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Synchronization, stability and normal hyperbolicity

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Abstract: Synchronization is studied in the framework of invariant manifold theory. Normal hyperbolicity and its persistence are applied to give general results on synchronization and its stability. Simple numerics illustrate the importance of the stability issue.

1 Introduction

Synchronization phenomena of oscillators and coupled oscillators have been studied by physicists, engineers and mathematicians [AVR][ACH][AR][H1-3][PC1] [PC2][CI] [HCP] [CO1][CO2][R][RT][RV1-2] etc.. While this brings many interesting questions to mathematicians, it has also been applied to many areas such as communications, signal processing, etc..

This phenomenon was studied for coupled oscillators by many authors. It was observed that, even individual oscillators may be chaotic, the coupled oscillators could exhibit synchronization, that is, the corresponding coordinates of the system approach each other as time evolves. The phenomenon often results from the diffusive coupling, but more likely with the interactions of the dissipation of the system. For systems possessing strong partial dissipations, master-slave synchronization and synchronization from partial coupling are expected (see [ACH][H1-3][PC1]). Pecora and Carroll observed that for a class of chaotic systems, it can be decomposed into two subsystems: a drive (master) and a stable response (slave) subsystem. The latter one synchronizes when coupled with a common driven signal.

Concerning the phenomena of synchronization, two basic questions are particularly interested and important. The first one is: given a coupled chaotic system or a chaotic system, when the system possesses synchronization or self-synchronization. For coupled identical systems, the diagonal of the system is invariant. Synchronization is equivalent to the attracting property of the diagonal, which in turn is determined by the Lyapunov exponents normal to the diagonal. More precisely, if all the Lyapunov exponents normal to the diagonal are negative, then the coupled oscillators are synchronized.

For master-slave synchronization the question is more delicate. The appearing of master-slave synchronization depends on the product structure of the system. For some product structure it may happen that no coordinate is synchronizing coordinate, while for other product structure some coordinates may be synchronized coordinates. Hence, instead of asking simply that if the system has synchronizing coordinates, one should ask, from global dynamical point of view, that if there exists product structure such that the system is synchronized with some coordinates as synchronizing coordinates. We like to bring the attention on the following observation. Due to the dependence of the synchronization on the product structure, the dynamic of the system does not play the whole role for the synchronization. For example, in the Lorenz equation with chaotic attractor, there are one negative, one zero and one positive Lyapunov exponents. Nevertheless, the Lorenz equation still possesses two types of master-slave synchronization (see [TWB]). Hence, for master-slave synchronization, one of the goals is to relate the dynamic properties to the existence of a product structure which produces master-slave synchronization.

Another question which is more important arises from the following consideration. In practice, for example the implementation of a system by designing circuits, various perturbations are unavoidable. This naturally addresses the question about the stability or robustness of the synchronization. While many literatures appear on the synchronizations, it seems to us that the stability property has not been paid much attention. After setting the problem in the framework of dynamical systems, we study this aspect based on the invariant manifold theory (see Theorem 2). It turns out that the stability depends not only on the normal Lyapunov exponents, but also on the generalized Lyapunov exponents (see Definition 3) which measure the comparison of the normal Lyapunov exponents and the parallel ones along trajectories in the attractor. Application to fully coupled systems is discussed, in particular on the strength of the coupling in Section 3. This issue is also important in numerical simulation. Some numerics are carried along this direction to support our idea. The formulation also allows general forms of dissipative coupling.

Master-slave synchronization and partial coupling are two widely used methods to detect strong partial dissipative property of a system. Relation between those two was discussed here (see Th.3 and also [H1-3]), by using two different applications.

2 Definitions and results

2.1 Consider the following system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{1}$$

where $x \in R^n$, $y \in R^m$. Suppose that the system is dissipative (see [H1-3]), then there exists a global attractor $\mathcal{A}$. Let $\pi_1 : R^{m+n} \to R^n$, $\pi_2 : R^{m+n} \to R^n$ be the projections form R^{m+n} onto R^n and R^m.

Many systems have a natural decomposition as (1), for example systems obtained by coupling identical oscillators. Even though the systems may be complicated, certain degree of coherence may exhibit. Among others, synchronization is one of the most studied lately. Here we give a formulation closely related to the invariant manifold theory and apply results there to the study of synchronization. The following definition is motivated by [AVR][H2].

Definition 1. The system (1) is *synchronized* for y with respect to x, if there exists C^1 map $H : R^n \to R^m$ such that the graph of H, denoted $graph(H)$, is invariant and globally attracting.

