Resenhas IME-USP 1997, Vol. 3, No. 2, 173 - 209.

Driven Tracer Particle and Einstein Relation in One Dimensional Symmetric Simple Exclusion Process[1]

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Abstract: We investigate the behavior of a tagged particle under the action of an external constant driving force in an infinite system of particles evolving in a one dimensional lattice according to symmetric random walks with hard core interaction. We prove that the position diffusively rescaled $\epsilon X(\epsilon^{-2}t)$ of the test particle converges in probability, as $\epsilon \to 0$, to a deterministic function $v(t)$, for a large class of initial distributions of the random environment. The function $v(\cdot)$ depends only on the initial distribution of the random environment through a non linear parabolic equation. An Einstein relation is satisfied asymptotically when the external force is small. This law of large numbers for the position of the tracer particle is deduced from the hydrodynamical limit of an inhomogeneous one dimensional symmetric zero range process with an asymmetry at the origin. This result is connected also with the evolution of the interfaces in a Potts model in two dimension under a Glauber dynamics at infinite temperature, for some particular initial conditions.

Key words: Tagged particle, Einstein relation, Exclusion processes, Hydrodynamic limit, interface dynamics

Introduction.

The motion of a tagged particle in an equilibrium fluid is one of the most studied questions in statistical mechanics. It is a standard example of a classical problem : deduce from a large system, typically infinite, evolving according to Newton's equations, a simple stochastic behavior of a small subsystem. In the case of a tagged particle, the question resumes to derive the motion of a single particle, through a macroscopic rescaling of space and time, from the dynamics of the entire system.

Although the popularity of Brownian motion, besides some particular cases (cf. [Spi1], [DGL1,2], [Spo], [So] and references therein), a general deduction of the macroscopic behavior of a tagged particle from the underlying microscopic dynamics is far from being well understood. Several simplifications have been introduced to investigate this problem rigorously : either by considering models with known invariant measures (reducing the phase space, for instance [GLR1,2], [Spo]) or introducing some randomness in the evolution (particles that dies after the collision with the tagged particle [So], particles with random exponentiál life time [PSV], particles evolving according to some stochastic dynamics [KV], [Spo]). We refer to [So] for a recent review on the subject.

[1] Mathematics Subject Classification: Primary 60K35; secondary 82A05

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

We consider in this article the one dimensional, nearest neighbor symmetric simple exclusion process. It can be described as follows. Particles evolve on the one dimensional lattice $\mathbb{Z}$ with an exclusion rule that prevents more than one particle per site. Each particle waits a mean one exponential time at the end of which it attempts to jump to the right or left with probability 1/2. If the chosen site is already occupied, the jump is suppressed to conform to the exclusion rule. We add to this system a tagged particle submitted to the same exclusion rule that forbids more than one particle per site and that, in contrast with the other particles, experiences the action of a constant external driving force. In result, the tagged particle jumps with probability $1/2 < p \leq 1$ to the right and $q = 1 - p$ to the left.

Without the presence of the environment, the tagged particle would behave as an asymmetric random walk. In particular, if X_t stands for its position at time t, $t^{-1}(X_t - X_0)$ would converge almost surely to $p - q$ as $t \uparrow \infty$. The presence of the symmetric environment affects dramatically the behavior of the tagged particle. Since the un tagged particles behave as symmetric random walks, we expect an accumulation of particles at the right of the tagged particle and a rarefection at the left. The environment decelerates thus the motion of the tagged particle and tends to confine it. In fact the main result of this article states that for a large class of initial states,

$$\lim_{t\to\infty} \frac{X_t - X_0}{\sqrt{t}} = v \tag{0.1}$$

in probability, where v is a real number depending only on the macroscopic profile of the initial state through the solution of a non linear parabolic equation with boundary conditions. Burlatsky et al. ([BOMM], [BOMR]) derived (0.1) heuristically in the case where the initial state is a Bernoulli product measure with some fixed density α.

The diffusive scale $\sqrt{t}$, already obtained by Harris [H] in the case of Brownian particles with hard core interaction in dimension 1, is peculiar to the nearest neighbor assumption that restrains the tagged particle to jump over the symmetric particles. In higher dimension or in dimension 1 without the nearest neighbor assumption, one would expect the tagged particle to move in the scale t.

In the context of interacting particle systems, the asymptotic behavior of a tagged particle has been continuously investigated. The question was already present in Spitzer [Spi2]. For general symmetric simple exclusion processes, it follows from a general result on additive functionals of reversible Markov processes due to Kipnis and Varadhan [KV] that $t^{-1/2}(X_t - X_0)$ converges in distribution to a non degenerate Gaussian variable. More precisely Kipnis and Varadhan establish an invariance principle, i.e., if the system is in equilibrium at a density of particles α the distribution of the rescaled process $\epsilon(X_{\epsilon^{-2}t} - X_0)$ converges to the law of a Brownian motion with diffusion coefficient $D(\alpha)$. In general $D(\alpha)$ is a complicate function of α. In dimension 3 or more, Varadhan [V1] proves that $D(\alpha)$ is Lipschitz continuous. More recently Asselah, Brito and Lebowitz have proven some bounds of $D(\alpha)$ in terms of the size or the range of the jumps [ABL]. In the case of the one dimensional nearest neighbor model it happens that $D(\alpha) = 0$. In fact here the tagged particle is trapped between its neighbors and the fluctuations of its position will depend directly on the fluctuations of the density of the particles around. In this case, Arratia showed that $t^{-1/4}(X_t - X_0)$ converges in distribution, as $t \uparrow \infty$, to a Gaussian variable with variance

$$\frac{1-\alpha}{\alpha}\sqrt{\frac{2}{\pi}}\,. \tag{0.2}$$

In fact (cf. [RV]) a corresponding invariance principle could be established, i.e. the convergence of the properly rescaled process $\epsilon(X_{\epsilon^{-4}t} - X_0)$ to a fractional Brownian motion of parameter 1/2, i.e. a Gaussian process with covariance

$$\sqrt{\frac{2}{\pi}}\frac{1-\alpha}{\alpha}\left(\sqrt{t}+\sqrt{s}-\sqrt{|t-s|}\right)\,. \tag{0.3}$$

This behavior should be characteristic of every one dimensional nearest neighbor model [Spo].

In section 6 we prove that if we start with an constant profile of density α then

$$\lim_{p-q\to 0}\frac{v}{p-q} = \frac{1-\alpha}{\alpha}\sqrt{\frac{2}{\pi}}$$

that means the Einstein relation between the *mobility* v given by (0.1) and the diffusivity given by (0.2) is verified. This is in agreement with the heuristic results of [BOMR].

Einstein relations can be established for a large class of weak-asymmetric models (i.e. the asymmetry is rescaled with the parameter ϵ relating the microscopic and the macroscopic scales (cf. [LR])). If the asymmetry is strong (i.e. not rescaled in the macroscopic limit) rigorous results on the Einstein relations are rare, essentially for the difficulty to compute the stationary state of the environment as seen from the tagged particle. The case studied in the present paper is non-stationary, but somehow the *local equilibrium* established around the particle is responsible for the validity of the Einstein relation.

Here is the idea of our approach. The first point is to understand that this is a non-stationary problem: the tracer will start to push the particles in front and generate an inhomogeneous density profile that will evolve deterministically under a diffusive rescaling of space and time. The proper way to formulate the problem is thus to prove that

$$\epsilon(X_{\epsilon^{-2}t} - X_0) \to v(t)$$

where $v(t)$ is a deterministic function of the (macroscopic) time t. This suggest that the problem is basically a hydrodynamic limit (cf. [KL]) with a moving boundary. We prove in fact that this model has a hydrodynamic behavior described by the solution of a Stefan problem (cf. (6.3)). The idea is to introduce Lagrangian coordinates : there is a natural map that transforms a one dimensional nearest neighbor exclusion process in a zero range process. This map transforms the moving boundary problem in a fixed boundary problem. Then we need to prove the hydrodynamic limit for a zero range process with boundary conditions. The hydrodynamic limit for systems with boundary conditions is an interesting problem in itself: dissipative boundary conditions (like here in the totally asymmetric case $p = 1, q = 0$) are a source of irreversibility of the system. Most hydrodynamic limit proved until now are for systems with nice stationary measures.

Most interesting is the connection between this problem and the evolution of the random interfaces in a 3-state Potts model at zero temperature under a Glauber dynamics. Herbert Spohn made us notice that our results permits to deduce the macroscopic evolution of these interfaces for some particular initial conditions.

The article is divided as follows. In section 1 we state the main results, explain why the diffusive scale is the correct scale to investigate the motion of the tagged particle and give the connection to the evolution of interfaces in the 2-dimensional Potts model at zero temperature. In section 2 we introduce the terminology and review some basic results concerning zero range processes used throughout the article. In section 3 and 4 we prove the hydrodynamic limit for the associated zero range processes in the cases $p = 1$ and $p < 1$. In section 5 we deduce the asymptotic behavior of the tagged particle from the hydrodynamical limit. In section 6 we obtain an explicit formula for the position of the tagged particle when the initial state is associated to a constant profile and deduce Einstein's relations. In section 7 we investigate the evolution of a surface model in a two dimensional Potts model. In the appendix we prove the uniqueness of the weak solutions of the non-linear diffusive equations involved in this work.

This paper is a detailed version of [LOV].

1. Statements of the results.

Consider a family of indistinguishable particles moving according to continuous time, symmetric, nearest neighbor random walks on $\mathbb{Z}$ with an exclusion rule that prevents more than a particle per site. To this system we add a tagged particle that moves according to an asymmetric random walk, jumping with probability p to the right, probability q to the left, and that respects the exclusion rule. The configuration of the system is denoted by (X, ξ), where $X \in \mathbb{Z}$ is the position of the tagged asymmetric particle, and $\xi \in \{0,1\}^{\mathbb{Z}}$ is the configuration of all other particles. Clearly $\xi(X) = 0$, because that site is already occupied by the asymmetric particle. The system just described is a Markov process whose generator acts on local functions $F: \mathbb{Z} \times \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{L}F(X,\xi) = (1/2) \sum_{z \neq X-1, X} & [F(X,\xi^{z,z+1}) - F(X,\xi)] \\ & + p(1-\xi(X+1))[F(X+1,\xi) - F(X,\xi)] \\ & \quad + (1-p)(1-\xi(X-1))[F(X-1,\xi) - F(X,\xi)] , \end{aligned} \tag{1.1}$$

where $\xi^{z,z+1}$ is the configuration obtained from ξ, exchanging the occupation variables $\xi(z)$, $\xi(z+1)$:

$$\xi^{z,z+1}(y) = \begin{cases} \xi(y) & \text{if } y \neq z, z+1, \\ \xi(z) & \text{if } y = z+1, \\ \xi(z+1) & \text{if } y = z\,. \end{cases}$$

To fix ideas set $p > 1/2$ and $X_0 = 0$. Here X_t stands for the position of the asymmetric tagged particle at time t. We prove in this article that for each $t \geq 0$ and for a class of initial states of the random environment ξ_0, as $N \uparrow \infty$, X_{tN^2}/N converges in probability to a real number v_t that depends only on the distribution of ξ_0. We start describing the initial states of the random environment ξ_0.

Denote by $\mathbb{Z}_*$ the set of integers distinct from 0. For $0 \leq \alpha \leq 1$, denote by μ_α the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}_*}$ with density α :

$$\mu_\alpha\{\xi\,, \xi(x) = 1\} = \alpha\,,$$

for every x in $\mathbb{Z}_*$. More generally, for a positive integer N and a profile $\kappa_0 \colon \mathbb{R} \to [0,1]$, denote by $\mu^N_{\kappa_0(\cdot)}$ the Bernoulli product measure associated to κ_0 :

$$\mu^N_{\kappa_0(\cdot)}\{\xi\,, \xi(x) = 1\} = \kappa_0(x/N)$$

for x in $\mathbb{Z}_*$ and by $P_{\mu^N_{\kappa_0(\cdot)}}$ the probability measure on the path space $D(\mathbb{R}_+, \mathbb{Z} \times \{0,1\}^{\mathbb{Z}})$ induced by the Markov process with generator $\mathcal{L}$ defined in (1.1) and the the initial measure $\delta_0 \times \mu^N_{\kappa_0(\cdot)}$.

Before stating the theorem, we introduce some notation required to define the limit v_t. Fix a strictly positive profile κ_0. Denote by $\mathcal{H} \colon \mathbb{R} \to \mathbb{R}$, $\mathcal{F} \colon \mathbb{R} \to \mathbb{R}$ the functions defined by

$$\mathcal{H}(A) = \int_0^A \kappa_0(u)\, du\,, \quad \mathcal{F}(B) = \frac{1}{\kappa_0(\mathcal{H}^{-1}(B))} - 1\,. \tag{1.2}$$

Here $\mathcal{H}^{-1}$ stands for the inverse of the strictly increasing, absolutely continuous function $\mathcal{H}$.

Consider the non-linear parabolic equation with boundary condition on $\mathbb{R}_+ \times \mathbb{R}_+$

$$\begin{cases} \partial_t \rho = (1/2)\Delta \Phi_0(\rho) \\ \rho(t,0) = 0 \\ \rho(0,\cdot) = \mathcal{F}_+(\cdot)\,, \end{cases} \tag{1.3}$$

where $\mathcal{F}_+$ stands for the restriction of $\mathcal{F}$ on $\mathbb{R}_+$ and $\Phi_0(\rho) = \rho/(1+\rho)$; and the nonlinear parabolic equation on $\mathbb{R}_+ \times \mathbb{R}$ with boundary condition at the origin

$$\begin{cases} \partial_t \rho = (1/2)\Delta \Phi_0(\rho) \\ p\Phi_0(\rho(t,0+)) = q\Phi_0(\rho(t,0-)) \\ \partial_u \Phi_0(\rho(t,0+)) = \partial_u \Phi_0(\rho(t,0-)) \\ \rho(0,\cdot) = \mathcal{F}(\cdot)\,. \end{cases} \tag{1.4}$$

A precise definition of solutions of these differential equations is given in sections 3 and 4. In the Appendix we prove a uniqueness result and in section 6 we show, introducing Lagrangian coordinates, that this equation can be transformed in a linear Stefan problem so that the original exclusion process with an asymmetric particle has a hydrodynamic behavior described by the solution of a Stefan problem.

