satisfy the

separation condition. However the source $x = 5$ is separating. In this case, $c(5) = 1, t(\nu) = 1, \pi_1(Q_A, I_\nu) = 1, l(1) = 1, t(5) = 1$ and $\pi_1(Q^{(1)}, I^{(1)}) \cong \mathbb{Z}$ yield that the last morphism in the exact sequence of our theorem is bijective.

3.4.

COROLLARY [AP](2.6). Let A be a connected triangular algebra, and assume there exists a non-trivial abelian group G such that, for every presentation (Q_A, I_ν) of A, we have $Hom(\pi_1(Q_A, I_\nu), G) = 0$. Then all sources in A are separating. In particular, if A is simply connected, then all sources in A are separating.

Proof. Let x be a source in A. By (3.2)(b), there exists a presentation (Q_A, I_μ) of A such that $t(\mu) = t(x)$. By (3.3), there exists an injective group morphism $G^{t(\mu)-c(x)} \to Hom(\pi_1(Q_A, I_\mu), G)$. Since the latter vanishes, we have $t(\mu) = c(x)$. Consequently, $c(x) = t(x)$ and x is separating. ∎

In view of this corollary, we may ask whether a statement similar to (2.6) holds for simply connected algebras, namely whether, if $A = B[M]$, where $M = rad\ P(x)$, then A is simply connected if and only if B is simply connected and x is separating. This is not the case. Indeed, by (2.3), if B is simply connected and x is separating, then A is simply connected. So the condition is sufficient. On the other hand, it is not necessary: while the above corollary says that the simple connectedness of A implies that x is separating, B is generally not simply connected, as is shown by the algebra of example (2.2)(e).

4. The Hochschild cohomology spaces and simple connectedness.

4.1. Let A be an algebra. We denote by $H^i(A)$ the i^{th} Hochschild cohomology space of A with coefficients in the bimodule ${}_AA_A$, as defined in [CE]. We need some facts from [H2], which we collect in the following theorem (these are, respectively, [H2](1.6), [H2](4.2) and [H2](5.3)). We recall that $A-$module T is a *tilting module* if $pdT_A \leq 1$, $Ext^1_A(T,T) = 0$ and the number of isomorphism classes of indecomposable summands of T equals the number of objects in A_0 (see [HR]).

THEOREM.

(a) Let Q be a finite, connected and acyclic quiver, then $H^0(kQ) = k$, $H^1(kQ) = 0$ if and only if Q is a tree, and $H^i(kQ) = 0$ for all $i \geq 2$.

(b) Let A be an algebra, T_A be a tilting module and $B = End\, T_A$, then $H^i(A) \cong H^i(B)$ for all $i \geq 0$.

(c) Let $A = B[M]$ be a one-point extension algebra. Then there exists an exact sequence $0 \to H^0(A) \to H^0(B) \to End\, M/k \to H^1(A) \to H^1(B) \to Ext^1_B(M,M) \to H^2(A) \to \ldots \to Ext^{i-1}_B(M,M) \to H^i(A) \to H^i(B) \to Ext^i_B(M,M) \to \ldots$ ∎

4.2. The exact sequence of (4.1)(c) yields a direct relationship between the first

Hochschild cohomology space and the separation condition.

LEMMA. Let $A = B[M]$, where $M = rad\ P(x)_A$.

(a) The morphism $H^1(A) \to H^1(B)$ of (4.1)(c) is injective if and only if x is separating, and M is the direct sum of pairwise orthogonal bricks.

(b) $H^i(A) \cong H^i(B)$, for all $i \geq 1$, if and only if x is separating, and M is the direct sum of pairwise orthogonal bricks such that $Ext^i_B(M, M) = 0$ for all $i \geq 1$.

Proof.

(a) Let $B = B_1 \times \ldots \times B_t$, and $M = M_1 \oplus \ldots \oplus M_t$, where each B_i is connected, and each M_i is a B_i-module. The morphism $H^1(A) \to H^1(B)$ is injective if and only if

$$0 \to H^0(A) \to H^0(B) \to End\, M/k \to 0$$

is a short exact sequence, or, equivalently, if and only if $dim_k H^0(B) = dim_k H^0(A) + dim_k End\, M - 1$. We have $dim_k H^0(A) = 1$ because ($H^0(A)$ is the centre of A, and k is algebraically closed, hence) $H^0(A) = k$. Similarly, $dim_k H^0(B) = t$. Thus, the morphism $H^1(A) \to H^1(B)$ is injective if and only if $dim_k End\, M = t$. Now, $dim_k End\, M = dim_k End(\oplus_{i=1}^t M_i) \geq t$, and equality holds if and only if, for each i, we have $End\, M_i = k$ and, for $i \neq j$, we have $Hom_B(M_i, M_j) = 0$.

(b) this is trivial. ∎

4.3. The above lemma entails several important consequences. The following easy one should be compared with (3.4).

COROLLARY [S2](3.2). Let A be such that $H^1(A) = 0$, and x be a source in A. Then x is separating.

Proof. This follows directly from (4.2)(a). ∎

4.4. Let $A = B[M]$. If both A and B are connected, then $H^0(A) \cong k \cong H^0(B)$. Hence, if M is a brick, the morphism $H^1(A) \to H^1(B)$ is injective. As a consequence, we obtain the following corollary. For tubular algebras ,we refer the reader to [R1]. An algebra A is called *derived tubular* if there exists a tubular canonical algebra C and an equivalence of triangulated categories between the derived categories of bounded complexes $D^b(mod\ A) \cong D^b(mod\ C)$. It is shown in [AS1] that derived tubular algebra is always simply connected.

COROLLARY. Let A be a derived tubular algebra, then $H^1(A) = 0$.

Proof. Assume first that A is tubular canonical, then $A = B[M]$, where B is a connected tame hereditary algebra whose quiver is a tree, and M is a simple homogeneous $B-$module. In particular, M is a brick. Therefore, the morphism

$H^1(A) \to H^1(B)$ is injective. Since Q_B is a tree, $H^1(B) = 0$ (by (4.1)(a)). Hence $H^1(A) = 0$.