In this case, for any $(x_0, y_0) \in R^{m+n}$ one has

$$||y(t; x_0, y_0) - H(x(t; x_0, y_0))|| \to 0, \ t \to \infty;$$

In particular, if $(x_0, y_0) \in \mathcal{A}$, then $y_0 = H(x_0)$ and $y(t; x_0, y_0) = H(x(t; x_0, y_0))$ for any $t \in R^1$.

In the case $m = n$, (1) is called *mutually synchronized* if H is invertible.

For systems with $D = \{(x, y) : x = y\}$ invariant, for example, in coupled identical systems or any system symmetric with respect to D, the synchronization occurs for $H = I$.

In many applications the local synchronization is interested, that is, instead of requiring global attraction of $graph(H)$, one asks for *locally* attracting. In this case, we will say the system is *locally synchronized.*

In [PC1][TWB] etc., a master-slave or self synchronization phenomenon was introduced. System (1) is said to have *master-salve synchronization* with x as synchronizing coordinate if for any solution $(x_0(t), y_0(t))$ of (1), the solution $Y(t; Y_0)$ of

$$Y' = g(x_0(t), Y)$$

converges to $y_0(t)$ for any initial condition Y_0, that is, $|Y(t; Y_0) - y_0(t)| \to 0$, as $t \to \infty$.

In terms of skew-product flow, this can be treated as a special case of the synchronization in Definition 1. To see this, consider the skew-product flow induced by (1):

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \\ Y' &= g(x, Y) \end{aligned} \tag{2}$$

$((x, y), Y) \in (R^n \times R^m) \times R^m$.

If system (1) has master-slave synchronization with x as synchronizing coordinate, then $D_x := \{(x, y, Y) : y = Y\}$ is attracting and hence $\mathcal{A} \subset D_x$ for (2). Choose $H : \ R^n \times R^m \to R^m$, $(x, y) \mapsto y$, then $D_x = graph(H)$, and for $(x_0, y_0, Y_0) \in \mathcal{A}$, $Y_0 = y_0$, and

$$H(x(t; (x_0, y_0), Y_0), y(t; (x_0, y_0), Y_0)) = Y(t; (x_0, y_0), Y_0).$$

By Definition 1 the system (2) is synchronized for Y with respect to (x, y).

Definition 2. Suppose (1) is locally synchronized with map H. The synchronization is C^1 *stable* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $\tilde{f}$, $\tilde{g}$, $|\tilde{f} - f|_{C^1} < \delta$ and $|\tilde{g} - g|_{C^1} < \delta$, the system

$$\begin{aligned} x' &= \tilde{f}(x, y) \\ y' &= \tilde{g}(x, y) \end{aligned}$$

is locally synchronized with map $\tilde{H}$ and $|\tilde{H} - H|_{C^1} < \epsilon$.

2.2 Very often in the investigation of synchronization, system has a natural invariant submanifold. The synchronization and its stability relate to the attraction and the persistence of this manifold. The invariant manifold theory naturally comes into play an important role. Before introducing the results, we recall some elements of the invariant manifold theory for the purpose of this topic.

Suppose, for system (1), there exists an invariant manifold $M = graph(H)$, where $H : R^n \to R^m$. Consider the linearization along M

$$z' = A(z(t; z_0))z \tag{3}$$

where $z(t; z_0)$ is the solution of (1) with $z(0; z_0) = z_0 \in M$ and $A(z) = Jf(z)$ is the Jacobian matrix of f at z. Let $\Phi(t; z_0)$ be the fundamental matrix solution of (3). Assume that there exists an invariant splitting with respect to $\Phi(t; z_0)$, that is, $T_z R^{m+n} = T_z M \oplus N_z$ for $z \in M$, and $\Phi(t; z_0) T_{z_0} M = T_{z(t;z_0)} M$ and $\Phi(t; z_0) N_{z_0} = N_{z(t;z_0)}$ for all $t \in R^1$. Denote $\Phi_c(t; z_0)$ and $\Phi_s(t; z_0)$ the restrictions of $\Phi(t; z_0)$ on $T_{z_0} M$ and N_{z_0}, respectively.

Definition 3. *The generalized Lyapunov exponents* for $z_0 \in M$ are defined as

$$\alpha(z_0) = \limsup_{t\to\infty} \frac{1}{t} \ln ||\Phi_s(t; z_0)||,$$

and

$$\beta(z_0) = \limsup_{t\to\infty} \frac{\ln m(\Phi_c(t; z_0))}{\ln ||\Phi_s(t; z_0)||},$$

where, for a linear operator L, $m(L) := \min\{|Lx|;\ |x| = 1,\ x \in D(L)\}$.