Theorem 1.1 . *Assume $p = 1$. Fix a profile $\kappa_0 : \mathbb{R}_+ \to [0,1]$ such that $\sigma \leq \kappa_0 \leq 1-\sigma$ for some $\sigma > 0$. Then, for every $\delta > 0$,*

$$\lim_{N\to\infty} P_{\mu^N_{\kappa_0(\cdot)}} \Big[\Big| \frac{X_{tN^2}}{N} - v_t \Big| > \delta \Big] \; = \; 0 \; ,$$

where

$$v_t \; = \; \int_0^\infty \Big\{ \mathcal{F}(u) - \rho(t,u) \Big\} du \tag{1.5}$$

and ρ is the solution of equation (1.3).

Theorem 1.2 . *Assume $p < 1$ and for $\alpha < 1$ define let $\psi_\alpha(u) = \alpha \mathbf{1}\{u < 0\} + (q\alpha/p)\mathbf{1}\{u > 0\}$. Fix a profile $\kappa_0 : \mathbb{R} \to [0,1]$ such that $\psi_\alpha \leq \kappa_0 \leq 1 - \sigma$ for some $\sigma > 0$, $0 < \alpha < 1$. Then, for every $\delta > 0$,*

$$\lim_{N\to\infty} P_{\mu^N_{\kappa_0(\cdot)}} \Big[\Big| \frac{X_{tN^2}}{N} - v_t \Big| > \delta \Big] \; = \; 0 \; ,$$

where v_t is given by (1.5) and ρ is the solution of equation (1.4).

The integral defining v_t in (1.5) must be understood in the following sense : consider the sequence $\{H_n, \, n \geq 1\}$ of real functions defined by

$$H_n(u) \; = \; (1 - un^{-1})^+ \; . \tag{1.6}$$

It follows from the equation satisfied by ρ that $\int_0^{+\infty} H_n(u)\{\mathcal{F}(u) - \rho(t,u)\}du$ converges as $n \uparrow \infty$. This limit defines the right hand side of (1.5).

In the case where the initial state is a Bernoulli product measure with a fixed density α, more explicit computations can be made:

Theorem 1.3 . *Fix $\alpha > 0$ and recall that μ_α stands for the Bernoulli product measure with density α. If the initial state is μ_α, then*

$$\lim_{p-q\to 0} \frac{v_t}{p-q} \; = \; \frac{(1-\alpha)}{\alpha} \sqrt{\frac{2t}{\pi}} \; .$$

Theorems 1.1 and 1.2 are proven in section 5. Theorem 1.3 and more asymptotic results are proven in section 6.

We now explain why in Theorems 1.1 and 1.2 the asymmetric tagged particle moves at scale $\sqrt{t}$ and why is the displacement related to the solution of the differential equations (1.3), (1.4).

We start labeling all particles. The tagged asymmetric particle is labeled 0. For $j \geq 1$, we label the j-th particle at the right (left) of the tagged particle by j $(-j)$. For x in $\mathbb{Z}$, denote by $\eta(x)$ the number of holes between particle x and particle $x+1$. In this way we transformed a configuration of $\{0,1\}^{\mathbb{Z}}$ with a particle at some site X into a configuration $\{\eta(x),\, x \in \mathbb{Z}\}$ of $\mathbb{N}^{\mathbb{Z}}$. Denote by $\mathcal{T}: \mathbb{Z} \times \{0,1\}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ the transformation just described. $\mathcal{T}$ induces a transformation on the space of continuous functions (resp. probability measures) of $\mathbb{Z} \times \{0,1\}^{\mathbb{Z}}$ to the space of continuous functions (resp. probability measures) of $\mathbb{N}^{\mathbb{Z}}$ still denoted by $\mathcal{T}$.

The dynamics of the process (X_t, ξ_t) induces a dynamics for η_t that can be informally described as follows. For every $x \neq -1$, if there is at least one particle at site x, at rate $1/2$ one of them jumps to site $x+1$ and, symmetrically, if there is at least one particle at site $x+1$, at rate $1/2$ one of them jumps to site x. The picture is slightly different between sites -1 and 0 due to the behavior of the asymmetric tagged particle. A particle jumps at rate q from site -1 to site 0 if there is a particle at -1 and a particle jumps at rate p from site 0 to site -1 if there is a particle at the origin.

This process is the so called zero range process with an asymmetry at the origin. The position at time t of the asymmetric tagged particle corresponds in the zero range model to the total number of jumps between 0 and t from 0 to -1 minus the total number of jumps in the same interval from -1 to 0 :

$$X_t = \sum_{x \geq 0} \{\eta_0(x) - \eta_t(x)\}\ .$$

Notice that though the sums $\sum_{x \geq 0} \eta_0(x)$ and $\sum_{x \geq 0} \eta_t(x)$ might be both infinite, we can give a precise meaning for the sum : the right hand side is to be understood in the same sense as the right hand side of (1.5) by the use of the functions (1.6) (with the limit in the L^2 sense) (cf. [RV]).

Since in the zero range process the jumps of particles over all bonds, except the bond $\{-1,0\}$, are symmetric, we expect the process to have a diffusive hydrodynamic behavior, i.e., that for a large class of initial profiles, the process accelerated by N^2 is such that for all continuous functions with compact support G,

$$N^{-1} \sum_{x} G(x/N) \eta_{tN^2}(x) \tag{1.7}$$

converges in probability to $\int_{\mathbb{R}} G(u)\rho(t,u)du$, where ρ is the solution of a nonlinear heat equation.

The zero range processes we consider in this article have an asymmetry at the origin. This asymmetry is reflected in the hydrodynamic equations (1.3) and (1.4) by a boundary condition at the origin.

In particular, approximating $\mathbf{1}\{u > 0\}$ by the sequence defined in (1.6), it follows from (1.7) that

$$\frac{X_{tN^2}}{N} = N^{-1} \sum_{x \geq 0} \{\eta_0(x) - \eta_{tN^2}(x)\}$$

converges in probability to v_t given by (1.5). This explains the renormalization in $\sqrt{t}$ and the relation between v_t and the differential equations (1.3) and (1.4).

We investigate also the evolution of a surface in a two dimensional Potts model. On $\mathbb{Z}^2$, consider a spin system $\{\sigma(x),\ x \in \mathbb{Z}^2\}$ taking three possible values : $\sigma(x) \in \{-1, 0, 1\}$ for x in $\mathbb{Z}^2$ and the Hamiltonian $\mathfrak{H}$ defined by

$$\mathfrak{H}(\sigma) = - \sum_{\substack{x,y \in \mathbb{Z}^2 \\ |x-y|=1}} \iota(\sigma(x)) \delta_{\sigma(x),\sigma(y)} ,$$

where $\delta_{a,b}$ is equal to 1 if $a = b$ and 0 otherwise and, say, $\iota(-1) < \iota(0) = \iota(1)$. This assumption on ι states that 0-spins or 1-spins stick together strongerly that -1-spins. It will be explained below.

We consider a spin flip dynamics where a spin is allowed to change at rate 1/2 if and only if it does not increase the energy. More precisely, consider the Markov process whose generator acts on cylinder functions as

$$(\mathcal{L}_{sp} f)(\sigma) = (1/2) \sum_{x \in \mathbb{Z}^2} \sum_{j=-1}^{1} \mathbf{1}\{(\Delta_{x,j}\mathfrak{H})(\sigma) \le 0\}[f(\sigma^{x,j}) - f(\sigma)] , \tag{1.8}$$

where $\sigma^{x,j}$ stands for the configuration where the spin at x is fixed to be equal to j :

$$(\sigma^{x,j})(y) = \begin{cases} \sigma(y) & \text{for } y \neq x , \\ j & \text{for } y = x . \end{cases}$$

and $(\Delta_{x,j}\mathfrak{H})(\sigma)$ is the modification of the Hamiltonian due to the flip of the spin at x to j :

$$(\Delta_{x,j}\mathfrak{H})(\sigma) = \mathfrak{H}(\sigma^{x,j}) - \mathfrak{H}(\sigma) .$$

The total energy may therefore only decrease for the dynamics just introduced.

Denote by $\mathcal{M} = \mathcal{M}(\mathbb{N})$ the collection of non decreasing functions on $\mathbb{N}$ and by $\mathcal{A}$ the set of configurations σ for which there exists a function $\mathfrak{f}$ in $\mathcal{M}$ such that

(a) $\sigma(x) = -1$ if and only if $x_1 < 0$,

(b) $\sigma(x) = 0$ if and only $x_1 \ge 0$ and $x_2 \le \mathfrak{f}(x_1)$,

(c) $\sigma(x) = 1$ if and only $x_1 \ge 0$ and $x_2 > \mathfrak{f}(x_1)$.

The configurations of $\mathcal{A}$ are thus characterized by the monotone functions $\mathfrak{f}$ in $\mathcal{M}$: there is a one to one correspondence between $\mathcal{M}$ and $\mathcal{A}$. Moreover, it is easy to check that the set $\mathcal{A}$ is stable under the dynamics induced by the generator $\mathcal{L}_{sp}$ defined in (1.8) because we assumed $\iota(-1) < \iota(1)$. More precisely, for a configuration σ, denote by P_σ^{sp} the probability on the paths space $D(\mathbb{R}_+, \{-1,0,1\}^{\mathbb{Z}^2})$ induced by the Markov process σ_t with generator L_{sp} defined in (1.8) starting from the configuration σ. It is not difficult to show that for every σ in $\mathcal{A}$,

$$P_\sigma^{sp}\Big[\sigma_t \in \mathcal{A} \quad \text{for all } t \ge 0\Big] = 1 .$$

For each configuration σ in $\mathcal{A}$, denote by $\mathfrak{f}_\sigma$ the monotone function in $\mathcal{M}$ associated to σ. Since $\mathcal{A}$ is stable under the dynamics and the correspondence between $\mathcal{A}$ and $\mathcal{M}$ is one to one, to investigate the evolution of σ_t starting from a configuration σ in $\mathcal{A}$ we may as well examine the evolution of the process $\mathfrak{f}_t = \mathfrak{f}_{\sigma_t}$.

We now introduce some notation in order to state the fourth main theorem of this article. Denote the configurations of the space $\mathbb{N}^{\mathbb{N}}$ by the symbol η. For each $\alpha \geq 0$, denote by ν_α^+ the product measure on $\mathbb{N}^{\mathbb{N}}$ whose marginals are given by

$$\nu_\alpha^+\{\eta, \eta(x) = k\} \;=\; \frac{1}{1+\alpha}\Big(\frac{\alpha}{1+\alpha}\Big)^k \,.$$

For each probability measure m on $\mathcal{M}$, denote by $\mathfrak{A}m$ the probability measure on $\mathbb{N}^{\mathbb{N}}$ that corresponds to the distribution of $\{\mathfrak{f}(x+1) - \mathfrak{f}(x),\, x \geq 0\}$. Fix a sequence of probability measures $\{m^N,\, N \geq 1\}$ on $\mathcal{M}$. We shall assume that

(A0) For every $N \geq 1$, $m^N\{\mathfrak{f},\, \mathfrak{f}_0 = 0\} = 1$,

(A1) The sequence $(\mathfrak{A}m^N)$ is bounded above (resp. below) by ν_α^+ (resp. ν_λ^+) for some $0 < \lambda < \alpha < \infty$.

(A2) There exists a bounded function $\lambda_0 \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for each continuous function $G \colon \mathbb{R}_+ \to \mathbb{R}$ with compact support and each $\delta > 0$,

$$\lim_{N\to\infty} m^N\Big[\Big|N^{-1}\sum_x G(x/N)N^{-1}\mathfrak{f}(x) - \int du\, G(u)\lambda_0(u)\Big| \geq \delta\Big] \;=\; 0\,.$$

Assumption (A0) is just a normalization. Assumption (A1) requires the distribution of the increments $\{\mathfrak{f}(x+1) - \mathfrak{f}(x),\, x \geq 0\}$ to be bounded above and below by some product measure in order to be able to use coupling techniques. The third one just imposes a law of large number for the surface at time 0.

For each probability measure m on $\mathcal{M}$, denote by $\mathbb{P}_m^{sp,N}$ the probability measure on the path space $D(\mathbb{R}_+, \mathcal{M})$ induced by the Markov process $\mathfrak{f}_t = \mathfrak{f}_{\sigma_t}$ with generator (1.8) accelerated by N^2 and the initial measure m.

Theorem 1.4 . *Fix a sequence of initial measures $\{m^N,\, N \geq 1\}$ satisfying assumptions (A0), (A1), (A2). For every $\delta > 0$*

$$\lim_{N\to\infty} \mathbb{P}_{m^N}^{sp,N}\Big[\Big|N^{-1}\mathfrak{f}_t(0) \;-\; v_t\Big| > \delta\Big] \;=\; 0\,,$$

where v_t is given by (1.5) and ρ is the solution of (1.3) with initial condition λ_0 instead of $\mathcal{F}_+$. Moreover, for any continuous function $G \colon \mathbb{R}_+ \to \mathbb{R}$ with compact support and any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{m^N}^{sp,N}\left[\Big|N^{-1}\sum_x G(x/N)N^{-1}\{\mathfrak{f}_t(x) - \mathfrak{f}_t(0)\} - \int du\, G(u)\lambda(t,u)\Big| \geq \delta\right] \;=\; 0,$$

where λ is the unique solution of

$$\begin{cases} \partial_t \lambda = (1/2)\partial_u \Phi_0(\partial_u \lambda)\,, \\ \partial_u \lambda(t,0) = 0\,, \\ \lambda(0,\cdot) = \lambda_0(\cdot)\,. \end{cases} \tag{1.9}$$

2. Zero range processes.

We have seen that the motion of the asymmetric tagged particle is related to zero range processes. In this section we establish some notation and recall some basic facts.

