Resenhas IME-USP 1997, Vol. 3, No. 2, 159 - 171.

Martingale Approach for Markov Processes in Random Environment and Branching Markov Chains [1]

M.V. Menshikov

Abstract: In this paper we study random walks with branching (BRW), and two examples of countable Markov chains in random environment which are a one dimensional random walk and a random string. We introduce the notion of recurrence and transience for BRW and provide criteria for them. For many dimensional BRW we find the critical (for transience vs. recurrence) speed of the decay of the average number of the off-springs at a point with respect to the distance from it to the origin. We show also that the Lyapunov function method is powerful for investigating a random walk in a random environment.

Key words: random walk, random string, branching random walk, Lyapunov function, random environment.

Introduction

The purpose of this paper is to demonstrate a martingale approach for a qualitative investigation of Markov processes in a random environment and branching Markov chains. In a more detailed way these problems were considered by the author together with Francis Comets, Serguei Popov and Stanislav Volkov in [4, 16]. Today the theory of countable Markov chains is developing in several directions. One of these directions is a qualitative analysis of Markov chains in a countable state space with a complicated structure. We can mention several such models. Multidimensional Markov chains with partial linear non homogeneities are considered in [1, 7]. Markov chains with the state space being equal to all finite sequences of letters from some alphabet (the so-called *strings*) are studied in [8, 9]. The subject of the papers [10, 11, 16] is the so-called *branching random walk*, which is also a Markov chain in some complicated state space. Note, that for these problems a martingale method (method of Lyapunov functions) was developed.

Another important branch of this theory is the theory of countable Markov chains in a random environment. In this paper we study various Markov chains in a random environment using Lyapunov functions. Let us briefly describe what we mean by the "random environment". A given countable, time-homogeneous Markov chain $\mathcal{L} = \{\eta_n; n \geq 0\}$ could be defined by its state space $X = \{x_i\}$ and by the collection of transition probabilities $P_{ij} = \mathsf{P}\{\eta_{n+1} = x_j \mid \eta_n = x_i\}$. Assume that on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$ we are given a collection of random variables $P_{ij}(\omega)$, $i, j \in \mathbf{N}$, $\omega \in \Omega$. Suppose also that for any fixed ω (which we call the *realization* of the random environment) the numbers $P_{ij}(\omega)$ are transition

[1] AMS 1991 subject classification: Primary 60J15, 60J80

probabilities of some countable Markov chain. Together with the state space X, these transition probabilities define a Markov chain $\mathcal{L}(\omega)$. For such model with quenched disorder there arise some natural questions: what is the probability (with respect to probability measure $\mathbf{P}$) for the Markov chain $\mathcal{L}(\omega)$ of being recurrent (transient, ergodic, or some other property)? However, in this general formulation this problem can hardly be solved.

Here we study several classes of such models. In the papers [1, 7, 8, 16] it was shown that the method of Lyapunov functions is very useful (and sometimes unique) for investigating countable Markov chains and branching random walks. This method, sometimes called martingale method, amounts to study the chain through some one-dimensional projections cleverly selected. In the present paper we apply this method to three models: random walk in a random environment, random strings in a random environment, and a branching random walk. The main idea of our approach for random environment is the following: for a given $\omega \in \Omega$ consider the Markov chain $\mathcal{L}(\omega)$. For this Markov chain we construct a Lyapunov function $f(x) = f(x;\omega)$. It turns out that for the models under consideration this function is a *spatially homogeneous* random field, so we can investigate its asymptotical behavior. Constructing a Lyapunov function is not an easy matter. It is much easier and standard to analyze its asymptotic behavior, when the first step can be performed. Then, knowing the properties of the Lyapunov function for fixed ω and using criteria for countable Markov chains, we thus obtain the qualitative classification of the Markov chain $\mathcal{L}(\omega)$.

It is very important to note that the models under consideration have the following common property:

$$\mathbf{P}\{\omega : \mathcal{L}(\omega) \text{ is recurrent}\} = 0 \text{ or } 1.$$

The same is true for the ergodicity, transience, and some other characteristics of Markov chains.

1 General criteria for countable Markov chains

In this section we formulate martingale criteria for countable irreducible Markov chains. They are given in a rather simplified form which is nevertheless sufficient for our purpose. More general criteria can be found in [7] and in [1].