The generalized Lyapunov exponents were introduced for the study of normally hyperbolic invariant manifolds (see [HPS][F][He]) and center manifold theory for invariant manifolds (see [CLY]).

Note that the generalized Lyapunov exponents for $z_0 \in M$ is completely determined by those in $\mathcal{A}$. The Uniformity Lemma in [F] states that α and β achieve their maximums on $\mathcal{A}$.

The results we have are

Theorem 1 *Consider system (1). Suppose that $graph(H)$ is invariant. If $\alpha(z_0) < 0$ for $z_0 \in graph(H)$, then $graph(H)$ is attracting, and hence (1) is locally synchronized.*

Theorem 2 *Suppose that $graph(H)$ is locally synchronized. The synchronization is C^1 stable if and only if $\alpha(z_0) < 0$ and $\beta(z_0) < 1$ for $z_0 \in graph(H)$.*

These results are consequences of well-known theorems in the invariant manifold theory regarding persistence of invariant manifolds, see for example, [HPS][CY][He] for the first one, and [M][F] for the second. The latter is particularly important in applications due to the requirement on the stability of real designs.

3 Synchronization of fully coupled systems

In this part, the theorems are applied for the study of synchronization of fully coupled systems, in particular for the stability property . Let

$$x' = f(x) \tag{4}$$

$x \in R^n$, be a dissipative process and $\mathcal{A}$ be the attractor. Denote λ_M, λ_m the maximal and the minimal Lyapunov exponents over $\mathcal{A}$.

Consider the coupled system

$$\begin{aligned} x' &= f(x) + A(x - y) \\ y' &= f(y) + B(y - x) \end{aligned} \tag{5}$$

The diagonal $D = \{(x, y) : x = y\}$ is invariant under (5), and the coupled system is synchronized if D is attracting.

Make the change of variables,

$$u = \frac{y - x}{2},\ v = \frac{y + x}{2},$$

the equation is then written as

$$\begin{aligned} u' &= \frac{1}{2}[f(v + u) - f(v - u) + 2(A + B)u] \\ v' &= \frac{1}{2}[f(v + u) + f(v - u) + 2(B - A)u]. \end{aligned} \tag{6}$$

Now the synchronization is equivalent to that $\{u = 0\}$ is attracting.

Linearizing (6) along $\{u = 0\}$, say along $(0, v_0(t))$,

$$\begin{aligned} u' &= [Jf(v_0(t)) + (A + B)]u \\ v' &= (B - A)u + Jf(v_0(t))v \end{aligned} \tag{7}$$

and noticing that $v_0(t)$ is a solution of (4).

For symmetric coupling, that is, A=B, (7) has a natural invariant splitting $R^n(u) \oplus R^n(v)$. Now $\{u = 0\}$ is locally attracting if

$$u' = [Jf(v_0(t)) + 2A]u \tag{8}$$

has zero as an exponentially stable solution.

In the case $A = -kI$, if $(H1) : k > \lambda_M$, then the Lyapunov exponents of (8) are less than zero, by Theorem 1, $\{u = 0\}$ is attracting and the coupled system is locally synchronized. But, to insure the stability of the synchronization, by Theorem 3, one needs $k > \lambda_M$ and $-k + \lambda_M < \lambda_m$, that is, $(H2) : k > \max\{\lambda_M, \lambda_M - \lambda_m\}$.

To illustrate the importance of the results, we consider the coupled Lorenz equation. For the choices of the parameters $\sigma = 10$, $r = 28$ and $b = 2.667$, in the Lorenz equation, it is known (see [L][MM][HT][HHTZ], etc.) the attractor is chaotic: there is a Lyapunov exponent closed to 0.912, one to 0 and one to -14.577 (this is obtained by using the numerical algorithm in [NY]).

First we consider the weakly coupled Lorenz system with $k = 1$, which satisfies (H1) but not (H2),

$$\begin{aligned}
x_1' &= \sigma(y_1 - x_1) - k(x_1 - x_2) \\
y_1' &= rx_1 - y_1 - x_1 z_1 - k(y_1 - y_2) \\
z_1' &= x_1 y_1 - bz_1 - k(z_1 - z_2) \\
x_2' &= \sigma(y_2 - x_2) - k(x_2 - x_1)2 \\
y_2' &= rx_2 - y_2 - x_2 z_2 - k(y_2 - y_1) \\
z_2' &= x_2 y_2 - bz_2 - k(z_2 - z_1).
\end{aligned}$$