Let now A be an arbitrary derived tubular algebra. Then there exist a tubular canonical algebra C and an equivalence of triangulated categories $D^b(mod\ A) \cong D^b(mod\ C)$. By [ASO], there exist a sequence of algebras $A = A_0, A_1, \ldots, A_{m+1} = C$ and a sequence of modules $T^i_{A_i} (0 \leq i \leq m)$ such that $A_{i+1} = End\ T^i_{A_i}$ and either T^i is a tilting A_i-module, or $DT^{(i)}$ is a tilting $A_i^{op}-$module. Applying (4.1)(b), we deduce that $H^1(A) \cong H^1(C) = 0$. ∎

4.5. The first application of (4.2) to the study of simple connectedness has been in the representation-directed case. We recall that an algebra A is *representation-directed* whenever its module category contains no cycle of non-zero non-isomorphisms between indecomposables. Such an algebra is always representation-finite [R1].

COROLLARY [H2](5.5). Let A be a representation-directed algebra, then A is simply connected if and only if $H^1(A) = 0$.

Proof. Since A is representation-finite, it suffices to show that A satisfies the separation condition if and only if $H^1(A) = 0$. We use induction on the number of objects of A_0. Let $A = B[M]$, where $M = rad\ P(x)$, and $B = B_1 \times \ldots \times B_t$, $M = M_1 \oplus \ldots \oplus M_t$, where each B_i is connected and each M_i is a B_i-module.

Assume first that A satisfies the separation condition. Then x is separating, hence each M_i is indecomposable. Since *mod* A has no cycles, each M_i is a brick. By (4.2)(a), the morphism $H^1(A) \to H^1(B)$ is injective. On the other hand, each B_i satisfies the separation condition, hence, by the induction hypothesis, $H^1(B) = 0$. Therefore $H^1(A) = 0$.

Conversely, assume $H^1(A) = 0$. By (4.2)(a), x is separating. On the other hand, for each i, B_i is representation-directed (because it is a full convex subcategory of A) hence, since M_i is an indecomposable B_i-module, we have $Ext^1_{B_i}(M_i, M_i) \subseteq D\ Hom_{B_i}(M_i, \tau M_i) = 0$. Thus $Ext^1_B(M, M) = 0$. By (4.1)(c), $H^1(B) = 0$. By the induction hypothesis, B satisfies the separation condition. By (2.6), so does A. ∎

4.6. In general, we have the following relation between the first Hochschild cohomology space $H^1(A)$ and the fundamental groups $\pi_1(Q_A, I_\nu)$ of A. This theorem was first shown for triangular algebras in [AP](3.2), then in general in [PS](3), [FGM](2).

THEOREM. Let A be a non-necessarily triangular algebra, and (Q_A, I_ν) be a presentation of A. There exists an injective group morphism

$$Hom\ (\pi_1(Q_A, I_\nu), k^+) \to H^1(A)$$

(where k^+ denotes the additive group of the field k). ∎

This theorem yields an alternative proof of (4.3). Indeed, $H^1(A) = 0$ implies that $Hom\ (\pi_1(Q_A, I_\nu), k^+) = 0$ for any presentation (Q_A, I_ν) of A. By (3.4), x is separating.

4.7.

COROLLARY [BM]. Let A be an algebra having a monomial presentation (Q_A, I). If $H^1(A) = 0$, then Q_A is a tree (hence A is simply connected).

Proof. Since I is generated by monomial, the group $\pi_1(Q_A, I)$ is free and it is trivial if and only if Q_A is a tree (by (1.3)(a)). If Q_A is not a tree, then $Hom(\pi_1(Q_A, I), k^+) \neq 0$. By the theorem, $H^1(A) \neq 0$. This shows the first statement. The second follows from (1.3)(a). ∎

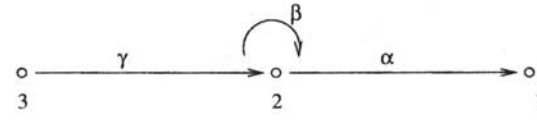
4.8. EXAMPLES.

(a) It is easy to construct examples of simply connected algebras, or of algebras satisfying the separation condition and having a non-zero first Hochschild cohomology space. Let B be a simply connected algebra (or an algebra satisfying a separation condition) such that $H^1(B) = 0$, and M be an indecomposable $B-$module which is not a brick. Then $A = B[M]$ is simply connected (or satisfying the separation condition): this follows from the indecomposability of M (which implies that the extension point is separating) and our assumption on B, by (2.3) (or (2.6), respectively). On the other hand, in the exact sequence

$$0 \to H^0(A) \to H^0(B) \to End\ M/k \to H^1(A) \to 0$$

we have $H^0(A) \cong k \cong H^0(B)$, $End\ M/k \neq 0$. Hence $H^1(A) \neq 0$.
For instance, let B be a hereditary algebra whose quiver is a tree, and M be an indecomposable regular module of regular length m, lying in a tube of rank $< m$. Then $B[M]$ is simply connected, and even satisfies the separation condition, but $H^1(B[M]) \neq 0$.

(b) If $H^1(A) = 0$, it is generally not true that A is triangular. If Q is the quiver



then, for any field k, there exists an admissible ideal I of kQ such that $H^1(kQ/I) = 0$ (see [BL]). If $char\ k = 0$, we may take I to be the ideal generated by $\beta^7, \gamma\alpha, \gamma e^{\beta}\alpha, \gamma e^{3\beta}\alpha$ (notice indeed that, since $\beta^7 \in I, e^{\beta}$ and $e^{3\beta}$ are expressed by polynomials in β).

(c) The algebra A of example (2.2)(e) does not satisfy the separation condition, since the points 3 and 4 are not separating. On the other hand,

$H^1(A) = 0$. Indeed, A is a one-point extension of a hereditary algebra of type $\tilde{\mathbf{A}}_{2,2}$ by a simple homogeneous module then, by [R1](4.9), it is a tilted algebra of tubular type $(2,2,2)$, that is, there exist a quiver Q, of underlying graph $\tilde{D}_5$, and a tilting $kQ-$module T such that $A = End\, T$. By (4.1)(a), $H^1(kQ) = 0$, hence, by (4.1)(b), $H^1(A) \cong H^1(kQ) = 0$. Notice however that A is simply connected.
Similarly, the algebra of example (1.3)(d) does not satisfy the separation condition, but is simply connected, and its first Hochschild cohomology space is zero.