Proposition 1.1 *The Markov chain η_t is transient if and only if there exist a set M and a positive function $f : X \mapsto \mathbf{R}$ such that*

$$\mathsf{E}\left[f(\eta_{t+1}) - f(\eta_t) \mid \eta_t = x\right] \le 0, \qquad \textit{for all } x \notin M;$$

$$f(x_1) < \inf_{x \in M} f(x), \quad \textit{at least for one } x_1 \notin M.$$

Proposition 1.2 *Let η_t be a discrete time-homogeneous Markov chain with a countable state space X, and a point $0 \in X$. Suppose that there exists a function $f : X \mapsto \mathbf{R}$ with the following properties:*

- $f(x) \neq \text{const}$;
- *there exists some* M_0 *such that* $|f(x)| < M_0$ *for all* $x \in X$;
- *for all* $x \neq 0$
$$\mathsf{E}(f(\eta_{t+1}) - f(\eta_t) \mid \eta_t = x) = 0. \tag{1}$$

Then Markov chain η_t *is transient (non-recurrent).*

This is a consequence of Theorem 2.2.2 of [7]. The details are omitted. The next propositions are just simplified forms of Theorems 2.2.1 and 2.2.3 of [7] correspondingly.

Proposition 1.3 *Let* η_t *be a discrete time-homogeneous Markov chain with a countable state space* X, $0 \in X$. *Suppose that there exists a function* $f : X \mapsto \mathbf{R}$ *such that* $f(x) \to +\infty$ *as* $x \to \infty$ *and for all* $x \neq 0$

$$\mathsf{E}(f(\eta_{t+1}) - f(\eta_t) \mid \eta_t = x) \leq 0. \tag{2}$$

Then Markov chain η_t *is recurrent.*

Proposition 1.4 (Foster) *Let* η_t *be a discrete time-homogeneous Markov chain with a countable state space* X, $0 \in X$. *Suppose that there exists a function* $f : X \mapsto \mathbf{R}$, $f(x) \geq 0$, *and* $\delta > 0$ *such that for all* $x \neq 0$

$$\mathsf{E}(f(\eta_{t+1}) - f(\eta_t) \mid \eta_t = x) \leq -\delta; \tag{3}$$

and $\mathsf{E}(f(\eta_{t+1}) \mid \eta_t = 0) < \infty$. *Then Markov chain* η_t *is ergodic (positive recurrent).*

2 Branching Markov chains

Let us introduce the notion of a branching random walk (BRW). Suppose that for every x there exists a sequence of non-negative numbers $g_0(x), g_1(x), \ldots$ such that

$$\sum_{k=0}^{\infty} g_k(x) = 1$$

and the value

$$\bar{k}(x) = \sum_{k=0}^{\infty} k g_k(x)$$

is finite. Hence, $g(x)$ is for every x the distribution function of a random variable $k(x)$, which has a finite expectation.

The state of the process $\omega(t)$ at time $t \in \{0, 1, 2, \ldots\}$ is a set of coordinates $\{x_1(t), x_2(t), \ldots, x_{n(t)}(t)\}$. In such a situation we will also say that there exist $n(t)$

particles having coordinates $x_i(t)$ (which need not be different). The process runs as follows. From the particle situated at point x with probability $g_k(x)$ k new particles arise and, if $k \neq 0$, each of these jumps to some point y with probability p_{xy}, independently of the other particles and the previous history of the process. If $k = 0$, we will say that the particle did not produce descendants. Note that two or more particles having the same coordinates do not influence each other. This procedure is conducted simultaneously for all $n(t)$ particles and the coordinates of the new-born ones form the state of BRW at time $t+1$. The obtained process is a Markov chain with countable state space

$$\mathcal{X} = \{\oslash\} \cup X \cup X^2 \cup X^3 \cup \ldots,$$

where $\oslash$ denotes the state where the process has died out (at some point in time none of the particles produced a descendant). The obtained processes are rather complicated and we will try to classify them.