The fourth-order Runge-Kutta numerical integration of the trajectory with initial condition $(1, 2, 3, 4, 5, 6)$ up to 17000 iterations is plotted. The differences of $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ approach zero rapidly (see Fig. 1.1-1.3). This agrees with the theoretical prediction that this weakly coupled system is locally synchronized. Now we add a small perturbation to obtain the perturbed weakly coupled Lorenz system

$$\begin{aligned}
x_1' &= \sigma(y_1 - x_1) - k(x_1 - x_2) + 0.01x_1 - 0.02z_1 \\
y_1' &= rx_1 - y_1 - x_1 z_1 - k(y_1 - y_2) - 0.01x_1 z_3 \\
z_1' &= x_1 y_1 - bz_1 - k(z_1 - z_2) \\
x_2' &= \sigma(y_2 - x_2) - k(x_2 - x_1) - 0.03z_1 x_2 \\
y_2' &= rx_2 - y_2 - x_2 z_2 - k(y_2 - y_1) \\
z_2' &= x_2 y_2 - bz_2 - k(z_2 - z_1).
\end{aligned}$$

Applying the same numerical integration for the same initial condition $(1, 2, 3, 4, 5, 6)$. The differences of $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ exhibit a chaotic behavior and the magnitudes are rather large (see Fig. 2.1-2.3). This is because that without $(H2)$, the persistence of D is not guarante ed under small perturbations.

Consider now a strong coupling which satisfies (H2), say $k = 15$, with the same perturbation and apply the numerical scheme for the same initial condition, one sees that although the differences are rather chaotic, their magnitudes are very small (see Fig. 3.1-3.3), which reflects a slight perturbation of the diagonal. Nevertheless, the diagonal is persistent and hence the synchronization is stable for this stronger coupling.

4 Master-slave synchronization and partially coupled system

4.1 The master-slave synchronization reflects strong partial dissipation in a system. It is realized by decomposing the system into two subsystems and driving one by the other. Intuitively, this can also be realized by coupling two identical systems only along the driven subsystem. The following result gives a relation between those two.

Theorem 3 *Suppose*

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{9}$$

has the master-slave synchronization with y as synchronizing coordinates, then the partially coupled systems

$$\begin{aligned} x_1' &= f(x_1,y_1) \\ y_1' &= g(x_1,y_1) - k(y_1 - y_2) \\ x_2' &= f(x_2,y_2) \\ y_2' &= g(x_2,y_2) - k(y_2 - y_1), \end{aligned} \tag{10}$$

and

$$\begin{aligned} x_1' &= f(x_1,y_1) \\ y_1' &= g(x_1,y_1) + ky_2 \\ x_2' &= f(x_2,y_2) \\ y_2' &= g(x_2,y_2) + ky_1, \end{aligned} \tag{11}$$

are both synchronized for k large.

Proof. Here we provide the proof for the first coupling system. To show this, we make the change of variables

$$u_1 = \frac{x_2 - x_1}{2}, u_2 = \frac{x_2 + x_1}{2}, v_1 = \frac{y_2 - y_1}{2}, v_2 = \frac{y_2 + y_1}{2}.$$

Under the new variables (u_1, v_1, u_2, v_2), (10) becomes

$$\begin{aligned} u_1' &= \frac{1}{2}f(u_2+u_1, v_2+v_1) - \frac{1}{2}f(u_2-u_1, v_2-v_1) \\ v_1' &= \frac{1}{2}g(u_2+u_1, v_2+v_1) - \frac{1}{2}g(u_2-u_1, v_2-v_1) - 2kv_1 \\ u_2' &= \frac{1}{2}f(u_2+u_1, v_2+v_1) + \frac{1}{2}f(u_2-u_1, v_2-v_1) \\ v_2' &= \frac{1}{2}g(u_2+u_1, v_2+v_1) + \frac{1}{2}g(u_2-u_1, v_2-v_1). \end{aligned} \tag{12}$$

Note that (10) is synchronized, if the plane $D = \{u_1 = v_1 = 0\}$ is attracting (it is invariant already) under (12).