5. Strongly simply connected algebras.

5.1. As one sees, it is not easy to recognise whether a given algebra is simply connected or not. The following subclass, introduced in [S2](2.2), is much more accessible.

DEFINITION. A triangular algebra A is called *strongly simply connected* if every full convex subcategory of A is simply connected.

Schurian strongly simply connected algebras were already introduced in [D], where they are called completely separating algebras.

Using the above definition, it does not seem easy to verify whether a given algebra is strongly simply connected or not. The following theorem gives our first criteria of strong simple connectedness.

THEOREM. Let A be a triangular algebra. The following conditions are equivalent:

(a) A is strongly simply connected.
(b) There exists a presentation (Q, I) of A such that, for any connected full convex bound subquiver (Q', I') of (Q, I), the group $\pi_1(Q', I')$ is trivial.
(c) Every connected full convex subcategory of A satisfies the separation condition.
(d) Every connected full convex subcategory of A satisfies the coseparation condition.
(e) For every connected full convex subcategory B of A, we have $H^1(B) = 0$.

Proof. The equivalence of (a) and (b) follows from [AL1](1.3), and the equivalence of (a) with the remaining conditions from [S2](4.1). ∎

5.2. EXAMPLES.

(a) It follows from the definition that if A is an algebra whose quiver is a tree, then A is strongly simply connected.
(b) Similarly, a hereditary algebra is strongly simply connected if and only if it is simply connected, or if and only if its quiver is a tree.

(c) Let A be as in example (1.3)(d) (or (2.2)(e)), then A is simply connected, but not strongly simply connected.

(d) The algebra A of example (2.2)(d) is strongly simply connected.

(e) Let A be representation-finite, then A is strongly simply connected if and only if A is simply connected [BrG] (2.8).

(f) Let A be a schurian algebra all of whose indecomposable projective modules are directing. Then it is shown in [AP](5.4) that the following conditions are equivalent:

i) A is simply connected.

ii) A is strongly simply connected.

iii) A satisfies the separation condition.

(g) Let R be a representation-finite algebra, and A be its Auslander algebra, then it is shown in [AB] that the following conditions are equivalent:

i) R is simply connected.

ii) A is simply connected.

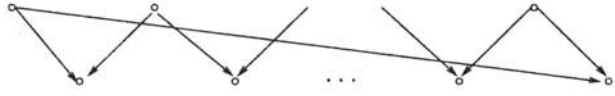
iii) A is strongly simply connected.

iv) A satisfies the separation condition.

v) $H^1(A) = 0$.

In fact, the equivalence of i) and v) follows from [H3](4).

(h) It is shown in [D] that the incidence algebra of a (finite) partially ordered set is strongly simply connected if and only if it does not contain a full subcategory whose quiver is a crown, that is, is of the form



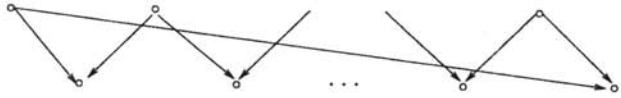
As we shall now see, such subcategories play an important rôle in the study of strongly simply connected algebras.

5.3. We now give criteria allowing to recognise from its bound quiver whether a given algebra is strongly simply connected or not. We need a few definitions. Let Q be an acyclic quiver. A *cycle* C in Q is a subquiver such that each point in C is an endpoint of exactly two arrows in C and there exists an enumeration $\{x_0, x_1, \ldots, x_{n-1}, x_n = x_0\}$ of the points of C such that, for each i with $1 \leq i \leq n$, there exists an edge between x_{i-1} and x_i in C.

A *contour* (p, q) in Q from x to y is a pair of paths of positive length from x to y. A contour (p, q) is called *interlaced* if p and q have a common point besides x and y. A contour (p, q) is called *irreducible* if there exists no sequence of paths $p = p_0, p_1, \ldots, p_m = q$ in Q from x to y such that, for each i, the contour (p_i, p_{i+1}) is interlaced. A cycle C in Q is *irreducible* if, either C is an irreducible contour, or C is not a contour, but satisfies the following condition and its dual: for each

source x in C, no proper successor of x in Q is also a source in C, and exactly two proper successors of x in Q are sinks in C.

Thus, a typical example of an irreducible cycle which is not a contour is provided by a *crown*, that is, a subquiver of the form:



Finally, let I be an admissible ideal in kQ. A contour (p,q) from x to y is called *naturally contractible* in (Q,I) if there exists a sequence of paths $p = p_0, p_1, \dots, p_m = q$ in Q such that, for each i, the paths p_i and p_{i+1} have subpaths q_i and q_{i+1}, respectively, such that there exists a minimal relation $\rho_i = \sum_{j=1}^{m} \lambda_{ij} w_{ij}$ satisfying $q_i = w_{i1}$ and $q_{i+1} = w_{i2}$.

For instance, in the example (1.3)(d), bound by the ideal I', the arrows β, γ are homotopic in (Q,I'), but the contour (β,γ) is not naturally contractible. Similarly, in the bound quiver of example (1.3)(e), the arrows δ, ϵ are homotopic, but the contour (δ,ϵ) is not naturally contractible. On the other hand, in the bound quiver of example (2.2)(d), all shown contours are naturally contractible.

We have the following criterion.

THEOREM. Let A be a triangular algebra. The following conditions are equivalent:

(a) A is strongly simply connected.

(b) There exists a presentation (Q,I) of A such that each irreducible cycle in Q is an irreducible contour, and each irreducible contour is naturally contractible in I.