Definition 2.1 *The branching random walk is called recurrent, if for all $x \in X$ and the starting position $\omega(0) = \{x\}$ (i.e. exactly one particle comes out from x) at least one of the descendants reaches the point α_0 with unit probability*

$$\mathbf{P}\{\exists t \geq 0,\ \exists i \in \{1, 2, \ldots, n(t)\} \ :\ x_i(t) = \alpha_0\} = 1.$$

Otherwise the process is called transient.

Note that the above definition does not depend on the selection of starting position. Moreover, it is obvious that there are two reasons for the process to be transient: first, all the particles may "run away to infinity"; second, the process may die out before any particle reaches α_0.

Now let us introduce a modified version of our BRW. Suppose that the point α_0 is an *absorbinis an* absorbing point, i.e. when a particle hits this point, it stops producing descendants and stays in this point forever. We denote the obtained process BRW$*$ and note that it is similar to the initial process except that

$$\begin{aligned} p_{\alpha_0 \alpha_0} &= 1; \\ p_{\alpha_0 x} &= 0, \quad x \neq \alpha_0; \\ g_1(\alpha_0) &= 1; \\ g_k(\alpha_0) &= 0, \quad k \neq 1. \end{aligned}$$

Definition 2.2 *The branching random walk is called finite on average, if in the modified process for any starting position $\omega(0) = \{x\}$ the expectation of the number of particles hitting α_0 is finite; it is called infinite on average otherwise.*

The introduced notion is not stable, in the sense that changing the probabilities even in a finite number of points may transform a finite on average process into an infinite on average one and vice versa.

Besides, none of the notions above is "stronger" than the other since there exist pairs $(P, g(x))$ such that the process is, for example, recurrent but infinite on average. All four combinations are possible.

3 Criteria

First, we present the criterion for transience. Obviously, if some $g_0(x) > 0$ then BRW is transient by definition (when the particle starts at x, it may produce no descendant with positive probability). So, as far as transience is concerned, we will assume that the process does not die out for sure, which is equivalent to saying that

$$g_0(x) = 0 \quad \text{for all } x. \tag{4}$$

Consider the moment-generating function for $k(x)$

$$\psi(x,z) = \sum_{k=0}^{\infty} z^k g_k(x), \tag{5}$$

and note that

$$\frac{d}{dz}\psi(x,1) = \bar{k}(x). \tag{6}$$

Moreover, let E_1 be the probability operator, such that for any function $f(x)$

$$E_1 f(x) = \sum_{y \in L} p_{xy} f(y).$$

It is important that this linear operator preserves constant functions and also that the image of a function that is less than 1 everywhere, after applying E_1 has the same property.

Theorem 3.1 *Let* (4) *be satisfied. The branching random walk is transient if and only if there exists a finite set M and a non-negative function $f(x) \le 1$, such that*

$$\begin{aligned} \psi\left(x, E_1 f(x)\right) &\ge f(x), \quad \textit{for all } x \notin M; & (7)\\ f(x_1) &> \max_{x \in M} f(x), \quad \textit{for at least one } x_1 \notin M. & (8) \end{aligned}$$

(Expression (7) *is well-defined since $E_1 f(x) \le 1$ and the derivative given by* (5) *exists when $z \le 1$).*

Let us also formulate a corollary that is much easier to check.

Corollary 3.1 *If* (4) *holds and there exist a set M and a non-negative bounded function $h(x)$, such that*

$$E_1 h(x) \le \frac{h(x)}{\bar{k}(x)}, \quad \textit{for all } x \notin M \tag{9}$$

and

$$h(x_1) < \min_{x \in M} h(x), \quad \textit{for at least one } x_1 \notin M,$$

then BRW is transient.

Below we present a criterion for finiteness on average. It looks similar to the one given in [10].

Theorem 3.2 *The branching random walk is finite on average if and only if there exists a non-negative function $f(x)$ such that $f(\alpha_0) > 0$ and, when $x \neq \alpha_0$,*

$$E_1 f(x) \leq \frac{f(x)}{\bar{k}(x)}, \tag{10}$$

where $\bar{k}(x)$ is the average number of direct descendants of a particle at x.

4 Branching random walks in $\mathbf{Z}^d$

Let us consider a transient simple random walk without branching ξ_t on $\mathbf{Z}^d$, $d \geq 3$ (with equiprobable jumps to adjacent points within Euclidean distance (denoted by $\|\cdot\|$) of 1).