The nearest neighbor, symmetric, space homogeneous zero-range process $(\eta_t)_{t\geq 0}$ is one of the simplest interacting particle systems that describes the evolution of particles on the lattice $\mathbb{Z}$. It can be informally regarded as follows. We fix a jump rate $g:\mathbb{N}\to\mathbb{R}_+$ such that $g(0)=0<g(k)$ for $k\geq 1$. If there are n particles at some site x, at rate $g(n)$ one of them, independently of the number of particles at other sites, jump with probability 1/2 to one of its neighbors. This is a Markov process on $\mathbb{N}^{\mathbb{Z}}$ whose generator acts on functions that depend only on a finite number of coordinates as

$$(Lf)(\eta) \;=\; (1/2)\sum_{x\in\mathbb{Z}} g(\eta(x))[f(\sigma^{x,y}\eta)-f(\eta)]\,, \tag{2.1}$$

where, for configurations η with at least one particle at x, $\sigma^{x,y}\eta$ stands for the configuration obtained from η moving a particle from x to y :

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if } z\neq x,y,\\ \eta(x)-1 & \text{if } z=x,\\ \eta(y)+1 & \text{if } z=y\,. \end{cases} \tag{2.2}$$

To guarantee the existence of such process (cf. [A]), we assume the jump rate to be Lipschitz : $G_1^* = \sup_{n\geq 0}|g(n+1)-g(n)|<\infty$. Denote by $Z:\mathbb{R}_+\to\mathbb{R}_+$ the partition function defined by

$$Z(\varphi) \;=\; \sum_{k\geq 0}\frac{\varphi^k}{g(1)\cdots g(k)}$$

and by $0<\varphi^*\leq\infty$ the radius of convergence of Z. In order to avoid some degeneracy we assume that the partition function Z diverges as approaching its domain of definition :

$$\lim_{\varphi\to\varphi^*} Z(\varphi) \;=\; \infty\,. \tag{2.3}$$

For $0\leq\varphi<\varphi^*$, let $\bar\nu_\varphi$ be the translation invariant product measure on $\mathbb{N}^{\mathbb{Z}}$ with marginals given by :

$$\bar\nu_\varphi\{\eta;\eta(x)=j\} \;=\; \frac{1}{Z(\varphi)}\frac{\varphi^j}{g(1)\cdots g(j)} \tag{2.4}$$

for $j\geq 0$. Let $R(\varphi)$ be the density of particles for the measure $\bar\nu_\varphi$:

$$R(\varphi) \;=\; E_{\bar\nu_\varphi}[\eta(0)]\,. \tag{2.5}$$

From assumption (2.3) it follows that $R: [0, \varphi^*) \to [0, \infty)$ is a smooth strictly increasing bijection. Denote by $\Phi: \mathbb{R}_+ \to [0, \varphi^*)$ the inverse of $R(\cdot)$. Since $R(\varphi)$ has a physical meaning as the density of particles, instead of parameterizing the above family of measures by φ, we use the density ρ as parameter and we write :

$$\nu_\rho = \bar{\nu}_{\Phi(\rho)}$$

for $\rho \geq 0$. With this convention,

$$\Phi(\rho) = E_{\nu_\rho}[g(\eta(0))] .$$

Moreover, Φ is a smooth function whose derivative is bounded above by G_1^* and below by a strictly positive constant on each compact set of $\mathbb{R}_+$ (cf. [KL]).

The hydrodynamic limit for the zero range in infinite volume with generator given by (2.1) is presented in [LM]. The proof relies on the so called one and two block estimates, that allow to replace cylinder functions by functions of the empirical density (cf. [GPV], [KL]). In the two blocks estimate, a cut off to avoid large densities must be introduced. In the context of zero range processes, to justify this cut off, we need to assume either that the product invariant measures have all exponential moments finite (i.e. that $Z(\cdot)$ is finite on $\mathbb{R}_+$) or that the jump rate $g(\cdot)$ is non decreasing and that the initial state is bounded above by an invariant measure. In the latter case the entropy arguments are replaced by coupling arguments (that are in force because the process is attractive). Since we are mainly interested in this paper in the case where $g(k) = \mathbf{1}\{k \geq 1\}$, we shall assume throughout this article, that the jump rate is non decreasing and bounded (if it were unbounded, it would belong to the first class of models).

3. The case p=1.

In the case where the asymmetric tagged particle jumps only to the right, the evolution of the medium on its left is irrelevant for its motion. For the corresponding zero range dynamics, $p = 1$ means that at rate 1 a particle at the origin jumps to -1 and no particle jumps from -1 to 0. We may therefore assume that there is at -1 an infinite reservoir or an absorption point to which particles from the origin jump at rate 1 and from which no particle jumps. Moreover, the position of the tagged particle at time t corresponds in the zero range process to the number of particles that left the system before time t.

Since the techniques required to prove the hydrodynamic behavior of such zero range process apply to a large class of systems, we introduce a general set up. Fix a jump rate $g : \mathbb{N} \to \mathbb{R}_+$ satisfying the assumptions of section 2. Consider the zero–range process on $\mathbb{N}$ whose generator acts on cylinder functions as

$$L = L_b + \sum_{x \geq 0} \Big\{ L_{x,x+1} + L_{x+1,x} \Big\} , \qquad (3.1)$$

where

$$L_{x,y} f(\eta) = (1/2) g(\eta(x)) [f(\sigma^{x,y}\eta) - f(\eta)]$$

and

$$L_b f(\eta) \;=\; g(\eta(0))[f(\eta - \mathfrak{d}_0) - f(\eta)]\;.$$

Here, for a site x, $\mathfrak{d}_x$ stands for the configuration with no particles but one at x, summation is performed site by site and $\sigma^{x,y}\eta$ is defined in (2.2).

To state the hydrodynamic behavior of the zero range process with absorption at -1, we need to introduce some terminology on weak solutions of non linear parabolic equations. Fix a bounded function $\rho_0 : \mathbb{R}_+ \to \mathbb{R}$. A bounded function $\rho : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is said to be a weak solution of the partial differential equation

$$\begin{cases} \partial_t \rho = (1/2)\Delta\Phi(\rho) \\ \rho(t,0) = 0 \\ \rho(0,\cdot) = \rho_0(\cdot) \end{cases} \tag{3.2}$$

in the layer $[0,T] \times \mathbb{R}_+$ if

(a) $\Phi(\rho(t,u))$ is absolutely continuous in the space variable and for every $t > 0$,

$$\int_0^t ds \int_{\mathbb{R}_+} du\, e^{-u} \big\{ \partial_u \Phi(\rho(s,u)) \big\}^2 \;<\; \infty\,,$$

(b) $\rho(t,0) = 0$ for almost every $0 \le t \le T$ and

(c) For every smooth function with compact support $G : \mathbb{R}_+ \to \mathbb{R}$ vanishing at the origin and for every $0 \le t \le T$,

$$\int du\, \rho(t,u) G(u) \;-\; \int du\, \rho_0(u) G(u) = -\,(1/2) \int_0^t ds \int_{\mathbb{R}_+} du\, G'(u) \partial_u \Phi(\rho(s,u)).$$

We prove in the appendix the uniqueness of weak solutions of (3.2). The existence for special initial conditions ρ_0 follows from the tightness of the sequence $\mathbb{Q}_{\mu^N}$ defined below in Lemma 3.5 and the regularity of the limit points of this sequence proved in Proposition 3.6.

We now describe the initial states considered in this section. For $\varphi < \varphi^*$, denote by $\bar\nu_\varphi^+$ the marginal on $\mathbb{N}^{\mathbb{N}}$ of the product measure $\bar\nu_\varphi$ defined in (2.4). Fix a sequence of probability measures $\{\mu^N,\, N \ge 1\}$ on $\mathbb{N}^{\mathbb{N}}$. To prove the hydrodynamic behavior of the system, we shall assume that

(H1) The sequence μ^N is bounded above (resp. below) by $\bar\nu_\alpha^+$ (resp. $\bar\nu_\lambda^+$) for some $0 < \lambda < \alpha < \varphi^*$.

(H2) There exists a bounded function $\rho_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each continuous function $G : \mathbb{R}_+ \to \mathbb{R}$ with compact support and each $\delta > 0$,

$$\lim_{N\to\infty} \mu^N \Big[\Big| N^{-1} \sum_x G(x/N)\eta(x) - \int du\, G(u) \rho_0(u) \Big| \ge \delta \Big] \;=\; 0\,.$$

We already explained at the end of section 2 the first assumption. The second one just imposes a hydrodynamic behavior (a law of large number for the empirical measure defined below in (3.3)) at time 0.

For each probability measure μ on $\mathbb{N}^{\mathbb{N}}$, denote by $\mathbb{P}_\mu^N$ the probability measure on the path space $D(\mathbb{R}_+, \mathbb{N}^{\mathbb{N}})$ induced by the Markov process with generator (3.1) accelerated by N^2 and the initial measure μ. Expectation with respect to $\mathbb{P}_\mu^N$ is denoted by $\mathbb{E}_\mu^N$.

Theorem 3.1 . *Fix a sequence of initial measures satisfying assumptions (H1), (H2). For any continuous function* $G: \mathbb{R}_+ \to \mathbb{R}$ *with compact support and any* $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}^N \left[\left| N^{-1} \sum_x G(x/N)\eta_t(x) - \int du\, G(u)\rho(t,u) \right| \geq \delta \right] = 0$$

where ρ *is the unique solution of (3.2).*

For each positive integer N and each configuration η, define the empirical distribution $\pi^N = \pi^N(\eta)$ as the positive Radon measure on $\mathbb{R}_+$ obtained by assigning a mass N^{-1} to each particle :

$$\pi^N = N^{-1} \sum_{z\geq 0} \eta(z)\delta_{z/N} \tag{3.3}$$

and set $\pi_t^N = \pi^N(\eta_t)$. Fix $T > 0$. Theorem 3.1 follows from the convergence in distribution of the process $\{\pi_t^N, 0 \leq t \leq T\}$, stated below in Theorem 3.2, and some standard topology arguments (cf. Chap IV of [KL]). To state the convergence in distribution of the empirical measure, we need some notation. Denote by $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{R}_+)$ the space of positive Radon measures on $\mathbb{R}_+$ endowed with the vague topology, a metrizable topology. For each probability measure μ on $\mathbb{N}^{\mathbb{N}}$, denote by $\mathbb{Q}_\mu^N$ the probability measure on the path space $D([0,T], \mathcal{M}_+)$ induced by $\mathbb{P}_\mu^N$ and the empirical measure π^N defined in (3.3).

Theorem 3.2 . *The sequence* $\mathbb{Q}_{\mu^N}^N$ *converges to the probability measure concentrated on the absolutely continuous path* $\pi(t,du) = \rho(t,u)du$ *whose density is the solution of (3.2).*

Guo, Papanicolaou and Varadhan introduced in [GPV] a method, well known by now, to prove Theorem 3.2 provided one has a bound on the *entropy* and on the *Dirichlet form* of the system with respect to some invariant measure. These bounds are usually obtained computing the time derivative of the entropy of the distribution of particles at time t relative to the equilibrium distribution. In the present context, however, there is only one invariant measure : the trivial one $\delta_{\underline{0}}$ concentrated on the configuration $\underline{0}$ with no particles. Since all other probability measures on $\mathbb{N}^{\mathbb{N}}$ are orthogonal with respect to this one, the entropy of any reasonable measure with respect to $\delta_{\underline{0}}$ is infinite and the entropy method does not apply straightforwardly. To overcome this problem, we compute the relative entropy with respect to an inhomogeneous product measure that is not invariant but *close* to the invariant measure.

The proof of Theorem 3.2 may be decomposed in four distinct steps. We first obtain an estimate of the entropy and the Dirichlet form of the system with respect to some inhomogeneous product measure (Proposition 3.3). Then, in Lemma 3.5 we show that the sequence $\mathbb{Q}^N_{\mu^N}$ is tight and that all limit points are concentrated on absolutely continuous measures. This part is relatively easy and does not differ from the conservative case in infinite volume (cf. [KL]). In the third step, postponed to the appendix, we prove the uniqueness of weak solutions of equation (3.2). Finally, in the last step, we prove that all limit points of the sequence $\mathbb{Q}^N_{\mu^N}$ are concentrated on weak solutions of (3.2). This is done introducing a class of martingales associated to the empirical measure.

To obtain an estimate on the entropy and on the Dirichlet form, we first assume that the initial state satisfies the more restrictive condition :

(H3) There exists a parameter $\beta < \varphi^*$ for which the relative entropy of μ^N with respect to $\bar\nu^+_\beta$ is bounded by $C_0 N$ for some finite constant C_0.

This assumption implies that the initial state is a local perturbation of the product measure $\bar\nu^+_\beta$. In particular, it forces the initial profile ρ_0 to be asymptotically constant.