(c) For any presentation (Q,I) of A, each irreducible cycle in Q is an irreducible contour, and each irreducible contour is naturally contractible in I. ∎

5.4. In the schurian case, the situation is easier. As we now see, a triangular algebra is schurian and strongly simply connected if and only if it has a presentation such that all irreducible cycles are commutative contours. This implies that such an algebra has a multiplicative basis [BGRS]. A presentation of a schurian strongly simply connected algebra, such as in (b) below, is called a *normed presentation.*

THEOREM [AL1](2.4). Let A be a triangular algebra. The following conditions are equivalent:

(a) A is schurian and strongly simply connected.

(b) There exists a presentation (Q,I) of A such that all irreducible cycles in Q are irreducible contours and, for each irreducible contour (p,q), we have $p, q \notin I$ but $p - q \in I$.

(c) For any presentation (Q, I) of A, all irreducible cycles in Q are irreducible contours and, for each irreducible contour (p, q), we have $p, q \notin I$ and there exists a non-zero scalar λ such that $p - \lambda q \in I$. ■

5.5. We now give a construction for strongly simply connected algebras as iterated one-point extensions which resembles the one given in (2.6) for algebras satisfying the separation condition. We thus seek a criterion in order that a one-point extension of a strongly simply connected algebra be also strongly connected.

Let B a triangular algebra, and M be a $B-$module. An enumeration $\{x_1, \ldots, x_m\}$ of the points of *supp* M is called an *admissible order of sinks* (or of *sources*) if $j > i$ implies that x_j is not a sucessor (or predecessor, respectively) of x_i. Since B is triangular, then, for each $B-$module M, there exists at least one admissible order of sinks (or of sources) of the points of *supp* M. With each such order is associated a filtration of B by a sequence of full convex subcategories. Indeed, let $\{x_1, \ldots, x_m\}$ be an admissible order of sinks (or of sources) of the points of *supp* M. We let $B^{(0)} = B$ and, for each i such that $0 < i < m$, we let $B^{(i)}$ be the full subcategory of B generated by the non-successors (or non-predecessors, respectively) of the points $x_1, \ldots, x_i$. Clearly, each $B^{(i)}$ is convex and we have $B = B^{(0)} \supseteq B^{(1)} \supseteq \ldots \supseteq B^{(m-1)}$.

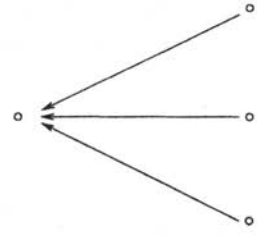
DEFINITION. Let B a triangular algebra, and M be a $B-$module.

(a) M is called *completely coseparated* if, for any admissible order of sinks of the points of *supp* M and, for each i such that $0 \leq i < m$, the restriction $M_{|_{B^{(i)}}}$ of M to $B^{(i)}$ is a separated $B^{(i)}-$module.

(b) M is called *completely separated* if, for any admissible order of sources of the points of *supp* M and, for each i such that $0 \leq i < m$, the restriction $M|_{B^{(i)}}$ of M to $B^{(i)}$ is a separated $B^{(i)}-$module.

For instance, any universal module is completely coseparated and completely separated.

There exist completely coseparated modules which are not completely separated, as is shown by the following example: let B be the hereditary algebra given by the quiver



and M be the indecomposable module of dimension-vector $2\begin{smallmatrix}1\\1\end{smallmatrix}$.

On the other hand, any completely cosepareted (or completely separated) module is separated, thus, if the algebra is connected, is indecomposable. In fact, it will follow from the theorem below and [S2](4.2) that such a module is even a brick. There exist however bricks which are not completely coseparated: take the hereditary algebra given by the quiver opposite to the one above, and consider again the indecomposable module of dimension-vector $2\begin{smallmatrix}1\\1\end{smallmatrix}$. We then have the following result:

THEOREM [AL1](3.4). Let B be a strongly simply connected algebra, and M be a $B-$module. Then

(a) $A = B[M]$ is strongly simply connected if and only if M is a completely coseparated $B-$module.

(b) $A = [M]B$ is strongly simply connected if and only if M is a completely separated $B-$module. ∎

As a consequence, one can show that a connected algebra A is strongly simply connected if and only if there exists a sequence of algebras $A_0, A_1, \ldots, A_n = A$, with $A_0 = k$ and, for each i with $0 \leq i < n$, an A_i-module M_i, such that either M_i is completely coseparated and $A_{i+1} = A_i[M_i]$, or M_i is completely separated and $A_{i+1} = [M_i]A_i$.

5.6. The preceding theorem (5.5) raises the question of how to construct the completely coseparated modules. In the schurian case, these modules are completely classified. Let Q be an acyclic quiver. An enumeration $\{x_1, \ldots, x_m\}$ of the points of Q is called an *admissible order of sinks* (or of *source*) if $j > i$ implies that x_j is not a successor (or predecessor, respectively) of x_i. Let $\{x_1, \ldots, x_m\}$ be an admissible order of sinks (or of sources) of the points of Q, then let $Q^{(0)} = Q$ and, for each i such that $0 < i < m$, let $Q^{(i)}$ be the full subquiver of Q generated by the non-successors (or non-predecessors, respectively) of $x_1, \ldots x_i$ in Q. Clearly, each $Q^{(i)}$ is convex in Q, and we have $Q = Q^{(0)} \supseteq Q^{(1)} \supseteq \ldots \supseteq Q^{(m-1)}$.

DEFINITION. Let Q be an acyclic quiver. A full subquiver Q' of Q is called *completely coseparated* (or *completely separated*) if, for each admissible order of sinks (or of sources, respectively) of the points of Q, and each i such that $1 \leq i \leq m$, the intersection of Q' with each of the connected components of $Q^{(i)}$ is empty or connected.

It is easily shown that if A is strongly simply connected, and M_A is a completely coseparated (or completely separated) module, then *supp* M is a completely coseparated (or completely separated) subquiver of Q_A.