Next theorem from [16] shows that if the branching is intensive enough then BRW is recurrent and if not then it is transient

Theorem 4.1 *Let BRW have the same transition probabilities as the simple random walk in $\mathbf{Z}^d$, $d \geq 3$. Let condition (4) be fulfilled, and let the average number of off-springs at point $a \in \mathbf{Z}^d$ be $\bar{k}(a) = 1 + \epsilon(a)$ and $Var(k(a)) \leq A_1 D/\|a\|^2$. Then there exist constants $D > m > 0$, such that if*

$$\epsilon(a) > \frac{D}{\|a\|^2},$$

then BRW is recurrent and if

$$\epsilon(a) < \frac{m}{\|a\|^2},$$

then it is transient.

5 Random walk on $\mathbf{Z}_+$ in a random environment

5.1 Description of the process

Here we describe the model which was first considered by M.V. Kozlov [14] and F. Solomon [19].

Let $\{\xi_i\}_{i=1,2,\ldots}$ be a sequence of i.i.d. random variables on Ω with values in $[0, 1]$. Suppose also that

$$\mathbf{P}\{\xi_1 = 0\} = \mathbf{P}\{\xi_1 = 1\} = 0.$$

Then, for fixed environment, i.e. for given realization of the sequence of i.i.d. random variables $\{\xi_i\}$, consider a Markov chain η_t on $\mathbf{Z}_+$ defined as follows: $\eta_0 = 0$,

$$
\begin{aligned}
p_n &= \mathsf{P}\{\eta_{t+1} = n-1 \mid \eta_t = n\} = \xi_n, \\
q_n &= \mathsf{P}\{\eta_{t+1} = n+1 \mid \eta_t = n\} = 1 - \xi_n, \ n = 1, 2 \ldots,
\end{aligned}
$$

and $\mathsf{P}\{\eta_{t+1} = 1 \mid \eta_t = 0\} = 1$. In fact, η_t can be described as a birth and death process with birth, respectively death parameters p_n and $q_n = 1 - p_n$.

This model has been extensively investigated and many of its properties are known, see, for example, [12], or [13] for possible generalizations. In this part of the paper our goal is not to obtain some new results, but to illustrate how Lyapunov functions method works and to show that results may be obtained in a very short way using this method. All through the paper we will use $\mathbf{P}, \mathbf{E}$ to denote probability and expectation for the random environment ω, keeping the notations P, E for the Markov chain η_t itself.

5.2 Recurrence, transience, and asymptotical behavior of the process

Let η_t be the random walk in a random environment introduced in Section 5.1. Denote $\zeta_n = \log(p_n/q_n)$. The next theorem is due to Solomon [19]. But we prove this theorem using the method of Lyapunov functions.

Theorem 5.1 *Assume* $\mathbf{E}|\zeta_1| < \infty$.

- *If* $\mathbf{E}\zeta_1 < 0$, *then the random walk is transient for almost all* ω *(for almost all environments).*
- *If* $\mathbf{E}\zeta_1 \geq 0$, *then the random walk is recurrent for almost all* ω.
- *Moreover, if* $\mathbf{E}\zeta_1 > 0$, *then the random walk is ergodic.*

Sketch of proof. First, we prove the recurrence and the transience. Let us try to construct a function $f(x)$ satisfying (1) for fixed ω. Denote $\Delta_i = f(i) - f(i-1)$ and let $\Delta_0 = 1$. Then we have

$$\mathsf{E}(f(\eta_{t+1}) - f(\eta_t) \mid \eta_t = x) = -p_x\Delta_x + q_x\Delta_{x+1}, \tag{11}$$

so

$$\Delta_{x+1} = \frac{p_x}{q_x}\Delta_x, \tag{12}$$

and, consequently,

$$\Delta_{x+1} = \exp\Big\{\sum_{i=1}^{x}\zeta_i\Big\}.$$

We set $f(0) = 0$. Then, since

$$f(x) = \sum_{i=1}^{x} \Delta_i,$$

it is clear that the function $f(x)$ is positive and that if $\mathbf{E}\zeta_1 \geq 0$, then for almost all ω we have $f(x) \to +\infty$, and if $\mathbf{E}\zeta_1 < 0$, then $f(x)$ is bounded. Applying Propositions 1.2 and 1.3 we complete the proof of first two statements of Theorem 5.1.