To deduce an estimate on the entropy of the system, we need to introduce a class of inhomogeneous product measures. Recall that we assumed the entropy of μ^N with respect to $\bar\nu^+_\beta$ to be bounded by $C_0 N$. For $x \ge 0$, define γ_x by $\gamma_x = \beta(1+x)/N$ for $0 \le x \le N-1$ and $\gamma_x = \beta$ for $x \ge N$. Denote by $\bar\nu^N_{\gamma(\cdot)}$ the product measure on $\mathbb{N}^{\mathbb{N}}$ with marginals given by

$$\bar\nu^N_{\gamma(\cdot)}\{\eta, \eta(x) = k\} \;=\; \frac{1}{Z(\gamma_x)} \frac{(\gamma_x)^k}{g(k)!} \tag{3.4}$$

for all $x \ge 0$.

First of all, we claim that the entropy of μ^N with respect to $\bar\nu^N_{\gamma(\cdot)}$ is bounded by $C_1 N$ for some finite constant C_1 depending only on C_0, α and β :

$$H(\mu^N \,|\, \bar\nu^N_{\gamma(\cdot)}) \;\le\; C_1 N \,. \tag{3.5}$$

Indeed, by the explicit formula for the entropy,

$$H(\mu^N \,|\, \bar\nu^N_{\gamma(\cdot)}) \;=\; H(\mu^N \,|\, \bar\nu^+_\beta) \;+\; \int \log \frac{d\bar\nu^+_\beta}{d\bar\nu^N_{\gamma(\cdot)}} d\mu^N \,.$$

By assumption (H3), the first term is bounded by $C_0 N$. Since $\gamma_x = \beta$ for $x \ge N$, the Radon–Nikodym derivative $d\bar\nu^+_\beta / d\bar\nu^N_{\gamma(\cdot)}$ is a cylinder function. Moreover, since $\gamma_x \le \beta$ for all x, $\log Z(\gamma_x)/Z(\beta) \le 0$ so that the second term on the right hand side is bounded above by

$$-\sum_{x=0}^{N-1} \log \frac{\gamma_x}{\beta} \int \eta(x) \mu^N(d\eta) \,.$$

Since $\gamma_x \le \beta$ and since by assumption (H1) $\mu^N \le \bar{\nu}_\alpha^+$, this expression is bounded above by

$$-R(\alpha)\sum_{x=0}^{N-1}\log\frac{\gamma_x}{\beta}$$

that divided by N converges, as $N \uparrow \infty$, to a finite constant because $\log u$ is integrable in $[0,1]$. This proves claim (3.5).

For each probability density f with respect to $\bar{\nu}^N_{\gamma(\cdot)}$, define the Dirichlet form $D_\gamma(f)$ by

$$D_\gamma(f) \;=\; D_{\gamma,b}(f) \;+\; D_{\gamma,i}(f) \;=\; D_{\gamma,b}(f) \;+\; \sum_{x\ge 0} D_{x,x+1}(f)\;,$$

where,

$$\begin{aligned} D_{\gamma,b}(f) \;&=\; (1/2)\int g(\eta(0))\left[\sqrt{f(\eta-\mathfrak{d}_0)}-\sqrt{f(\eta)}\right]^2\, d\bar{\nu}^N_{\gamma(\cdot)}\;,\\ D_{x,x+1}(f) \;&=\; (1/2)\int g(\eta(x))\left[\sqrt{f(\eta+\mathfrak{d}_{x+1}-\mathfrak{d}_x)}-\sqrt{f(\eta)}\right]^2\, d\bar{\nu}^N_{\gamma(\cdot)}\;. \end{aligned} \tag{3.6}$$

Proposition 3.3 . *Let S_t^N be the semi-group associated to the generator L introduced in (3.1) accelerated by N^2. Denote by $f_t = f_t^N$ the Radon–Nikodym derivative of $\mu^N S_t^N$ with respect to $\bar{\nu}^N_{\gamma(\cdot)}$. There exists a finite constant $C = C(\beta)$ such that*

$$\partial_t H(\mu^N S_t^N | \bar{\nu}^N_{\gamma(\cdot)}) \;\le\; -N^2 D_\gamma(f_t) \;+\; CN\;.$$

Proof. Denote by L^*_γ the adjoint operator of L with respect to $\bar{\nu}^N_{\gamma(\cdot)}$. It is easy to check that f_t is the solution of the forward equation

$$\begin{cases} \partial_t f_t = N^2 L^*_\gamma f_t \\ f_0 = (d\mu^N)/(d\bar{\nu}^N_{\gamma(\cdot)}) \end{cases} \tag{3.7}$$

The explicit formula for the entropy gives that $H(\mu^N S_t^N|\bar{\nu}^N_{\gamma(\cdot)}) = \int f_t \log f_t d\bar{\nu}^N_{\gamma(\cdot)}$. Therefore, since f_t is the solution of the forward equation (3.7),

$$\begin{aligned} \partial_t H(\mu^N S_t^N | \bar{\nu}^N_{\gamma(\cdot)}) \;&=\; \int N^2 L^*_\gamma f_t \,\log f_t\, d\bar{\nu}^N_{\gamma(\cdot)} \;+\; \int N^2 L^*_\gamma f_t \, d\bar{\nu}^N_{\gamma(\cdot)}\\ &=\; \int f_t\, N^2 L \log f_t\, d\bar{\nu}^N_{\gamma(\cdot)}\\ &=\; N^2\int f_t (L\log f_t - \frac{Lf_t}{f_t})\, d\bar{\nu}^N_{\gamma(\cdot)} \;+\; N^2 \int L f_t\, d\bar{\nu}^N_{\gamma(\cdot)}\;. \end{aligned} \tag{3.8}$$

Notice that the last term would vanish if $\bar{\nu}^N_{\gamma(\cdot)}$ was an invariant measure.

Since for every a, $b > 0$, $a\log(b/a)-(b-a)$ is less than or equal to $-(\sqrt{b}-\sqrt{a})^2$, for every x, $y \geq 0$, we have that

$$f_t\, L_{x,y}\log f_t \;-\; L_{x,y}f_t \;\leq\; -(1/2)g(\eta(x))\left[\sqrt{f_t(\eta+\mathfrak{d}_y-\mathfrak{d}_x)}-\sqrt{f_t(\eta)}\right]^2$$

$$f_t\, L_b\log f_t \;-\; L_b f_t \;\leq\; -g(\eta(0))\left[\sqrt{f_t(\eta-\mathfrak{d}_0)}-\sqrt{f_t(\eta)}\right]^2 .$$

Recall the definition of the Dirichlet form $D_\gamma(\cdot)$ introduced in (3.6). The previous estimate shows that the first term on the rightmost expression of (3.8) is bounded above by $-2N^2D_\gamma(f_t)$.

To estimate the term $N^2\int Lf_t d\bar{\nu}^N_{\gamma(\cdot)}$, which correspond to the price we are paying for not using an invariant distribution as reference measure, let us write it explicitly

$$N^2\int Lf_t d\bar{\nu}^N_{\gamma(\cdot)} = N^2\sum_{x\geq 0}\int (L_{x,x+1}f_t + L_{x+1,x}f_t)\, d\bar{\nu}^N_{\gamma(\cdot)} + N^2\int L_bf_td\bar{\nu}^N_{\gamma(\cdot)} \quad (3.9)$$

Performing the change of variable $\xi = \eta - \mathfrak{d}_x + \mathfrak{d}_y$, the measures change as

$$\frac{d\bar{\nu}^N_{\gamma(\cdot)}(\eta)}{d\bar{\nu}^N_{\gamma(\cdot)}(\xi)} \;=\; \frac{\gamma_x g(\xi(y))}{\gamma_y g(\eta(x))} \,.$$

In particular, we have that

$$\int (L_{x,x+1}f_t + L_{x+1,x}f_t)\; d\bar{\nu}^N_{\gamma(\cdot)} \;=\; (1/2)\Big(\frac{\gamma_x}{\gamma_{x+1}}-1\Big)\int g(\eta(x+1))f_t(\eta)\, d\bar{\nu}^N_{\gamma(\cdot)}$$
$$+\; (1/2)\Big(\frac{\gamma_{x+1}}{\gamma_x}-1\Big)\int g(\eta(x))f_t(\eta)\, d\bar{\nu}^N_{\gamma(\cdot)} \,.$$

We may thus rewrite the right hand side of (3.9) as

$$\begin{aligned}
&(1/2)\sum_{x\geq 1}\frac{(\Delta_N\gamma)(x)}{\gamma_x}\int g(\eta(x))f_t(\eta)\, d\bar{\nu}^N_{\gamma(\cdot)}\\
&\quad+\; (N^2/2)\Big(\frac{\gamma_1}{\gamma_0}-1\Big)\int g(\eta(0))f_t(\eta)\, d\bar{\nu}^N_{\gamma(\cdot)} \qquad (3.10)\\
&\quad+\; N^2\int g(\eta(0))[f_t(\eta-\mathfrak{d}_0)-f_t(\eta)]\, d\bar{\nu}^N_{\gamma(\cdot)} \,.
\end{aligned}$$

In this formula, $(\Delta_N\gamma)(x)$ stands for $N^2\{\gamma_{x+1}+\gamma_{x-1}-2\gamma_x\}$. By definition of γ, $(\Delta_N\gamma)(x)=0$ for all x except at $x=N-1$, where $(\Delta_N\gamma)(N-1)=N^2(\gamma_{N-2}-\gamma_{N-1})$ which is negative because γ is non decreasing. The first line of (3.10) is therefore negative. A change of variables $\xi=\eta-\mathfrak{d}_0$ permits to write the second term of the second line as

$$N^2\int[\gamma_0 - g(\eta(0))]f_t(\eta)\, d\bar{\nu}^N_{\gamma(\cdot)} \,.$$

The second line of (3.10) is therefore equal to

$$\beta N + (1/2)N^2\Big(\frac{\gamma_1}{\gamma_0} - 3\Big) \int g(\eta(0)) f_t(\eta)\, d\bar{\nu}^N_{\gamma(\cdot)} \le \beta N$$

because $\gamma_0 = \beta/N$, f is a density and $\gamma_1/\gamma_0 = 2$. This concludes the proof of the proposition. □

For $0 \le t \le T$, let $\bar{f}_t = t^{-1}\int_0^t f_s ds$. By Gronwall inequality, the convexity of the Dirichlet form and of the entropy, we have the following estimates.

Corollary 3.4 . *There exists a finite constant C_2 depending only on α, β and C_0 such that*

$$H(t^{-1}\int_0^t S^N_s \mu^N | \bar{\nu}^N_{\gamma(\cdot)}) + N^2 D_\gamma(\bar{f}_t) \le C_2 N t^{-1} .$$

With the previous estimate on the entropy and on the Dirichlet form, we are in a position to apply the classical entropy method to prove the hydrodynamic behavior of the system. We start showing that the sequence $\mathbb{Q}^N_{\mu^N}$ is tight.

Lemma 3.5 . *The sequence $\mathbb{Q}^N_{\mu^N}$ is tight. Moreover, all limit points are concentrated on weakly continuous paths $\pi(t,du)$ that are absolutely continuous and whose density are bounded above by $R(\alpha)$:*

$$\pi(t,du) = \rho(t,u)du \quad and \quad \rho(t,u) \le R(\alpha) .$$

Proof. The proof that the sequence is tight and that all limit points are concentrated on weakly continuous paths is similar to the proof of the same statement in the space homogeneous case in infinite volume (cf. [KL]). Details are left to the reader.

To prove that all limit points are concentrated on absolutely continuous measures with density bounded by $R(\alpha)$, consider a zero range process with reflexion at the origin. Its generator L^r is given by $\sum_{x\ge 0}\{L_{x,x+1} + L_{x+1,x}\}$. It is easy to check that the homogeneous product measures $\bar{\nu}^+_\varphi$, $\varphi < \varphi^*$, are invariant for this system. Since μ^N is bounded above by $\bar{\nu}^+_\alpha$ and the process is attractive, it is possible to couple the zero range process with absorption at the origin with a zero range process with reflexion at the origin starting from $\bar{\nu}^+_\alpha$ in such a way that the latter is always above the former. Since the latter is starting from an equilibrium state with density $R(\alpha)$, in the limit $N \uparrow \infty$, the empirical measure becomes absolutely continuous with density bounded by $R(\alpha)$. Since the zero range process with absorption at the origin is below the one with reflexion, the lemma is proved. □

We now show that all limit points of the sequence $\mathbb{Q}^N_{\mu^N}$ are concentrated on weak solutions of (3.2).

Proposition 3.6 . *All limit points $\mathbb{Q}$ of the sequence $\mathbb{Q}^N_{\mu^N}$ are concentrated on paths $\pi(t,du) = \rho(t,u)du$ such that* *(i) For every smooth function $G: \mathbb{R}_+ \to \mathbb{R}$ with compact support and vanishing at 0,*

$$< \rho_t, G > - < \rho_0, G > = (1/2) \int_0^t ds \int_0^\infty du\, (\Delta G)(u) \Phi(\rho(s,u)) ,$$

(ii) For each $0 \le t \le T$,

$$\int_0^t ds \int_0^\infty du \left\{ \partial_u \Phi(\rho(s,u)) \right\}^2 < \infty$$

(iii) For each $0 \le t \le T$,

$$\int_0^t ds\, \Phi(\rho(s,0)) = 0 .$$

An integration by parts in (i), taking advantage of (iii), shows that every limit point of the sequence $\mathbb{Q}^N_{\mu^N}$ is concentrated on weak solutions of equation (3.2).