Let now A be an algebra. Given a full subquiver Q of Q_A, we denote by $U(Q)$ (see [D](2.8)) the representation of Q_A defined by

$$U(Q)_x = \begin{cases} k & \text{if } x \in Q_0, \\ 0 & \text{if } x \notin Q_0. \end{cases}$$

$$U(Q)_\alpha = \begin{cases} 1 & \text{if } \alpha \in Q_1, \\ 0 & \text{if } \alpha \notin Q_1. \end{cases}$$

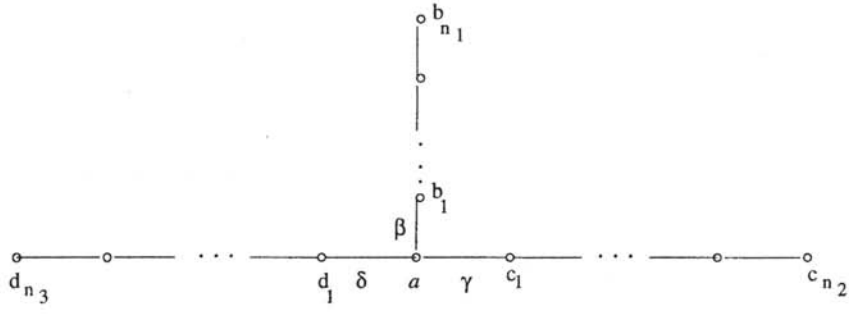
for $x \in (Q_A)_0$ and $\alpha \in (Q_A)_1$. We then have the following theorem.

THEOREM [AL1] (4.4). Let A be a schurian and strongly simply connected algebra, with normed presentation (Q_A, I), and let M be an $A-$module, then:

(a) $A[M]$ is schurian and strongly simply connected if and only if $M \cong U(Q)$, where Q is a completely coseparating convex subquiver of Q_A containing no path lying in I.

(b) $[M]A$ is schurian and strongly simply connected if and only if $M \cong U(Q)$, where Q is a completely separating convex subquiver of Q_A containing no path lying in I. ∎

5.7. We now give the classification of the completely coseparating modules over the hereditary algebras A whose quiver Q_A has for underlying graph a star with three branches Π_{n_1,n_2,n_3} where $n_1, n_2, n_3 \geq 1$



The point a is called the *node* of the star. Assume that the node is a sink, and that Q is a connected full subquiver of Q_A containing both the node and its three neighbours. We denote by $V(Q)$ the representation of Q_A defined by

$$V(Q)_x = \begin{cases} k^2 & \text{if } x = a, \\ k & \text{if } x \in Q_0 \backslash \{a\}, \\ 0 & \text{if } x \notin Q_0. \end{cases}$$

$$V(Q)_\alpha = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if } \alpha = \beta, \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if } \alpha = \gamma, \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if } \alpha = \delta, \\ 1 & \text{if } \alpha \in Q_1 \setminus \{\beta, \gamma, \delta\}, \\ 0 & \text{if } \alpha \notin Q_1. \end{cases}$$

for $x \in (Q_A)_0$ and $\alpha \in (Q_A)_1$. We then have the following result.

THEOREM [AC] (4.3). Let A be a hereditary algebra of type Π_{n_1,n_2,n_3}, and M be an $A-$module with connected support Q. Then M is completely coseparating if and only if $M \cong U(Q)$, or the node of Q_A is a sink contained in Q together with its three neighbours and $M \cong V(Q)$. ∎

The proof is done by induction on the number of objects in A, starting from the Dynkin diagram $\mathbb{D}_4$.

As a consequence, if A is a hereditary algebra whose quiver has for underlying graph a Dynkin or an euclidean diagram distinct from $\tilde{\mathbf{A}}_m$ or $\tilde{\mathbb{D}}_n$, and M is an $A-$module with connected support Q, then M is completely coseparating if and only if $M \cong U(Q)$, or else Q contains a sink and three neighbours of this sink, and $M \cong V(Q)$. This led to the complete classification of the completely coseparating modules over the tame hereditary algebras (see [AC]).

6. Tilting and simple connectedness.

6.1. Let A be an algebra, T_A be a tilting module and $B = End\ T$. Then it is known that there is a close connection between the representation theories of A and B. This connection is known as tilting theory, and we refer to [A2] for details. It has long been conjectured that simple connectedness is preserved under the tilting process, that is, if A, T, B are as above, then A is simply connected if and only if B is simply connected. It was first shown in [A1] (3.5) that, if A is a representation-finite simply connected algebra, and T_A is a splitting tilting module, then $B = End\ T$ is simply connected. The following generalisation of this statement was proved in [AS2].

THEOREM. Let A be a representation-finite simply connected algebra and T_A be a tilting module. Then $B = End\ T$ satisfies the separation condition.

Proof. Let $T = T_1 \oplus \ldots \oplus T_n$, with the T_i indecomposable. We clearly may assume that the T_i are pairwise non-isomorphic. Since A is simply connected and representation-finite, it is representation-directed, so the indecomposable summands T_i of T may be partially ordered in the order induced by the arrows in the Auslander-Reiten quiver $\Gamma(mod\ A)$ of A, that is, $Hom_A\ (T_j, T_i) \neq 0$ implies $j \leq i$. In particular, this implies that B is a triangular algebra. For each i with $1 \leq i \leq n$, we set

$$B_i = End\ (\oplus_{j=1}^{i} T_j)$$

(thus $B_n = B$). We prove by descending induction that, for each i, we have $H^1(B_i) = 0$. Since A is representation-directed and satisfies the separation condition, it follows from (4.5) that $H^1(A) = 0$. By (4.1)(b), this implies $H^1(B_n) = H^1(B) = 0$.