To prove ergodicity, we construct the Lyapunov function $f(s)$ in the reverse way. Let $f(x) = \sum_{i=1}^{x} \Delta_i$, where

$$\Delta_i = \frac{1}{p_i} + \frac{q_i}{p_i p_{i+1}} + \frac{q_i q_{i+1}}{p_i p_{i+1} p_{i+2}} + \frac{q_i q_{i+1} q_{i+2}}{p_i p_{i+1} p_{i+2} p_{i+3}} + \cdots \tag{13}$$

Then one can check that if $\mathbf{E}\zeta_1 > 0$, then

i) the series in the right-hand side of (13) converge, so the function $f(x)$ is well defined;

ii) $f(x)$ satisfies (3) with $\delta = 1$.

Remark. In fact, if $\mathbf{E}\zeta_1 = 0$, then it can be easily proved that the random walk is null recurrent (for almost all environments). This proof is based on writing down explicitly the stationary measure for this Markov chain.

Now we discuss the case $\mathbf{E}\zeta_1 = 0$ in a more detailed way. We are interested in the asymptotic behavior of the process as $t \to \infty$. The result of next theorem from [4] is slightly weaker than the result of [18], but its proof is simple and brief.

Theorem 5.2 *Let $\mathbf{E}\zeta_1 = 0$, and $0 < \mathbf{E}\zeta_1^2 < \infty$. Then for any integer $k \geq 1$ and for any $\varepsilon > 0$ we have*

$$\eta_t / (\log t \log_2 t \ldots \log_k^{1+\varepsilon} t)^2 \longrightarrow 0 \tag{14}$$

almost surely as $t \to \infty$, with $\log_1 t := \log t$, $\log_{m+1} t = \log(\log_m t)$, $m \geq 1$. Also, for any $\varepsilon > 0$ and for any $p > 0$ we have

$$\frac{\eta_t}{\log^{2+\varepsilon} t} \longrightarrow 0 \tag{15}$$

in L^p, as $t \to \infty$.

In the above statements, "almost surely" means "for $\mathbf{P}$-almost every environment it holds P-a.s." and "convergence in L^p" stands for "convergence in $L^p(\mathsf{P})$ for $\mathbf{P}$-a.e. ω". P. Deheuvels and P. Revész proved (14) in [6], Theorem 4, with a completely different approach, but martingale method is much shorter.

6 Strings in a random environment

6.1 Dynamics of the string

Consider a finite alphabet $\mathcal{S} = \{1, \dots, k\}$. A string is just a finite sequence of symbols from $\mathcal{S}$. We write $|s|$ for the length of the string s, i.e. if $s = s_1 \dots s_n$, $s_i \in \mathcal{S}$, then $|s| = n$.

Consider a time-homogeneous Markov chain with the state space equal to the set of all finite strings. We describe the transition matrix of this Markov chain as follows: let $s = s_1 \dots s_n$, and $s_n = i \in \mathcal{S}$. Then

- we erase the rightmost symbol of s with probability $r_i^{(n)}$;
- we substitute the rightmost symbol i by j with probability $q_{ij}^{(n)}$;
- and we add the symbol j to the right end of the string with probability $p_{ij}^{(n)}$.

Of course we assume that for all i and for all $n = 1, 2, \dots$

$$r_i^{(n)} + \sum_j q_{ij}^{(n)} + \sum_j p_{ij}^{(n)} = 1. \tag{16}$$

These parameters do not define the evolution when the string is empty (its length equals 0), but we simply assume that the jumps from the empty string are somehow defined and can only occur to strings of length 1. Clearly, these "boundary conditions" do not affect the asymptotic behavior of the string. So we see that the process is completely defined by the collection of numbers $\{r_i^{(n)}, q_{ij}^{(n)}, p_{ij}^{(n)}\}$, $n = 1, 2, \dots$, $i = 1, \dots, k$, and $j = 1, \dots, k$.