Proof. Fix a limit point $\mathbb{Q}$ and assume without loss of generality that the sequence $\mathbb{Q}^N_{\mu^N}$ converges to $\mathbb{Q}$. For each smooth function G of class $C^2_K(\mathbb{R}_+)$ vanishing at the origin, consider the martingale $M_t^{G,N} = M_t^G$ defined by

$$M_t^G = < \pi_t^N, G > - < \pi_0^N, G > - \int_0^t ds\, N^2 L < \pi_s^N, G > .$$

A simple computation using the fact that G vanishes at 0 and that the jump rate is bounded shows that the expected value of the quadratic variation of M_t^G vanishes as $N \uparrow \infty$. In particular, by Doob inequality, for every $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}^N_{\mu^N} \Big[\sup_{0 \le t \le T} |M_t^G| > \delta \Big] = 0 . \tag{3.11}$$

On the other hand, a summation by parts shows that the integral term of the martingale M_t^G can be rewritten as

$$(1/2) \int_0^t ds \Big\{ N^{-1} \sum_{z \ge 1} \Delta_N G(z/N) g(\eta_s(z)) + (\nabla_N G)(0) g(\eta_s(0)) \Big\} .$$

In this formula, for each positive integer N, ∇_N and Δ_N denote respectively the discrete derivative and discrete Laplacian :

$$(\nabla_N G)(u) = N \Big\{ G(u + N^{-1}) - G(u) \Big\} ,$$

$$(\Delta_N G)(u) = N^2 \Big\{ G(u + N^{-1}) + G(u - N^{-1}) - 2G(u) \Big\} .$$

For each positive integer ℓ, denote by $\eta^\ell(x)$ the density of particles in a box of size $2\ell+1$ centered at x :

$$\eta^\ell(x) \;=\; \frac{1}{2\ell+1} \sum_{|z-x|\le\ell} \eta(z)\;.$$

Since η is a configuration on $\mathbb{N}$, it is understood in the previous formula that summation is carried out only over sites z in $\mathbb{N}$.

Fix $0 \le t \le T$. By Lemma 3.5, $\mathbb{Q}$ is concentrated on weakly continuous paths. In particular, property (i) follows from (3.11) and from Lemmas 3.7 and 3.8 below.

Property (ii) is proved in a similar way as it is done for finite volume Ginzburg–Landau processes (cf. [GPV]), lattice gases or non gradient models in contact with stochastic reservoirs (cf. [ELS2], [KLO]).

Property (iii) follows from Lemma 3.9 below. □

Lemma 3.7 . *For every continuous function H with compact support,*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty} \mathbb{E}_{\mu^N}\Big[\int_0^T dt\Big|N^{-1}\sum_{z\ge 1} H(z/N)\Big\{g(\eta_t(z)) - \Phi(\eta_t^{N\epsilon}(z))\Big\}\Big|\Big] \;=\; 0\;.$$

Proof. The proof follows from the, by now standard, one and two blocks estimates made possible by the bounds obtained in Corollary 3.4 on the Dirichlet form and on the entropy (cf. [GPV]. [KL]). □

Lemma 3.8 . *For every $0 \le t \le T$,*

$$\lim_{N\to\infty} \mathbb{E}_{\mu^N}\Big[\int_0^t g(\eta_s(0))ds\Big] \;=\; 0\;.$$

Proof. Recall that we denote by f_t the Radon–Nikodym derivative of $\mu^N S_t^N$ with respect to $\bar\nu^N_{\gamma(\cdot)}$ and that $\bar f_t = t^{-1}\int_0^t f_s ds$. With this notation, the expectation in the statement writes

$$t\int \bar f_t(\eta) g(\eta(0)) d\bar\nu^N_{\gamma(\cdot)}\;.$$

Adding and subtracting $\bar f_t(\eta - \mathfrak{d}_0)$ and changing variables, we obtain that this integral is equal to

$$t\int g(\eta(0))[\bar f_t(\eta) - \bar f_t(\eta-\mathfrak{d}_0)] d\bar\nu^N_{\gamma(\cdot)} \;+\; t\gamma_0\;.$$

The second term vanishes as $N\uparrow\infty$ because $\gamma_0 = \beta/N$. The first one, by Schwarz inequality and a change of variables, is bounded above by

$$\frac{1}{A}\{\|g\|_\infty + \gamma_0\} + AD_{\gamma,b}(\bar f_t)\;.$$

for every $A > 0$. Choosing $A = \sqrt{N}$, we conclude the proof of the lemma in virtue of Corollary 3.4. □

Lemma 3.9 . *For every* $0 \le t \le T$,

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\Big[\int_0^t ds\,\Phi(2\eta_s^{N\epsilon}(0))\Big] = 0 .$$

Notice that in this last expression we multiply $\eta_t^{N\epsilon}(0)$ by 2 to obtain the density of particles on the box $[0,\epsilon N]$.

The proof of Lemma 3.9 is performed in three steps. We first show that we may replace the cylinder function $g(\eta(0))$ by an average over a small macroscopic box around the origin. We then replace this average by $\Phi(2\eta^{\epsilon N}(0))$ and recall Lemma 3.8 to conclude.

Lemma 3.10 . *For each* $0 \le t \le T$,

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\Big[\int_0^t ds\,G(s)\Big\{g(\eta_s(0)) - (N\epsilon)^{-1}\sum_{y=0}^{N\epsilon} g(\eta_s(y))\Big\}\Big] = 0 .$$

Proof. To keep notation simple, denote by $V(\eta_s)$ the expression inside braces in the previous formula :

$$V(\eta) = g(\eta(0)) - (\epsilon N)^{-1}\sum_{y=0}^{N\epsilon} g(\eta(y))$$

Recall that we denote by $\bar f_t$ the average density $t^{-1}\int_0^t f_s ds$. With this notation, we may rewrite the expectation in the statement of the Lemma as

$$t\int V(\eta)\bar f_t(\eta)\bar\nu^N_{\gamma(\cdot)}(d\eta) .$$

The idea of the proof is to use an integration by parts formula for $g(\eta(0))-g(\eta(y))$ to estimate the integral $\int f(\eta)V(\eta)\bar\nu^N_{\gamma(\cdot)}(d\eta)$ in terms of the Dirichlet form.

A change of variables $\xi = \eta - \mathfrak{d}_x$ gives that $\int V(\eta)\bar f_t(\eta)\bar\nu^N_{\gamma(\cdot)}(d\eta)$ is equal to

$$(N\epsilon)^{-1}\sum_{x=0}^{N\epsilon}\sum_{y=0}^{x-1}\Big\{\gamma_y\int\Big\{\bar f_t(\eta+\mathfrak{d}_y) - \bar f_t(\eta+\mathfrak{d}_{y+1})\Big\}$$

$$\bar\nu^N_{\gamma(\cdot)}(d\eta) + [\gamma_y - \gamma_{y+1}]\int \bar f_t(\eta+\mathfrak{d}_{y+1})\bar\nu^N_{\gamma(\cdot)}(d\eta)\Big\} .$$

Since γ_x is increasing in x, the second term is negative.

On the other hand, rewriting the difference $\{a-b\} = \{\bar f_t(\eta+\mathfrak{d}_y)-\bar f_t(\eta+\mathfrak{d}_{y+1})\}$ as $\{\sqrt{a}-\sqrt{b}\}\{\sqrt{a}+\sqrt{b}\}$ and applying Schwarz inequality, we bound the first term by

$$\frac{A}{2\epsilon N}\sum_{x=0}^{N\epsilon}\sum_{y=0}^{x-1}\gamma_y\int\Big\{\sqrt{\bar f_t(\eta+\mathfrak{d}_y)} - \sqrt{\bar f_t(\eta+\mathfrak{d}_{y+1})}\Big\}^2\bar\nu^N_{\gamma(\cdot)}(d\eta)$$

$$+ \frac{1}{A\epsilon N}\sum_{x=0}^{N\epsilon}\sum_{y=0}^{x-1}\gamma_y\int\Big\{\bar f_t(\eta+\mathfrak{d}_y) + \bar f_t(\eta+\mathfrak{d}_{y+1})\Big\}\bar\nu^N_{\gamma(\cdot)}(d\eta)$$

for every $A > 0$. Changing variables back, keeping in mind that γ_x is a non-decreasing function and inverting the order of summation, we show that this expression is bounded above by

$$\frac{A}{2}\sum_{y=0}^{N\epsilon-1} D_{y,y+1}(\bar{f}_t) \; + \; \frac{2\|g\|_\infty}{A}\epsilon N$$

for every positive A. Taking $A = \sqrt{\epsilon}N$, we conclude the proof of the lemma applying Corollary 3.4. □

In view of Lemma 3.8, to conclude the proof of Lemma 3.9, it remains to replace the average of the cylinder function $g(\eta(x))$ by $\Phi(2\eta^{\epsilon N}(0))$. This is the content of the next result.

Lemma 3.11 . *For every continuous function $G\colon [0,T] \to \mathbb{R}$,*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\Big[\Big|\int_0^T ds\, G(s)\Big\{(\epsilon N)^{-1}\sum_{x=0}^{\epsilon N} g(\eta_s(x)) - \Phi(2\eta_s^{\epsilon N}(x))\Big\}\Big|\Big] \;=\; 0 \;.$$

Proof. Recall the estimates of the entropy and the Dirichlet form obtained in Proposition 3.3. The proof of this result follows from these bounds and the, by now standard, proof of the one and two blocks estimate for homogeneous zero range processes that can be found in [KL], for instance. □

This concludes the proof of Theorem 3.2 under assumptions (H1), (H2), (H3). A coupling argument permits to remove Assumption (H3).

Consider a sequence μ^N satisfying assumptions (H1) and (H2). Fix $A > 0$ and let $\mu^{N,A}$ be the probability measure on $\mathbb{N}^{\mathbb{N}}$ defined by

$$\mu^{N,A} \;=\; \mu^N\Big|_{\Lambda_{AN}} \otimes \nu_\beta^+\Big|_{\Lambda_{AN}^c}$$

where $\Lambda_{AN} = \{0, \dots, AN\}$ and ν_Λ is the marginal of the probability measure ν on Λ.

Since $\nu_\lambda^+ \le \mu^N \le \nu_\alpha^+$ and since all cylinder functions can be decomposed as the difference of two monotone functions (cf. [KL]), a simple computation and the explicit formula for the relative entropy give that

$$H(\mu^{N,A}|\nu_\beta^+) \;\le\; \frac{1}{2}\Big\{H(\nu_\alpha^{+,AN}|\nu_\beta^{+,AN}) \;+\; H(\nu_\lambda^{+,AN}|\nu_\beta^{+,AN})\Big\}\;,$$

where $\nu_\gamma^{+,m}$ is the marginal of ν_γ^+ on $\{0, \dots, m\}$. In particular, the entropy $H(\mu^{N,A}|\nu_\beta^+)$ is bounded above by $C_0 N$ for some finite constant C_0 depending only on A, α and λ.

Let $\rho^A(t,u)$ denote the solution of (1.3) with initial condition $\rho_0^A(u) = \rho_0(u)\mathbf{1}\{u \le A\} + \beta\mathbf{1}\{u > A\}$. Investigating the time evolution of the integral

$\int_{\mathbb{R}_+} du e^{-u} \rho^A(t,u)^2$ we obtain uniform in A a priori estimates that show that ρ^A converges to the unique solution of (1.3) with initial condition ρ_0.

Since the jump rate g is non decreasing, we may couple a zero range starting from μ^N with another one starting from $\mu^{N,A}$ and show that as $A \uparrow \infty$ both behaves exactly in the same way on compact sets. This coupling, the hydrodynamic behavior of the empirical measure for a process starting from $\mu^{N,A}$ and the convergence of ρ^A to ρ, permit to extend Theorem 3.2 to sequence of measures satisfying assumptions (H1) and (H2).

4. The case $p < 1$.

We turn in this section to the case where the asymmetric tagged particle jumps at rate p to the right and at rate q to the left. The corresponding zero range process has jumps at rate (1/2) over all bonds but $\{-1, 0\}$. From the origin, particles jump at rate p to -1 and from -1 particles jump at rate q to 0. Recall that to fix ideas we assume $p > q$.

The purpose of this section is to deduce the hydrodynamic behavior of the just described space inhomogeneous process. Since the techniques required to derive the hydrodynamic behavior apply to a large class of systems, we introduce a general set up. Fix a jump rate $g : \mathbb{N} \to \mathbb{R}_+$ satisfying the assumptions of section 2 and consider the zero–range process on $\mathbb{Z}$ with generator given by

$$L = \sum_{x \neq -1} \{L_{x,x+1} + L_{x+1,x}\} + 2pL_{0,-1} + 2qL_{-1,0} , \tag{4.1}$$

where $L_{x,y}$ is the generator defined just after (3.1). In contrast with the previous section, this system possesses a one parameter family of invariant measures. For each $\varphi < \varphi^*/p$, denote by $\bar{\nu}^i_\varphi$ the product measure on $\mathbb{N}^{\mathbb{Z}}$ with marginals given by

$$\bar{\nu}^i_\varphi\{\eta, \eta(x) = k\} = \frac{1}{Z(\varphi_x)} \frac{\varphi_x^k}{g(k)!} , \tag{4.2}$$

where $\varphi_x = p\varphi$ for $x \leq -1$ and $\varphi_x = q\varphi$ for $x \geq 0$. A direct computation shows that the Markov process with generator given by (4.1) is reversible with respect to these product measures.