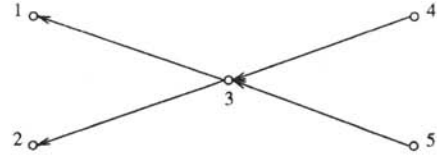
For a given i, the algebra B_i is the one-point extension of B_{i-1} by the radical X_i of the projective B_i-module $Hom_A\ (T, T_i)$, where X_i is considered as a $B_{i-1}-$module. By (4.1)(c), we have an exact sequence

$$0 \to H^0(B_i) \to H^0(B_{i-1}) \to End\ X_i/k \to H^1(B_i) \to$$

$$\to H^1(B_{i-1}) \to Ext^1_{B_{i-1}}(X_i, X_i) \to \dots$$

By the induction hypothesis, $H^1(B_i) = 0$. We claim that $H^1(B_{i-1}) = 0$. It suffices to prove that $Ext^1_{B_{i-1}}(X_i, X_i) = 0$. Since B_{i-1} is a full convex subcategory of B, we have $Ext^1_{B_{i-1}}(X_i, X_i) = Ext^1_B(X_i, X_i)$. Using methods of tilting theory, we can show that $Ext^1_B(X_i, X_i) = 0$ (we refer the reader to [AS2] for details). This shows our claim, and then it follows directly from (4.2) that the projective $B-$module $Hom_A(T, T_i)$ has a separated radical. ∎

The assumption that A is representation-finite is necessary for the validity of the theorem. Indeed, let A be the hereditary algebra given by the quiver



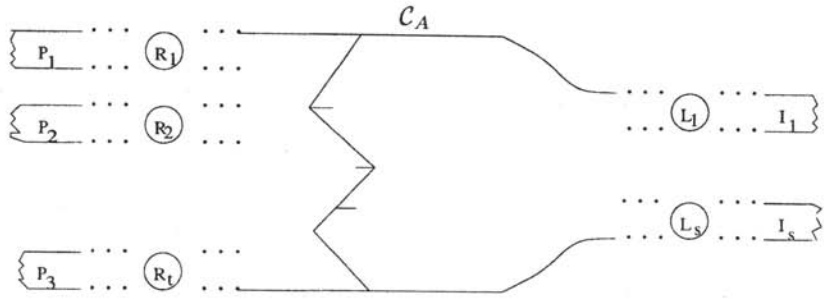
then A is representation-infinite and simply connected. Let $T_A = P(1) \oplus P(2) \oplus P(4) \oplus P(5) \oplus R$, where R is the simple regular with dimension-vector $\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix}$, then $B = End\ T$ is given by the bound quiver of example (2.2)(e), hence does not satisfy the separation condition (but is simply connected).

6.2. In view of (2.5)(a), we have the following corollary.

COROLLARY. Let A be a representation-finite simply connected algebra, and T_A be a tilting module. Then $B = End\ T$ is simply connected. ∎

6.3. We now consider the case of tilted algebras. Let Q be a finite acyclic quiver. An algebra A is called *tilted of type* Q if there exists a tilting $kQ-$module T such that $A = End\ T_{kQ}$ (see [HR]). Tilted algebras are characterised by the existence of *complete slices* in (at least) a component of their Auslander-Reiten quiver, called *connecting component(s)* [R1]. A tilted algebra has at most two connecting components and, if it has two, then it is *concealed,* that is, is the endomorphism algebra of a postprojective tilting module [R2]. The structure of the Auslander-Reiten quiver $\Gamma(modA)$ of a tilted algebra A is given in [K] as follows. If A is not concealed, and $\mathcal{C}_A$ is its unique connecting component, then the *left end algebra* ${}_\infty A$ of A is defined as ${}_\infty A = End(\oplus_{P(x) \notin \mathcal{C}_a} P(x))$. If A is concealed, then we define ${}_\infty A = A$. We have ${}_\infty A = \prod_{i=1}^t A_i$, where each A_i is a (connected) tilted algebra

having a complete slice in its preinjective component. The *right end algebra* A_∞ is defined dually. Thus if A is not concealed, then $\Gamma(modA)$ has the following shape.



It consists of postprojective components $P_1, \ldots, P_t$, preinjective components $I_1, \ldots, I_s$, the connecting component $\mathcal{C}_A$, families of right stable components $R_1, \ldots, R_t$ and families of left stable components $L_1, \ldots, L_s$.

Assume now that Q is a finite acyclic quiver. Then the hereditary algebra kQ is simply connected if and only if Q is a tree. Let T be a tilting $kQ-$module, then $A = End\ T$ is tilted of type Q. Thus the conjecture in (6.1) may be reformulated to say that a tilted algebra A is simply connected if and only if its type is a tree. As we shall see in (6.5) below, this conjecture is now proved in the case where A is tame. The first step towards proving this statement is the following.

LEMMA [AS1]. Let Q be a finite acyclic quiver whose underlying graph is a Dynkin or an euclidean quiver, and A be a tilted algebra of type Q, then A is simply connected if and only if the underlying graph of Q is not $\tilde{\mathbf{A}}_m$ for some $m \geq 1$. ∎

In fact, even more is proved: let Q be as above, and A be such that there exists an equivalence of triangulated categories $D^b(modA) \cong D^b(mod\ kQ)$ (then, A is *iterated tilted of type* Q, see [A2]), then A is simply connected if and only if the underlying graph of Q is not $\tilde{\mathbf{A}}_m$, for some $m \geq 1$.

6.4. An essential tool in the sequel is the orbit graph. We recall the definition. Let A be an algebra, and Γ be a connected component of $\Gamma(mod\ A)$. The *orbit graph* $\mathcal{O}(\Gamma)$ of Γ has as points the $\tau-$orbits M^τ of the $A-$modules M in Γ, and there exists an edge $M^\tau - N^\tau$ whenever there exist $m, n \in \mathbb{Z}$ and an irreducible morphism $\tau^m M \to \tau^n N$, or $\tau^n N \to \tau^m M$; in this case, the number of edges between M^τ and N^τ equals $dim_k Irr(\tau^m M, \tau^n N)$, or $dim_k\ Irr(\tau^n N, \tau^m M)$, respectively (here, $Irr(X, Y)$ denotes the space of irreducible morphisms from X to Y). The idea of using the orbit graph in the study of simple connectedness comes from a result in [BG] (4.2) saying that a representation-finite algebra is simply

connected if and only if the orbit graph of its Auslander-Reiten quiver is a tree. A first (easy) result on the orbit graphs of components of tilted algebras is the following.

LEMMA. Let A be a tilted algebra, and $\mathcal{C}_A$ be a connecting component of $\Gamma(mod\ A)$. Then $\mathcal{O}(\mathrm{C}_A)$ is a tree if and only if $H^1(A) = 0$.