For the case when the quantities $r_i^{(n)}$, $q_{ij}^{(n)}$ and $p_{ij}^{(n)}$ do not depend on n, this model was investigated by A.S. Gajrat, V.A. Malyshev, M.V. Menshikov and K.D. Pelih in [8]. In fact, they investigated even more general model, when the maximal value of jump may be greater than one. All these are models for LIFO (last in, first out) queuing systems [8], and they are random walks on trees. In the case $k = 1$ we recover the random walk from Section 5.

We consider here only *one-sided* evolution of the string. Two-sided *homogeneous* strings were studied by A.S. Gåjrat, V.A. Malyshev and A.A. Zamyatin in [9].

Let us now describe the random environment. We will view the vectors $x \in \mathbf{R}^{k+2k^2}$ as $x = (x^R, x^Q, x^P)$ with $x^R = (x_1^R, \dots, x_k^R)$, $x^Q = (x_{ij}^Q)_{1 \le i,j \le k}$, $x^P = (x_{ij}^P)_{1 \le i,j \le k}$. For $x = (x^R, x^Q, x^P)$ we will write $x \ge 0$, if all its components are non-negative. Consider a $2k^2$-dimensional manifold in $\mathbf{R}^{k+2k^2}$:

$$M = \left\{ x \in \mathbf{R}^{k+2k^2} : x \ge 0,\ x_i^R + \sum_{j=1}^{k} (x_{ij}^Q + x_{ij}^P) = 1 \text{ for all } i = 1, \dots, k \right\}.$$

and $\widetilde{Q}$ some probability measure on M. Let ξ_n, $n = 1, 2, \ldots$, be a sequence of i.i.d., $\mathbf{R}^{k+2k^2}$-valued random variables with distribution $\widetilde{Q}$ (this means that for $K \subset M$ we have $\mathbf{P}\{\xi_n \in K\} = \widetilde{Q}(K)$). Define the quantities $r_i^{(n)}$, $q_{ij}^{(n)}$ and $p_{ij}^{(n)}$ via the sequence of values of ξ_n, as follows: with $x = \xi_n$ we set

$$r_i^{(n)} = x_i^R,\ q_{ij}^{(n)} = x_{ij}^Q,\ p_{ij}^{(n)} = x_{ij}^P$$

Since $\xi_n = x \in M$, the condition (16) holds. Thus, for each fixed $\omega = \{\xi_1, \xi_2, \ldots\}$ (for fixed environment) we have defined the Markov chain describing the evolution of the string. We are interested in the properties of this process which hold for almost all ω, in particular, in transience or ergodicity of the process.

6.2 Lyapunov exponents and products of random matrices

In this section we review some properties of products of random matrices necessary to deal with the strings in a random environment.

Let $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbf{R}^k$. For $x \in \mathbf{R}^k$ define its norm by $\|x\| = \langle x, x \rangle^{1/2}$. The transpose of matrix A is denoted by A^*. Euclidean (operator) norm of a $k \times k$ real matrix A can be defined by any of the following formulas (it can be easily verified that they are equivalent):

$$\begin{aligned} \|A\| &:= \sup_{\|x\|=1} \|Ax\| \\ &= \sup_{\|x\|=\|y\|=1} \langle Ax, y \rangle \\ &= [\text{largest eigenvalue of } A^*A]^{1/2}. \end{aligned}$$

Consider a sequence of i.i.d. random matrices A_n. We assume that the matrices A_n satisfy the following condition:

Condition A. $\mathbf{E} \log^+ \|A\| < \infty$, where $\log^+ x = \max\{\log x, 0\}$.

Let $(A_1(\omega), A_2(\omega), \ldots)$ be a realization of a sequence of i.i.d. random matrices. Let $a_1(n) \geq a_2(n) \geq \cdots \geq a_k(n) \geq 0$ be the square roots of the (random) eigenvalues of $(A_n \ldots A_1)^*(A_n \ldots A_1)$. Then the following limit exists for almost all ω (and it is the same for almost all ω):

$$\lim_{n\to\infty} \frac{1}{n} \log a_j(n) = \gamma_j(A) \tag{17}$$

for $j = 1, \ldots, k$ (see Proposition 5.6 of [3], A does not need to be invertible). The numbers $\gamma_j, -\infty \leq \gamma_k \leq \cdots \leq \gamma_1 < \infty$ are called the *Lyapunov exponents* of the sequence of random matrices $\{A_n\}$. In particular,

$$\gamma_1(A) = \lim_{n\to\infty} \frac{1}{n} \log a_1(n) = \lim_{n\to\infty} \frac{1}{n} \|A_n \ldots A_1\| \qquad \text{a.s.} \tag{18}$$

is the *top Lyapunov exponent* .