Before stating the main result of this section, we introduce some terminology on weak solutions of non-linear parabolic equations. Fix a bounded function $\rho_0 : \mathbb{R} \to \mathbb{R}$. A bounded function $\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is said to be a weak solution of the partial differential equation

$$\begin{cases} \partial_t \rho = (1/2)\Delta\Phi(\rho) \\ p\Phi(\rho(t,0+)) = q\Phi(\rho(t,0-)) \\ \partial_u\Phi(\rho(t,0+)) = \partial_u\Phi(\rho(t,0-)) \\ \rho(0,\cdot) = \rho_0(\cdot) \end{cases} \tag{4.3}$$

if

(a) $\Phi(\rho(t,u))$ is absolutely continuous in the space variable and for every $t > 0$,

$$\int_0^t ds \int_{\mathbb{R}} du\, e^{-|u|}\{\partial_u \Phi(\rho(s,u))\}^2 < \infty ,$$

(b) $p\Phi(\rho(t,0+)) = q\Phi(\rho(t,0-))$ for almost every $t \geq 0$ and

(c) For every smooth function with compact support $G: \mathbb{R} \to \mathbb{R}$ and for every $t > 0$,

$$\int_{\mathbb{R}} du\, \rho(t,u)G(u) - \int_{\mathbb{R}} du\, \rho_0(u)G(u) = - \int_0^t ds \int_{\mathbb{R}} du\, G'(u)\partial_u \Phi(\rho(s,u)) .$$

Since $\rho(t,u)$ is only a measurable function, requirement (b) must be understood as

$$\lim_{\epsilon \to 0} \int_0^t ds \left\{ p\Phi\Big(\frac{1}{\epsilon}\int_0^\epsilon \rho(s,u)du\Big) - q\Phi\Big(\frac{1}{\epsilon}\int_{-\epsilon}^0 \rho(s,u)du\Big) \right\} = 0 \qquad (4.4)$$

for every $t \geq 0$. The third property in (4.3) just states that there is conservation of the total mass at the origin.

We prove in the appendix the uniqueness of weak solutions of (4.3). The existence for special initial conditions ρ_0 follows from the tightness of the sequence $\mathbb{Q}^N_{\mu^N}$ defined below and the regularity of the limit points of this sequence proved in Proposition 4.4.

For each probability measure μ on $\mathbb{N}^{\mathbb{Z}}$, denote by $\mathbb{P}^N_\mu$ the probability measure on the path space $D(\mathbb{R}_+, \mathbb{N}^{\mathbb{Z}})$ induced by the Markov process with generator (4.1) accelerated by N^2 and the initial measure μ. Expectation with respect to $\mathbb{P}^N_\mu$ is denoted by $\mathbb{E}^N_\mu$.

We now define the initial states we consider in the first main theorem of this section. Fix a sequence of initial measures μ^N on $\mathbb{N}^{\mathbb{Z}}$. We shall assume that

(IS1) The sequence μ^N is bounded above (resp. below) by some invariant state $\bar\nu^i_\alpha$ (resp. $\bar\nu^i_\lambda$) for some $0 < \lambda < \alpha < \varphi^*/p$.

(IS2) There exists a function $\rho_0: \mathbb{R} \to \mathbb{R}_+$ such that for each continuous function $G: \mathbb{R} \to \mathbb{R}_+$ and each $\delta > 0$,

$$\lim_{N\to\infty} \mu^N \Big[\Big| N^{-1} \sum_x G(x/N)\eta(x) - \int du\, G(u)\rho_0(u) \Big| \geq \delta \Big] = 0 .$$

It follows from assumption (IS1) that the function ρ_0 in (IS2) is necessarily bounded. Assumption (IS1) is relatively restrictive and is explained at the end of section 2. The second assumption just imposes a hydrodynamic behavior (a law of large number) at time 0.

Theorem 4.1 . *Consider a sequence of initial states μ^N satisfying assumptions (IS1), (IS2). For any continuous function $G: \mathbb{R} \to \mathbb{R}$ with compact support and any $\delta > 0$*

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu^N} \left[\Big| N^{-1} \sum_x G(x/N)\eta_t(x) - \int du\, G(u)\rho(t,u) \Big| \geq \delta \right] = 0$$

where ρ is the unique solution of (4.3).

Like in section 3 (cf. also Chap. IV of [KL]), we deduce this result from the convergence in distribution of the empirical measure $\pi^N = \pi^N(\eta)$ defined as the positive Radon measure on $\mathbb{R}$ obtained by assigning a mass N^{-1} to each particle :

$$\pi^N = N^{-1} \sum_{z \in \mathbb{Z}} \eta(z) \delta_{z/N} . \tag{4.5}$$

Set $\pi_t^N = \pi^N(\eta_t)$ and denote by $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{R})$ the space of positive Radon measures on $\mathbb{R}$ endowed with the vague topology, a metrizable topology. Fix $T > 0$. For each probability measure μ on $\mathbb{N}^{\mathbb{Z}}$, denote by $\mathbb{Q}_\mu^N$ the probability measure on the path space $D([0,T], \mathcal{M}_+)$ induced by $\mathbb{P}_\mu^N$ and the empirical measure π^N defined in (4.5).

Theorem 4.2 . *The sequence $\mathbb{Q}_{\mu^N}^N$ converges to the probability measure concentrated on the absolutely continuous path $\pi(t, du) = \rho(t, u)du$ whose density is the solution of (4.3).*

The proof of this result is divided in three steps. We first show that the sequence $\mathbb{Q}_{\mu^N}^N$ is tight. We then prove that all limit points are concentrated on weak solutions of (4.3). This is the content of Proposition 4.4 below. We conclude arguing that there exists a unique solution. This last part is left to the Appendix.

Coupling arguments similar to the ones presented at the end of the previous section show that it is enough to prove Theorem 4.2 under the assumption

(IS3) There exists a parameter $\beta < \varphi^*/p$ for which the relative entropy of μ^N with respect to $\bar{\nu}_\beta^i$ is bounded by $C_0 N$ for some finite constant C_0.

Lemma 4.3 . *The sequence $\mathbb{Q}_{\mu^N}^N$ is tight. Moreover, all limit points are concentrated on weakly continuous paths $\pi_t(du)$.*

The proof of the tightness of the sequence $\mathbb{Q}_{\mu^N}^N$ is similar to the proof in the space homogeneous case and is thus omitted (cf. Chap IV, V of [KL]).

Proposition 4.4 . *Every limit point $\mathbb{Q}^*$ of the sequence $\mathbb{Q}_{\mu^N}^N$ is concentrated on absolutely continuous paths $\pi(t, du) = \rho(t, u)du$ such that*

(a) $\rho_t(u) \le r(u)$, where $r(u) = R(p\alpha)\mathbf{1}\{u < 0\} + R(q\alpha)\mathbf{1}\{u > 0\}$, $R(\cdot)$ is the function defined in (2.5) and α the parameter introduced in assumption (IS1).

(b) Recall the constant C_0 introduced in assumption (IS3). Then,

$$E_{\mathbb{Q}^*}\Big[\int_0^T ds \int_{\mathbb{R}} du \left\{ \partial_u \Phi(\rho(s,u)) \right\}^2 \Big] < C_0 .$$

(c) $p\Phi(\rho(t, 0+)) = q\Phi(\rho(t, 0-))$ in the sense (4.4).

(d) For every smooth function $G: \mathbb{R} \to \mathbb{R}$ with compact support,

$$< \rho_t, G > - < \rho_0, G > = -(1/2) \int_0^t ds \int du \, (\partial_u G)(u) \partial_u \Phi(\rho(s,u)) .$$

Proof. Fix a limit point $\mathbb{Q}^*$ and assume without loss of generality that the sequence $\mathbb{Q}^N_{\mu^N}$ converges to $\mathbb{Q}^*$. Property (a) follows from assumption (IS1) and some elementary coupling arguments since the process is attractive.

To prove (b), we show that there exists a constant $C = C(g)$ such that

$$E_{\mathbb{Q}^*}\Big[\sup_G \Big\{ \int_0^T dt \int m(du)\,(\partial_u G)(t,u)\Phi(\rho(t,u)) - C(g)\|G\|_2^2 \Big\}\Big] \;\le\; C_0 \,.$$

In this formula, the supremum is carried over all smooth functions $G: [0,T]\times\mathbb{R} \to \mathbb{R}$ with compact support, $\|\cdot\|_2$ stands for the $L^2([0,T]\times\mathbb{R})$ norm and $m(u) = p\mathbf{1}\{u > 0\} + q\mathbf{1}\{u < 0\}$. The proof of this estimate follows closely the proof of a similar estimate in the space homogeneous case (cf. [KLO], [GPV]). We leave the details to the reader.

Property (c) follows straightforwardly from Corollary 4.9 below.

To prove (d), fix a smooth function $G: \mathbb{R} \to \mathbb{R}$ with compact support and consider the martingale $M_t^G = M_t^{G,N}$ defined by

$$M_t^G \;= <\pi_t^N, G> \;-\; <\pi_0^N, G> \;-\; \int_0^t N^2 L <\pi_s^N, G>\, ds \,.$$

A simple computation shows that the expected value of the quadratic variation of M_t^G vanishes as $N \uparrow \infty$. In particular, by Doob inequality, for every $\delta > 0$,

$$\limsup_{N\to\infty} \mathbb{P}^N_{\mu^N}\Big[\sup_{0\le t\le T} |M_t^G| > \delta\Big] \;=\; 0 \,. \tag{4.6}$$

On the other hand, the integral part of the martingale M_t^G writes

$$\int_0^t ds \Big\{ \frac{1}{2N}\sum_{x\in\mathbb{Z}} (\Delta_N G)(x/N) g(\eta_s(x)) + \frac{q-p}{2}(\nabla_N G)(-1/N) \\ \big\{ g(\eta_s(0)) + g(\eta_s(-1)) \big\} \Big\}.$$

Since g is bounded and G has compact support, we may replace in the above formula the discrete derivative and the discrete Laplacian by the continuous ones. For a fixed $\epsilon > 0$ and $0 \le t \le T$, let

$$V_\epsilon^G(t) \;=\; \int du \Big\{ (\Delta G)(u) \Phi\Big(\frac{\pi_t([u-\epsilon, u+\epsilon])}{2\epsilon}\Big) \\ +\; (q-p)(\nabla G)(0) \Big\{ \Phi\Big(\frac{\pi_t([0,\epsilon])}{\epsilon}\Big) + \Phi\Big(\frac{\pi_t([-\epsilon,0])}{\epsilon}\Big) \Big\} \,.$$

By (4.6) and Lemmas 4.5 and 4.6 below, for every $0 \le t \le T$ and $\delta > 0$,

$$\limsup_{\epsilon\to 0} \mathbb{Q}^*\Big[\Big| <\rho_t, G> - <\rho_0, G> - (1/2)\int_0^t ds\, V_\epsilon^G(s) \Big| > \delta \Big] \;=\; 0$$

because, by Lemma 4.3, $\mathbb{Q}^*$ is concentrated on weakly continuous paths π_t. Integrating by parts V_ϵ^G we obtain (d) applying parts (b) and (c) of the proposition. □

We proceed establishing the results needed in the proof of the previous proposition.

Lemma 4.5 . *For every continuous function $G: \mathbb{R} \to \mathbb{R}$ with compact support,*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt N^{-1}\sum_{z\in\mathbb{Z}} G(z/N)\{g(\eta_t(z)) - \Phi(\eta_t^{\epsilon N}(z))\}\Big|\Big] = 0 .$$

This result follows from the estimate on the entropy of μ^N and the, by now classical, one and two blocks arguments. Notice, however, that since g is assumed to be bounded, we need the measure μ^N to be bounded above by some invariant measure $\bar{\nu}^i_\varphi$ in order to introduce a cut off to avoid large densities in the two blocks estimate.

For a site x, a configuration η and a positive integer ℓ, denote by $M_\ell^\pm(x,\eta)$ the density of particles for the configuration η on a box of size ℓ at the right (left) of x :

$$M_\ell^+(x,\eta) = \frac{1}{\ell+1}\sum_{y=x}^{x+\ell}\eta(y) , \quad M_\ell^-(x,\eta) = \frac{1}{\ell+1}\sum_{y=x-\ell}^{x}\eta(y) .$$

Lemma 4.6 . *For every continuous function $H: [0,T] \to \mathbb{R}$,*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt\, H(t)\Big\{g(\eta_t(0)) - \Phi(M^+_{\epsilon N}(0,\eta_t))\Big\}\Big|\Big] = 0 .$$

The same result holds if $g((\eta_t(0))$ is replaced by $g((\eta_t(-1))$ and $M^+_{\epsilon N}(0,\eta_t)$ by $M^-_{\epsilon N}(-1,\eta_t)$.

This result follows from the next lemma and the statement of Lemma 3.11 translated to the present context.

Lemma 4.7 . *For every continuous function $H: [0,T] \to \mathbb{R}$,*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt\, H(t)\Big\{g(\eta_t(0)) - (\epsilon N)^{-1}\sum_{x=0}^{\epsilon N} g(\eta_t(x))\Big\}\Big|\Big] = 0 .$$

The same result holds if $g((\eta_t(0))$ is replaced by $g((\eta_t(-1))$ and the average over $\{0,\dots,\epsilon N\}$ is replaced by the average over $\{-\epsilon N,\dots,0\}$.

Proof. To keep notation simple, denote by $V(\eta_t)$ the expression inside braces in the previous formula :

$$V(\eta) = g(\eta(0)) - (N\epsilon)^{-1}\sum_{y=0}^{N\epsilon} g(\eta(y))$$

Inspired by the proof of Lemma 3.10, the idea here is to use an integration by parts formula for $g(\eta(0)) - g(\eta(y))$ to estimate the integral $\int f_t(\eta)V(\eta)\bar{\nu}^N_{\gamma(\cdot)}(d\eta)$ by the Dirichlet form.