Proof. Let Σ be a complete slice in $\mathcal{C}_A$, Then the underlying graph of Σ is $\mathcal{O}(\mathcal{C}_A)$ and A is tilted of type Σ^{op}. By (4.1)(b), $H^1(A) \cong H^1(k\Sigma^{op})$. By (4.1) (a), we infer that $H^1(A) = 0$ if and only if Σ is a tree. ∎

In view of (6.3), we infer that if A is tilted of Dynkin or euclidean type, then A is simply connected if and only if $H^1(A) = 0$.

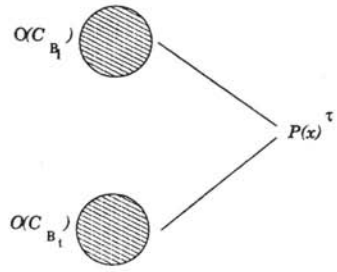
6.5. We now proceed to show that a tame tilted algebra A is simply connected if and only if its type is a tree. We notice that, in this case, each of ${}_\infty A$ and A_∞ is a direct product of tilted algebras of euclidean type.

THEOREM [AMP]. Let A be a tame tilted algebra. Then A is simply connected if and only if its type is a tree.

Proof. Since the statement clearly holds for concealed algebras, we may assume that A is representation-infinite and not concealed.

Suppose first that the type of A is a tree. By (6.4), this means that $H^1(A) = 0$. We must show that A is simply connected. If this is not the case, then we may assume that A is a counter example such that Q_A has a minimal number of points. We first observe that there is at least one projective in the connecting component $\mathcal{C}_A$: for, if this is not the case, then A is tilted of euclidean type, hence, by (6.3), A is simply connected, a contradiction. Let thus $P(x)_A$ be a projective in $\mathcal{C}_A$. We may assume that x is a source in Q_A, and thus $A = B[M]$, where $B = B_1 \times \ldots B_t$, with each B_i connected. Since $H^1(A) = 0$, then, by (4.4), the source x is separating. Also, for each i,, $\mathcal{O}(\mathcal{C}_{B_i})$ is a subgraph of $\mathcal{O}(\mathcal{C}_A)$, hence is a tree. By our minimality assumption, each B_i is simply connected. Applying (2.3), we get that A is simply connected, a contradiction.

Conversely, assume that A is simply connected. We must show that $H^1(A) = 0$. If the connecting component $\mathcal{C}_A$ does not contain projectives, then we are done by (6.3). If it does, let $P(x)_A$ be a projective in $\mathcal{C}_A$ which is maximal with respect to the order induced by the arrows. In particular, x is a source, so we can write $A = B[M]$, with $M = rad\, P(x)$ and $B = B_1 \times \ldots \times B_t$, where each B_i is connected. By (3.4), x is separating. It is then easy to see that $\mathcal{O}(\mathcal{C}_A)$ is obtained by glueing together the orbit graphs of the connecting components of the B_i as follows:



Therefore, we may assume that $t = 1$, and hence that M is indecomposable.

We want to apply (4.1)(c). Since A is tilted, we have $H^2(A) = 0$ (by (4.1)(a) and (b)). Also, x is separating, and M lies in $\mathcal{C}_A$, hence is a brick, so that (4.2)(a) yields a short exact sequence

$$0 \to H^1(A) \to H^1(B) \to ExT^1_B(M, M) \to 0.$$

Since M lies in $\mathcal{C}_A$, we have $Ext^1_B(M, M) = 0$, so that $H^1(A) \cong H^1(B)$. If B is simply connected, then $H^1(B) = 0$ by induction and we are done. Otherwise, let (Q_A, I_ν) be a presentation of A. Since B is not simply connected, we may assume that the restriction (Q_B, I'_ν) of (Q_A, I_ν) to B is such that $\pi_1(Q_B, I'_\nu) \neq 1$. Since $\pi_1(Q_A, I_\nu) = 1$, applying (3.3) yields that $l(1) \neq 0$, that is, the support of M contains a full subcategory which is a crown. One may then show that $Ext^1_B(M, M) \neq 0$, a contradiction which completes the proof. ∎

6.6. We now look at the strong simple connectedness of tilted algebras. In order to state our first result, we recall that a component Γ of the Auslander-Reiten quiver of an algebra A is called *directed* if, for each indecomposable M_0 in Γ, there is no sequence of non-zero non-isomorphisms between indecomposable modules of the form $M_0 \to M_1 \to \ldots M_t = M_0$ in *mod* A. If A is tilted, the directed components of $\Gamma(mod\ A)$ are the postprojective, preinjective and connecting components. The following result, first shown in [AL2](1.3), has since been generalised in [GPPRT] (4.2), where it is proved that, if A is a strongly simply connected (not necessarily tilted) algebra, then, for any directed component Γ of $\Gamma(mod\ A)$ such that, if $M_0 \to M_1 \to \ldots \to M_t$ is a sequence of non-zero non-isomorphism between indecomposables with M_0, M_t in Γ, all M_i lie in Γ, then the orbit graph $\mathcal{O}(\Gamma)$ of Γ is a tree.

PROPOSITION. Let A be a strongly simply connected tilted algebra. The orbit graph of each directed component of $\Gamma(mod\ A)$ is a tree.

Proof. We may assume that A is representation-infinite. Let Γ be a directed component of $\Gamma(mod\ A)$. We claim that it suffices to prove the statement in case Γ is a connecting component. Assume that Γ is a postprojective (or preinjective) component of $\Gamma(mod\ A)$, but is not connecting. Then Γ is a standard component without injective (or projective, respectively) modules. Let

$Ann\ \Gamma = \{a \in A | Ma = 0 \text{ for all } M \text{ in } \Gamma\}$. By [L](2.4), [S3](3.1), the algebra $B = A/Ann\Gamma$ is tilted, and Γ is a connecting component of $\Gamma(mod\ B)$. Since B is the support algebra of Γ, it is easily shown to be a full convex subcategory of A. Hence B is itself strongly simply connected. This shows our claim.