The following simple lemma establishes relations between Lyapunov exponents of A and A^{-1}.

Lemma 6.1 *Assume that A is a.s. invertible, and that both A and A^{-1} satisfy Condition A. Then for $j = 1, \ldots, k$*

$$\gamma_j(A^{-1}) = -\gamma_{k-j+1}(A). \tag{19}$$

We will also need the following theorem

Theorem 6.1 (Oseledec's Multiplicative Ergodic Theorem) *Let A_n, $n = 1, 2, \ldots$ be a stationary ergodic sequence of $k \times k$ real matrices on the probability space $(\Omega, \mathcal{B}, m)$ and suppose that $\mathsf{E} \log^+ \|A_1\| < \infty$. Let $\gamma_1 \geq \gamma_2 \geq \cdots \gamma_k$ be the Lyapunov exponents of the sequence A_n. Take a strictly increasing nonrandom sequence of integers*

$$1 = i_1 < i_2 < \cdots < i_p < i_{p+1} = k + 1$$

such that $\gamma_{i_{q-1}} > \gamma_{i_q}$, if $q = 2, 3, \ldots, p$, and $\gamma_i = \gamma_j$, if $i_q \leq i$, $j < i_{q+1}$ (the i_q's mark the points of decrease of the γ_i). Then for almost all $\omega \in \Omega$:

- *for every $v \in \mathbf{R}^k$*

  $$\lim_{n\to\infty} \frac{1}{n} \log \|A_n \ldots A_1 v\|$$

  exists or is $-\infty$;

- *for $q \leq p$*

  $$V(q, \omega) = \left\{ v \in \mathbf{R}^k : \lim_{n\to\infty} \frac{1}{n} \log \|A_n \ldots A_1 v\| \leq \gamma_{i_q} \right\}$$

  is a random vector subspace of $\mathbf{R}^k$ with dimension $k - i_q$;

- *if $V(0, \omega)$ denotes $\mathbf{R}^k$, then $v \in V(q-1, \omega) \ominus V(q, \omega)$ implies that*

  $$\lim_{n\to\infty} \frac{1}{n} \log \|A_n \ldots A_1 v\| = \gamma_q.$$

6.3 Main results

Introduce two sequences of random $k \times k$ matrices $\{B_n\}$ and $\{D_n\}$, $B_n = (p_{ij}^{(n)})_{i,j=1,\ldots,k}$ and $D_n = (d_{ij}^{(n)})_{i,j=1,\ldots,k}$, where $d_{ij}^{(n)} = -q_{ij}^{(n)}$ for $i \neq j$ and

$$d_{ii}^{(n)} = r_i^{(n)} + \sum_{j:j\neq i} q_{ij}^{(n)},$$

with $p_{ij}^{(n)}$, $q_{ij}^{(n)}$ and $r_i^{(n)}$ defined in Section 6.1. Important ingredients are the sequence of i.i.d. random matrices A_n,

$$A_n = D_n^{-1} B_n$$

and its Lyapunov exponents $\gamma_1, \ldots, \gamma_k$. Besides Condition A we will consider

Condition D. $\mathbf{E} \log(1/r_i^1) < \infty, i = 1, \ldots, k$

The interest of this condition appears in the

Proposition 6.1 *If Condition D holds then the matrix D is a.s. invertible, the Condition A holds and* $\mathbf{E}\log^{+}\|D^{-1}\| < \infty$

Now we are ready to formulate the main results. In the next one we use two different sets of assumptions, each one being of interest for applications.