By the entropy inequality,

$$\begin{aligned}&\mathbb{E}_{\mu^N}\Big[\Big|\int_0^T ds\, H(s)V(\eta_s)\Big|\Big]\\&\quad\le\ \frac{H(\mu^N|\bar\nu^i_\beta)}{NA}\ +\ \frac{1}{AN}\log\mathbb{E}_{\bar\nu^i_\beta}\Big[\exp\Big\{\Big|\int_0^T ds\, G(s)ANV(\eta_s)\Big|\Big\}\Big]\end{aligned}$$

for every $A>0$. By assumption, the first term on the right hand side is bounded by CA^{-1}. To prove the lemma it is therefore enough to show that the limit of the second one is less than or equal to 0 for every $\gamma>0$. Since $e^{|x|}\le e^x+e^{-x}$ and $\limsup_N N^{-1}\log\{a_N+b_N\}\ \le\ \max\{\limsup_N N^{-1}\log a_N, \limsup_N N^{-1}\log b_N\}$, replacing H by $-H$ we deduce that we only need to prove the previous statement without the absolute value in the exponent. By Feynman–Kac formula and the variational formula for the largest eigenvalue of an operator,

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{\gamma N}\log\mathbb{E}_{\bar\nu^i_\beta}\Big[\exp\Big\{\int_0^T ds\, G(s)ANV(\eta_s)\Big\}\Big]\\&\quad\le\ \int_0^T dt\,\sup_f\Big\{\int H(t)V(\eta)f(\eta)\bar\nu^i_\beta(d\eta)\ +\ A^{-1}ND(f)\Big\}\,.\end{aligned}\tag{4.7}$$

In this formula, the supremum is taken over all densities f with respect to $\bar\nu^i_\beta$ and $D(f)$ is the Dirichlet form

$$D(f)\ =\ \int\sqrt{f}L\sqrt{f}\,d\nu^i_\beta\,.$$

We are now ready to integrate by parts the cylinder function V. The rest of the proof is similar to the proof of Lemma 3.10 and omitted for this reason. □

The same argument permits to deduce the following result.

Lemma 4.8 . *For every continuous function $H:[0,T]\to\mathbb{R}$,*

$$\limsup_{N\to\infty}\mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt\, H(t)\{pg(\eta_t(0))-qg(\eta_t(-1))\}\Big|\Big]\ =\ 0\,.$$

The next result follows from Lemma 4.6 and Lemma 4.8.

Corollary 4.9 . *For every continuous function $H:[0,T]\to\mathbb{R}$,*

$$\limsup_{N\to\infty}\mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt\, H(t)\Big\{p\Phi(M^+_{\epsilon N}(0,\eta_t))-q\Phi(M^-_{\epsilon N}(-1,\eta_t))\Big\}\Big|\Big]\ =\ 0\,.$$

This concludes the proof of Theorem 4.2 under assumptions (IS1), (IS2) (IS3). Assumption (IS3) was needed only in the derivation of the estimates of the entropy and the Dirichlet form. The arguments presented in [LM] permit to deduce these estimates with assumption (IS2) in place of (IS3). This proves Theorem 4.2.

5. The asymmetric tagged particle.

We prove in this section Theorems 1.1 and 1.2 through the hydrodynamic behavior of the inhomogeneous zero range processes considered in the previous two sections.

We have seen in the first section that the displacement of the asymmetric tagged particle corresponds in the zero range process to the total flux of particles through the origin. For this reason, we start deducing the total flux through the origin from the hydrodynamic limit proved in the previous two sections.

Proposition 5.1 . *In the case $p = 1$, consider a sequence of probability measures μ^N satisfying assumptions (H1), (H2). Then, for every $t \geq 0$ and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu^N}\Big[\Big|N^{-1}\sum_{x\geq 0}\{\eta_t(x)-\eta_0(x)\} - \int_0^\infty du\,\{\rho(t,u)-\rho_0(u)\}\Big| > \delta\Big] \;=\; 0\ , \quad (5.1)$$

where ρ is the solution of (3.2). In the case $p < 1$, consider a sequence of probability measures μ^N satisfying assumptions (IS1), (IS2). Then, for every $t \geq 0$ and $\delta > 0$ (5.1) holds, where ρ is now the solution of (4.3).

In the previous statement, as explained in section 1, the sum and the integral must be interpreted correctly. Proposition 5.1 follows from the hydrodynamic behavior of the inhomogeneous processes considered in section 3, 4 and the definition of the sum and the integral.

Recall from section 1 that we denote by $\mathcal{T}$ the transformation that associates to each configuration (resp. continuous function, probability measure) of $\{0,1\}^{\mathbb{Z}_\bullet}$ (the set of configurations of $\{0,1\}^{\mathbb{Z}}$ with a particle at the origin) a configuration (resp. continuous function, probability measure) of $\mathbb{N}^{\mathbb{Z}}$. Theorem 1.1 and 1.2 follow from Proposition 5.1 if we prove the following proposition

Proposition 5.2 . *Fix a sequence of initial states $\mu^N_{\rho_0(\cdot)}$ satisfying the assumptions of Theorem 1.1 or 1.2. The sequence $\mathcal{T}\mu^N_{\rho_0(\cdot)}$ satisfy assumptions (H1), (H2) in the case $p = 1$ or (IS1), (IS2) in the case $p < 1$.*

Proof. We start with the case $p = 1$. A simple computation shows that $\mathcal{T}$ transforms the Bernoulli product measure μ_ρ in the product measure $\bar\nu^+_{1-\rho}$ defined by (2.4). Fix a profile $\rho_0 : \mathbb{R}_+ \to [0,1]$ for which there exists $\sigma > 0$ such that $\sigma \leq \rho_0 \leq 1-\sigma$. Recall that we denote by $\mu^N_{\rho_0(\cdot)}$ the inhomogeneous product measure associated to ρ_0 :

$$\mu^N_{\rho_0(\cdot)}\{\xi\,,\xi(x)=1\} \;=\; \rho_0(x/N)$$

for $x > 0$. Denote by $\nu^N_{\rho_0(\cdot)}$ the probability measure on $\mathbb{N}^{\mathbb{N}}$ associated to $\mu^N_{\rho_0(\cdot)}$ through the transformation $\mathcal{T}$. We shall now show that $\nu^N_{\rho_0(\cdot)}$ fulfills assumptions (H1), (H2).

We first claim that if μ is a product measure on $\{0,1\}^{\mathbb{N}_\bullet}$ bounded above (resp. below) by μ^+_ρ for some $0 < \rho < 1$, then $\mathcal{T}\mu$ is bounded below (resp. above) by

$\bar{\nu}^+_{1-\rho}$. Here μ^+_ρ (resp. $\bar{\nu}^+_\rho$) stand for the restriction on $\mathbb{N}$ of the measures μ_ρ (resp. $\bar{\nu}_\rho$). Notice that the inequalities are reversed by the application $\mathcal{T}$. To prove this claim, for $x \geq 1$, denote by γ_x the probability of finding a particle at x for the probability μ. Assume, without loss of generality that $\gamma_x \leq \rho$. For $j \geq 1$, denote by N_j the position of the j-th particle at the right of the origin. Since $\gamma_x \leq \rho$ for every x and μ, μ_ρ are product measures, it is possible to couple μ and μ_ρ in such a way that $N_1^\mu \geq N_1^\rho$ and $N_{j+1}^\mu - N_j^\mu \geq N_{j+1}^\rho - N_j^\rho$ for all $j \geq 1$. In this formula, N_j^μ (resp. N_j^ρ) stands for the position of the j-th particle under the distribution μ (resp. μ_ρ). Applying the transformation $\mathcal{T}$ to this coupling measure, we construct a measure on $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ with first marginal equal to $\mathcal{T}\mu$, second marginal equal to $\mathcal{T}\mu^+_\rho = \bar{\nu}^+_{1-\rho}$ and concentrated on configurations (η^1, η^2) below the diagonal. This shows that $\mathcal{T}\mu \geq \bar{\nu}^+_{1-\rho}$, what concludes the proof of the claim. In particular, $\bar{\nu}^+_\sigma \leq \nu^N_{\rho_0(\cdot)} \leq \bar{\nu}^+_{1-\sigma}$ for every $N \geq 1$ and assumption (H1) is verified.

Notice, however, that the claim "$\mu^1 \leq \mu^2$ implies $\mathcal{T}\mu^1 \geq \mathcal{T}\mu^2$" is not correct. Consider, for instance, the configuration ξ^1, ξ^2 such that

$$\xi^1(x) = 1 \quad \text{if and only if } x \neq 1,2,3 \text{ and } \quad \xi^2(x) = 1 \quad \text{if and only if } x \neq 1,3\ .$$

In this case the deterministic measures δ_{ξ^i} are such that $\delta_{\xi^1} \leq \delta_{\xi^2}$ but it is not correct that $\delta_{\mathcal{T}\xi^1}$ is above $\delta_{\mathcal{T}\xi^2}$.

We turn now to the second assumption (H2). Recall from section 1 that we denote by $\mathcal{H}: \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{F}: \mathbb{R}_+ \to \mathbb{R}_+$ the functions defined by

$$\mathcal{H}(A) \;=\; \int_0^A \rho_0(u)\, du\ , \quad \mathcal{F}(B) \;=\; \frac{1}{\rho_0(\mathcal{H}^{-1}(B))} \;-\; 1\ .$$

Here $\mathcal{H}^{-1}$ stands for the inverse of the strictly increasing, absolutely continuous function $\mathcal{H}$ It follows from this definition that

$$\int_0^B \mathcal{F}(u)\, du \;=\; \mathcal{H}^{-1}(B) - B \tag{5.2}$$

for every $B > 0$. In order to check (H2), we just need to show that under $\nu^N_{\rho_0(\cdot)}$,

$$N^{-1} \sum_{x=0}^{[BN]} \eta(x) \tag{5.3}$$

converges in probability to $\int_0^B \mathcal{F}(u) du$ for every $B > 0$. Fix a positive integer n. The following inequalities state that for the exclusion process the total number of sites in $\Lambda_n = \{0, \dots, n\}$ is equal to the total number of particles plus the total number of holes (that corresponds to the total number of particles for the zero range process) :

$$\sum_{x=0}^{n} \xi(x) \;+\; \sum_{y=0}^{-1+\sum_{x=0}^{n}\xi(x)} \eta(y) \;\leq n+1 \;\leq\; \sum_{x=0}^{n} \xi(x) \;+\; \sum_{y=0}^{\sum_{x=0}^{n}\xi(x)} \eta(y)\ .$$

The convergence (5.3) follows from these inequalities, the fact that under the measure $\mu^N_{\rho_0(\cdot)}$, $N^{-1}\sum_{0\le x\le[nN]}\xi(x)$ converges to $\int_0^n \rho_0(u)\,du$ and identity (5.2). Details are left to the reader.

In exactly the same way, assumptions (IS1), (IS2) can be checked in the case $p<1$. The only difference is that we assume in (IS1) that the sequence of initial measures is bounded below by an invariant measure $\bar\nu^i_\varphi$ which is inhomogeneous in space. This forces the initial profile ρ_0 to be bounded below by the function $\psi_\alpha(u) = (1-\alpha)\mathbf{1}\{u<0\} + [1-(q/p)\alpha]\mathbf{1}\{u>0\}$ for some $0<\alpha<1$. □

6. Einstein relation.

We consider in this section initial profiles for which the solution of equation (1.4) is self scaling. For two fixed densities ρ_- and ρ_+ consider, for instance, the initial condition $\rho_0(\cdot)$ given by

$$\rho_0(u) \;=\; \rho_+\mathbf{1}\{u\ge 0\} \;+\; \rho_-\mathbf{1}\{y<0\}\,.$$

The the solution of (1.4) takes the form $\rho(t,u) = \varphi(u/\sqrt{t})$, where $\varphi(\cdot)$ is the solution of

$$\begin{cases} -\,z\varphi'(z) = \partial_z^2\Phi_0(\varphi(z))\,, \\ p\Phi_0(\varphi(0+)) = q\Phi_0(\varphi(0-))\,, \\ \dfrac{\varphi'(0+)}{(1+\varphi(0+))^2} = \dfrac{\varphi'(0-)}{(1+\varphi(0-))^2}\,, \\ \varphi(\pm\infty) = \rho_\pm\,. \end{cases} \tag{6.1}$$

In this formula $\Phi_0(a) = a/(1+a)$.

It easy to see that in this case $v_t = v\sqrt{t}$ where v is given by

$$v \;=\; \int_0^{+\infty}\{\rho_+ - \varphi(y)\}\,dy\,.$$

Moreover, since $\rho_+ = \varphi(\infty)$, we may write the expression inside braces as $\int_{[y,\infty)}\partial_z\varphi(z)dz$. Performing an integration by parts and keeping in mind that φ is the solution of (6.1), we obtain that

$$v \;=\; \frac{\varphi'(0+)}{(1+\varphi(0+))^2}\,.$$

We now transform (6.1) in a linear equation through a Lagrangian change of coordinates. Define

$$x(z) \;=\; \int_0^z (1+\varphi(y))\,dy$$

and set

$$m(x) = \frac{1}{1+\varphi(z(x))}\,.$$

We leave to the reader to check that this transformation is in fact the inverse of the transformation described in (1.2). Moreover, a simple computation shows that $m(x)$ is the solution of the linear equation

$$\begin{cases} m''(x) \;=\; -(x+v)m'(x)\;, \\ -v = \dfrac{m'(0+)}{m(0+)} = \dfrac{m'(0-)}{m(0-)}\;, \\ p(1-m(0+)) = q(1-m(0-))\;, \\ m(\pm\infty) = \alpha_{\pm} = \dfrac{1}{1+\rho_{\pm}}\;. \end{cases} \tag{6.2}$$

In fact (6.2) describes the self scaling solution of the Stefan problem :

$$\begin{cases} \partial_t m^*(x,t) = \dfrac{1}{2}\partial_{xx} m^*(x,t)\;, \\ -v_t = \dfrac{\partial_x m^*(v_t+,t)}{m^*(v_t+,t)} = \dfrac{\partial_x m^*(v_t-,t)}{m^*(v_t-,t)}\;, \\ p\{1-m^*(v_t+,t)\} = q\{1-m^*(v_t-,t)\}\;, \\ m^*(x,0) = \alpha_+ \mathbf{1}\{x\geq 0\} + \alpha_- \mathbf{1}\{x<0\}\;. \end{cases} \tag{6.3}$$

In other words, $m(x/\sqrt{t})$ is the macroscopic profile of density as seen from the tagged asymmetric particle.