Let now Γ be a connecting component in $\Gamma(mod\ A)$. Since A is strongly simply connected, we have $H^1(A) = 0$ by (5.1). Then (6.4) implies that Γ is a tree. ∎

6.7.

COROLLARY [ALP](1.4). Let A be a tilted algebra, then the following conditions are equivalent:

(a) A is strongly simply connected.
(b) For each full convex subcategory B of A, the graph $\mathcal{O}(\mathcal{C}_B)$ is a tree.
(c) For each full convex suncategory B of A, and each directed component Γ of $\Gamma(mod\ B)$, the graph $\mathcal{O}(\Gamma)$ is a tree.

Proof. The equivalence of (a) and (B) follows from (6.5) and the fact that by [H1](III.6.5), any full subcategory of a tilted algebra is tilted. Since (c) implies (b) trivially, we show that (b) implies (c). Let B be a full convex subcategory of A, and Γ be a directed componet of $\Gamma(mod\ B)$. As in the proof of (6.5), the algebra $B/Ann\Gamma$ is a full convex subcategory of A having Γ as a connecting component. The result follows. ∎

6.8. We have the following characterisation of the strong simple connectedness of a tame tilted algebra.

THEOREM [ALP]. Let A be a tame tilted algebra. The following conditions are equivalent:

(a) A is strongly simply conected.
(b) The orbit graph of each directed component of $\Gamma(mod\ A)$ is a tree.
(c) $H^1(A) = 0$, and A contains no full convex subcategory which is hereditary of type $\tilde{\boldsymbol{A}}_m$, for some $m \geq 1$.
(d) A satisfies the separation condition and contains no full convex subcategory which is hereditary of type $\tilde{\boldsymbol{A}}_m$, for some $m \geq 1$. ∎

Each of the state conditions depends on A alone, and not on all of its full convex subcategories. The key step in the proof is the statement that, if A is a tame tilted algebra which is not concealed, then the postprojective and the preinjective components of $\Gamma(mod\ A)$ which are not connecting have tree orbit graphs if and only if A contains no full convex subcategory which is hereditary of type $\tilde{\boldsymbol{A}}_m$, for some $m \geq 1$.

6.9. We have even better statement in case A is furthermore *sincere*, that is, there exists a sincere indecomposable module M_A lying on no cycle of non-zero non-isomorphisms between indecomposables of the form $M = M_0 \to M_1 \to \ldots \to M_t = M$.

PROPOSITION [ALP](4.1). Let A be a sincere tame tilted algebra. The following conditions are equivalent:

(a) A is strongly simply connected.

(b) The postprojective and the preinjective components of $\Gamma(mod\ A)$ have tree orbit graphs.

(c) A contains no full convex subcategory which is hereditary of type $\tilde{\boldsymbol{A}}_m$, for some $m \geq 1$. ∎

7. Simply connected mesh algebras.

7.1. Orbit graphs make sense and are useful in another context as well. Let Γ be a finite translation quiver without oriented cycles (see [R1, BG]) and I_Γ denote the ideal of the path algebra $k\Gamma$ generated by the mesh relations. The algebra $A = k\Gamma/I_\Gamma$ is called *the mesh algebra* of Γ. Thus, a typical example of a mesh algebra is provided by the Auslander algebra of a representation-directed algebra. The following result generalises [CV] and (5.2)(g).

THEOREM [ACVT]. Let Γ be a finite tranlation quiver without oriented cycles, and A be its mesh algebra. The following conditions are equivalent:

(a) A is simply connected.

(b) A is strongly simply connected.

(c) A satisfies the separation condition.

(d) Every irreducible cycle in Γ is an irreducible contour, and every irreducible contour is a mesh.

(e) Γ is simply connected.

(f) $\mathcal{O}(\Gamma)$ is a tree.

If, moreover, *char* $k = 0$ and Γ has no multiple arrows, then the above conditions are equivalent to:

(g) $H^1(A) = 0$. ∎

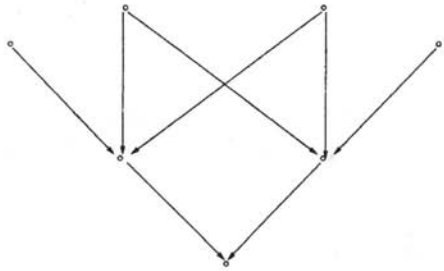
We briefly outline the scheme of the proof. It is trivial that (b) implies (c), which implies (a) by (2.5), which implies (e) trivially. That (d) implies (b) follows directly from (5.4), and the equivalence of (e) with (f) from [BG]. To prove that (a) implies (d), one first shows that, if a is a source in the quiver of a mesh algebra A, and if A is simply connected, then its full convex subcategory generated by all points of A except a is also simply connected. One next shows successively that, if a mesh algebra A contains an irreducible cycle which is not a contour, or an irreducible contour which is not a mesh, then A is not simply connected. For the proofs that (b) implies (d), that (f) implies (d), and the equivalence of these with (g), we refer the reader to [CV].

7.2. This result applies to a class of algebras arising naturally from the study of postprojective partitions, introduced by Auslander and Smalo [ASm]. We recall

that an algebra A is said to be an $\mathcal{H}_1-$*algebra* if the union of the first two postprojective classes $\mathcal{P}_0 \cup \mathcal{P}_1$ is closed under predecessors, or equivalently, is closed under submodules (see [AHC]). For such an algebra A, we let R denote the endomorphism algebra of the direct sum of one representative from each isomorphism class of indecomposable $A-$modules from $\mathcal{P}_0 \cup \mathcal{P}_1$. It is easily shown that R is a mesh algebra. Applying (7.1) yields the following theorem.

THEOREM [ACVT]. The algebra R is simply connected if and only if A satisfies the separation condition. ∎

It is worthwhile to observe that, while the statement implies that A is simply connected, it is generally not strongly simply connected. Indeed, the algebra A given by the quiver



bound by all possible commutativity relations, is an $\mathcal{H}_1-$algebra satisfying the separation condition, which is not strongly simply connected.

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