Theorem 6.2 *Assume that Condition A holds for the matrix* A_n, *and that*

- *either* B_n *is a.s. invertible and Condition A holds for* A_n^{-1}
- *or* $\mathbf{E}\log(1/p_{ii}) < \infty, i = 1, \ldots, k$

If $\gamma_1 > 0$, *then the Markov chain, describing the evolution of the string, is transient (for almost all* ω*).*

Theorem 6.3 *Assume Condition D for* A_n. *If* $\gamma_1 < 0$, *then the process is a.s. ergodic.*

Theorem 6.4 *Let* $\gamma_1 = 0$. *In addition to Condition D assume that* A_1 *is a.s. invertible, and that no finite union of proper subspaces of* $\mathbf{R}^k$ *is a.s stable by* A_1. *Then the process is a.s. recurrent.*

References

[1] S. Aspandiiarov, R. Iasnogorodski and M.V. Menshikov (1996) Passage-time moments for non-negative stochastic processes and an application to reflected random walk in a quadrant. *Ann. Prob.* **24**,2,932–960.

[2] P. Bougerol (1986) Oscillation de produits de matrices aléatoires dont l'exposant de Lyapunov est nul. In: *Lyapunov exponents* (L. Arnold and V. Wihstutz, ed.) *Lect. Notes Math.* **1186**, 27–36, Springer-Verlag, Berlin.

[3] P. Bougerol and J. Lacroix (1985) *Products of Random Matrices with Application to Schrödinger Operators.* Birkhäuser Boston.

[4] F. Comets, M.V. Menshikov, S. Yu. Popov (1997) Markov Processes in Random Environment: Lyapunov Function Approach. *Preprint of University Paris-7 of March 4, 1997.*

[5] E. Csáki (1978) On the lower limits of maxima and minima of Wiener process and partial sums. *Z. Wahrsch. verw. Geb.* 43, 205–221.

[6] P. Deheuvels and P. Revész (1986) Simple random walk on the line in random environment. *Probab. Th. Relat. Fields* 72, 215–230.

[7] G. Fayolle, V.A. Malyshev and M.V. Menshikov (1995) *Topics in the Constructive Theory of Countable Markov Chains.* Cambridge University Press.

[8] A.S. Gajrat, V.A. Malyshev, M.V. Menshikov and K.D. Pelih (1995) Classification of Markov chains describing the evolution of a string of characters. *Uspehi Mat. Nauk* **50** No. 2 (302), 5–24 (in Russian).

[9] A.S. Gajrat, V.A. Malyshev and A.A. Zamyatin (1995) Two-sided evolution of a random chain. *Markov Processes Relat. Fields* **1**, 281–316.

[10] F.I. Karpelevich, M.Ya. Kelbert and Yu.M. Suhov (1994) The boundedness of branching Markov processes. In: *The Dynkin Festschrift. Markov Processes and their Applications* (M. Freidlin, ed.). *Progress in Probability* **34**, 143–153. Boston, Birkhauser.

[11] F.I. Karpelevich and Y.M. Suhov (1996) A criterion of boundedness of discrete branching random walk. *Classical and Modern Branching Processes* (eds. K. Atreya and P. Jagers). IMA volumes in mathematics and its applications **84**, 141–156. N.Y., Springer-Verlag.

[12] H. Kesten, M.V. Kozlov and F. Spitzer (1975) A limit law for random walk in a random environment. *Compositio Mathematica* **30**, Fasc. 2, 145–168.

[13] E.S. Key (1984) Recurrence and transience criteria for random walk in a random environment. *Ann. Prob.* **12**, No. 2, 529–560.

[14] M.V. Kozlov (1973) A random walk on the line with stochastic structure. *Probab. Theory Appl.* **18**, 406–408.

[15] F. Ledrappier (1984) Quelques propriétés des exposants caractéristiques. *Lect. Notes Math.* **1097**, Springer, Berlin.

[16] M.V. Menshikov and S.E. Volkov (1997) Branching Markov Chains: Qualitative Characteristics. *Markov Processes and Related Fields.* **3**, No. 2, ???-???.

[17] I.V. Oseledec (1968) A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.* **19**, 197–231.

[18] Ya.G. Sinaï (1982) Limit behaviour of one-dimensional random walks in random environment. *Probab. Theory Appl.* **27**, 247–258.

[19] F. Solomon (1972) Random walks in a random environment. Ph.D. Thesis, Cornell University. See also *Ann. Prob.* **3** (1975), 1–31.

M.V. Menshikov
Laboratory LRS, Department of Probability
Faculty of Mechanics and Mathematics
State University, Moscow, 119899
menshikov@llrs.math.msu.su
Russia