The solution of (6.2) can be written as

$$m(x) = \begin{cases} A_+ + B_+ \displaystyle\int_0^x e^{-(1/2)y^2 - vy}\, dy & \text{for } x>0\;, \\ A_- + B_- \displaystyle\int_0^x e^{-(1/2)y^2 - vy}\, dy & \text{for } x<0\;, \end{cases}$$

where the parameters are related by the equations

$$p(1-A_+) \;=\; q(1-A_-)\;, \qquad -v \;=\; \frac{B_+}{A_+} \;=\; \frac{B_-}{A_-}\;, \qquad \alpha_{\pm} \;=\; A_{\pm} J(\pm v)$$

and

$$J(v) \;=\; 1 - v\int_0^{+\infty} e^{-(1/2)y^2 - vy}\, dy\;.$$

It follows from the previous identities that the parameters p, α_+, α_- and v satisfy the equation

$$p\left(1 - \frac{\alpha_+}{J(v)}\right) \;=\; q\left(1 - \frac{\alpha_-}{J(-v)}\right)\;. \tag{6.4}$$

This equation was obtained heuristically by [BDMO]. In particular, we cannot write v as an explicit function of p, α_+, α_-, but we can study some asymptotic relations. We consider three distinct asymptotics.

We first investigate the case of a constant initial profile : $\alpha_+ = \alpha_- = \alpha$. In this case we can write α as an explicit function of p and v :

$$\alpha \;=\; (p-q)\frac{J(v)J(-v)}{pJ(-v)-qJ(v)} \,\cdot$$

Elementary computations give the identity

$$(p-q)\frac{1-\alpha}{\alpha} = \frac{pJ(-v)-qJ(v)-(p-q)J(v)J(-v)}{J(v)J(-v)} \,. \tag{6.5}$$

For small asymmetry $p-q$, we have a small displacement v. Expanding $J(\cdot)$ around the origin we obtain that

$$J(v) \;=\; 1 \;-\; \sqrt{\frac{\pi}{2}}v + v^2 + O(v^3) \,.$$

Replacing in (6.5) $J(v)$ by its expansion gives, for fixed α and small $p-q$, that

$$v \;=\; (p-q)\sqrt{\frac{2}{\pi}}\frac{1-\alpha}{\alpha} \;+\; o(p-q) \,.$$

This proves the validity of Einstein relation for small drifts.

In the case $\alpha_+ \neq \alpha_-$ one can expand around the equilibrium, i.e., for small $p(1-\alpha_+)-q(1-\alpha_-)$. The same expansions show that

$$v \;=\; \frac{p(1-\alpha_+)-q(1-\alpha_-)}{p\alpha_+ - q\alpha_-}\sqrt{\frac{2}{\pi}} \;+\; o\left(p(1-\alpha_+)-q(1-\alpha_-)\right) \,.$$

A third possible asymptotics is given when the initial profile is constant and the density $\alpha = \alpha_+ = \alpha_-$ is small. In this case, for a fixed drift $p-q$, the displacement v is very large. Asymptotically, for $|v|$ close to ∞, a simple computation shows that

$$J(v) \sim \frac{1}{v^2} \,, \qquad J(-v) \sim ve^{v^2/2}\sqrt{2\pi} \,.$$

Using these expansions in (6.4) one obtains that

$$v \sim \sqrt{\frac{p-q}{\alpha_+}} \;+\; o\left(\frac{1}{\sqrt{\alpha_+}}\right) \,.$$

7. Surface motion in a Potts model.

We prove in this section Theorem 1.4. The spin flip dynamics induced by the generator $\mathcal{L}_{sp}$ on $\mathcal{M}$ coincides with the zero range dynamics investigated in section 3. We first define a correspondence between configurations $\mathfrak{f}$ such that $\mathfrak{f}(0) = 0$ and configurations of $\mathbb{N}^{\mathbb{N}}$. Fix a monotone function $\mathfrak{f}$ in $\mathcal{M}$ such that $\mathfrak{f}(0) = 0$. For $x \geq 0$, set

$$\eta(x) \;=\; \mathfrak{f}(x+1) - \mathfrak{f}(x) \,. \tag{7.1}$$

Recall that for $t \geq 0$, $\mathfrak{f}_t = \mathfrak{f}_{\sigma_t}$. It is easy to check that η_t, defined by $\eta_t(x) = \mathfrak{f}_t(x+1) - \mathfrak{f}_t(x)$, evolves according to the generator L introduced in (3.1) with jump rate $g(k) = \mathbf{1}\{k \geq 1\}$. (There is a slight difference. For the zero range process investigated in section 3, a particle at the origin is killed at rate $\mathbf{1}\{\eta(0) \geq 1\}$, while in the model we now consider it is killed at rate $(1/2)\mathbf{1}\{\eta(0) \geq 1\}$. This microscopic difference, however, does not modify the macroscopic behavior of the system. Details are left to the reader).

Fix a sequence of probability measures m^N satisfying the assumptions of Theorem 1.4. Denote by μ^N the sequence of $\mathfrak{A}m^N$ introduced in section 1. Clearly, by construction, the sequence μ^N satisfy assumptions (H1), (H2) of section 3. In particular, since it follows from (7.1) that

$$\mathfrak{f}_t(0) - \mathfrak{f}_0(0) = \sum_{x \geq 0} \{\eta_0(x) - \eta_t(x)\} ,$$

from Proposition 5.1 we obtain that for each $t \geq 0$, $N^{-1}\mathfrak{f}_{tN^2}(0)$ converges in probability to v_t defined by (1.5), where ρ is the solution of (1.3) with initial condition λ_0 instead of $\mathcal{F}_+$.

Fix now a smooth function $G: \mathbb{R}_+ \to \mathbb{R}$ with compact support. By definition of η_t,

$$N^{-1} \sum_{x \geq 0} G(x/N) N^{-1}[\mathfrak{f}_t(x) - \mathfrak{f}_t(0)] = N^{-1} \sum_{x \geq 0} \Big\{ N^{-1} \sum_{y \geq x+1} G(y/N) \Big\} \eta_t(x) .$$

It follows therefore from Theorem 3.1 that for each $t \geq 0$,

$$N^{-1} \sum_{x \geq 0} G(x/N) N^{-1}[\mathfrak{f}_{tN^2}(x) - \mathfrak{f}_{tN^2}(0)]$$

converges in probability, as $N \uparrow \infty$, to

$$\int_{\mathbb{R}_+} du\, G(u) \lambda(t,u) ,$$

where $\lambda(t,u) = \int_0^u \rho(t,v) dv$ and $\rho(t,u)$ is the solution of (3.2). To conclude the proof of Theorem 1.4, it remains to check that λ is the solution of (1.9) but this follows straightforwardly from the explicit form λ and the fact that $\rho(t,u)$ is the solution of (3.2).

8. Appendix : Uniqueness.

Case $p = 1$. This is an extension to infinite volume of an argument presented in [ELS2]. Fix a weak solution $\rho(t,u)$ of the differential equation (3.2). Since $\rho(t,\cdot)$ is in $L^1_{loc}(\mathbb{R}_+)$, we may define $R_t: \mathbb{R}_+^2 \to \mathbb{R}$ by

$$R_t(u,v) = \int_u^v \rho(t,w)\, dw . \tag{8.1}$$

Denote by $[\cdot,\cdot]$ the inner product in $L^2(\mathbb{R}_+^2)$. Fix a smooth function $H:\mathbb{R}_+^2 \to \mathbb{R}$ with compact support. Changing the order of summations we obtain that

$$[R_t, H] = \int_{\mathbb{R}_+} du\, \rho(t,w) h(w)\,, \tag{8.2}$$

where

$$h(w) = \int_0^w du \int_w^\infty dv\, H(u,v) - \int_w^\infty du \int_0^w dv\, H(u,v)\,.$$

Notice that h is a smooth function with compact support that vanishes at the origin. Moreover, its derivative is given by

$$h'(w) = \int_0^\infty du\, \{H(w,u) - H(u,w)\}\,.$$

Therefore, in virtue of (8.2), property (c) of weak solutions and a change of variables, for every smooth function H with compact support

$$[R_t, H] = [R_0, H] + \int_0^t ds \int_{\mathbb{R}_+} du \int_{\mathbb{R}_+} dv\, H(u,v)\big\{\partial_v \Phi(\rho(s,v)) - \partial_u \Phi(\rho(s,u))\big\}\,.$$

In particular, we have that

$$R_t(u,v) - R_0(u,v) = \int_0^t ds\, \{\partial_v \Phi(\rho(s,v)) - \partial_u \Phi(\rho(s,u))\} \tag{8.3}$$

for almost all (u,v) in $\mathbb{R}_+^2$.

Consider now two solutions ρ^1, ρ^2 of equation (3.2), denote by R_t^1, R_t^2 the respective functions associated to ρ^1, ρ^2, through (8.1) and set $W_t = R_t^1 - R_t^2$, $\bar\rho_t = \rho_t^1 - \rho_t^2$. Denote by $[\cdot,\cdot]_e$ the inner product on $L^2(\mathbb{R}_+^2)$ associated to the measure $e^{-(u+v)}dudv$. In view of property (a) of weak solutions and identity (8.3), $R:[0,T] \to L^2(\mathbb{R}_+^2, e^{-(u+v)}dudv)$ is almost everywhere differentiable. Therefore,

$$\frac{d}{dt}[W_t, W_t]_e = 2\int du \int dv\, e^{-(u+v)} W_t(u,v)\big\{\partial_v \bar\Phi_t(v) - \partial_u \bar\Phi_t(u)\big\}\,,$$

where $\bar\Phi_t(v)$ stands for $\Phi(\rho^1(t,v)) - \Phi(\rho^2(t,v))$. An integration by parts gives that the right hand side is equal to

$$-2\int_{\mathbb{R}_+} e^{-u} \bar\Phi_t(u) \bar\rho(t,u) + 2\int du \int dv\, e^{-(u+v)} W_t(u,v) \bar\Phi_t(v) \tag{8.4}$$

because $\int du \exp\{-u\} = 1$. By Schwarz inequality, the second term is bounded above by

$$\begin{aligned} &\|\Phi'\|_\infty [W_t, W_t]_e + \frac{1}{\|\Phi'\|_\infty} \int_{\mathbb{R}_+} du\, e^{-u} \big(\bar\Phi_t(u)\big)^2 \\ &\quad \le \|\Phi'\|_\infty [W_t, W_t]_e + \int_{\mathbb{R}_+} du\, e^{-u} \bar\Phi_t(u) \bar\rho(t,u) \end{aligned}$$

because Φ is an increasing function with a bounded first derivative. Adding this expression to the first term of (8.4), we obtain that the time derivative of $[W_t, W_t]_e$ is bounded above by

$$\|\Phi'\|_\infty [W_t, W_t]_e \ - \ \int_{\mathbb{R}_+} du\, e^{-u} \bar{\Phi}_t(u) \bar{\rho}(t,u) \ \leq \ \|\Phi'\|_\infty [W_t, W_t]_e$$

because Φ is non decreasing. By Gronwall inequality, we deduce that $[W_t, W_t]_e$ is bounded above by $[W_0, W_0]_e \exp\{\|\Phi'\|_\infty t\}$, what concludes the proof of the uniqueness of weak solutions of equation (3.2).

The case $p < 1$. The argument is similar to the one presented for $p = 1$. For $t \geq 0$, define $R_t : \mathbb{R}^2 \to \mathbb{R}_+$ as in (8.1). It can be shown that

$$R_t(u,v) - R_0(u,v) \ = \ \int_0^t ds \left\{ \partial_v \Phi(\rho(s,v)) - \partial_u \Phi(\rho(s,u)) \right\}$$

for almost all (u, v) in $\mathbb{R}^2$. Consider two solutions of equation (4.3). Denote by $m(du) = m(u)du$ the absolutely continuous measure with density $m(u) = p\mathbf{1}\{u < 0\} + q\mathbf{1}\{u > 0\}$ and fix a smooth function $\theta: \mathbb{R} \to \mathbb{R}_+$ such that $\theta(0) = 0$, $\theta(u) = |u|$ for u large enough and $\int m(du) \exp\{-\theta(u)\} = 1$. Let $[\cdot,\cdot]_m$ stand for the inner product in $L^2(\mathbb{R}^2)$ with respect to the measure $m(du)m(dv)\exp\{-\theta(u) - \theta(v)\}$. Fix two solutions ρ^1, ρ^2 of equation (4.3), denote by R_t^1, R_t^2 the respective functions associated to ρ^1, ρ^2, through (8.1) and set $W_t = R_t^1 - R_t^2$. With the same arguments presented above one can show that $[W_t, W_t]_m$ is bounded above by $[W_0, W_0]_m \exp\{C(\theta, \|\Phi'\|_\infty)t\}$. In this deduction the use of the measure $m(du)$ instead of the Lebesgue measure is fundamental in the integration by parts performed in (8.4) for the boundary term to cancel.

Acknowledgments. The authors would like to thank H. Rost and H. Spohn for fruitful discussions on the Einstein relations and the Potts model. We also thank G. S. Oshanin for pointing out a computational mistake in section 6 of a previous version of this paper. C. Landim and S. Volchan were partially supported by CNPq grant 300358/93-8 and PRONEX 41.96.0923.00 "Fenomenos Criticos em Probabilidade e Processos Estocasticos".

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