Now linearizing (12) along $u_1 = v_1 = 0$, that is, for any solution $(u_1^0(t), v_1^0(t), u_2^0(t), v_2^0(t))$ of (12) with $u_1^0(t) = v_1^0(t) = 0$, consider

$$X' = A(t)X \tag{13}$$

where $X = (u_1, v_1, u_2, v_2)$ and

$$A(t) = \begin{pmatrix} f_x(u_2^0(t), v_2^0(t)) & f_y(u_2^0(t), v_2^0(t)) & 0 & 0 \\ g_x(u_2^0(t), v_2^0(t)) & g_y(u_2^0(t), v_2^0(t)) - 2k & 0 & 0 \\ 0 & 0 & f_x(u_2^0(t), v_2^0(t)) & f_y(u_2^0(t), v_2^0(t)) \\ 0 & 0 & g_x(u_2^0(t), v_2^0(t)) & g_y(u_2^0(t), v_2^0(t)) \end{pmatrix}$$

Equation (13) has an invariant splitting: $D \oplus N$, where

$$D = \{(u_1, v_1, u_2, v_2) : u_1 = v_1 = 0\},$$

$$N = \{(u_1, v_1, u_2, v_2) : u_2 = v_2 = 0\}.$$

So D is locally attracting, if

$$\begin{aligned} u' &= f_x(u_2^0(t), v_2^0(t))u + f_y(u_2^0(t), v_2^0(t))v \\ v' &= g_x(u_2^0(t), v_2^0(t))u + (g_y(u_2^0(t), v_2^0(t)) - 2k)v \end{aligned} \tag{14}$$

has zero as a stable solution. Note that $(u_2^0(t), v_2^0(t))$ is a solution of (9), so if (9) has y as synchronizing coordinate, (14) is synchronized for large k. ∎

4.2 In what follows we apply Theorem 3 to two systems.
Example 1. Consider the Lorenz equation [L]

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= rx - y - xz \\ z' &= xy - bz. \end{aligned} \tag{15}$$

where σ, r, b are positive.

There are two mathematical models for Lorenz attractor: the geometric Lorenz attractor [GW] and Lorenz-type attractor [ABS]. Some work were carried out for the existence of those two types of attractors [Ry][ACL][AP]. For those models, the attractors have a two dimensional cross section and on the cross section there are stable foliations from which master-slave synchronization could be derived. But for the system (15), it is not known if the attractor is 'close' to the above models. It is first observed by Pecora and Carroll [PC1] that the system does possess master-slave synchronization with x (resp. y) as synchronized coordinate. This relys on a numerical computation of the Lyapunov exponents. Here we provide an analytic justification using Theorems 1 and 2.

In order for y being a synchronizing coordinate we consider the following

$$\begin{aligned} x' &= \sigma(y-x) \\ y' &= rx - y - xz \\ z' &= xy - bz \\ x_1' &= \sigma(y - x_1) \\ z_1' &= x_1 y - bz_1. \end{aligned} \tag{16}$$

The linearization in the directions normal to D is

$$\begin{aligned} x_1' &= -\sigma x_1 \\ z_1' &= yx_1 - bz_1. \end{aligned}$$

This can be solved explicitly by the variation of constant formula. In fact,

$$x_1(t; x_1(0), z_1(0)) = e^{-\sigma t} x_1(0),$$

and

$$z_1(t; x_1(0), z_1(0)) = e^{-bt} z_1(0) + \int_0^t e^{-b(t-s)} e^{-\sigma s} x_1(0) y(s) ds.$$

Hence,

$$\begin{aligned} |x_1(t; x_1(0), z_1(0))| &\le e^{-\sigma t}|x_1(0)|, \; |z_1(t; x_1(0), z_1(0))| \\ &\le e^{-bt}|z_1(0)| + C|x_1(0)||e^{-\sigma t} - e^{-bt}|. \end{aligned}$$

Both $x_1(t)$ and $z_1(t)$ approach zero exponentially at the rate not slower than e^{-4t} for the standard choices of the parameters σ, r, and b. So y is a synchronizing coordinate for (16). If the Lyapunov exponents in the directions of D are greater than -4, then the master-slave synchronization with y as the synchronizing coordinate is stable.

Similarly, for x being a synchronizing coordinate, we need look at

$$\begin{aligned} y_2' &= -y_2 - xz_2 \\ z_2' &= xy_2 - bz_2. \end{aligned}$$

Numerical computation indicates that both the Lyapunov exponents are negative (see [CO2]). The following approach we provided is special for this particular example, see also [H2].

Let $u = y_2^2 + z_2^2$. Then

$$u'(t) = -2y_2^2 - 2bz_2^2 < -2(y_2^2 + z_2^2) = -2u,$$

hence, $u(t; u(0)) \le e^{-2t} u(0)$ which implies that $y_2(t)$ and $z_2(t)$ approach zero not slower than e^{-t}. So x is a synchronizing coordinate for (16).

Again, if the Lyapunov exponents in the directions of D are greater than -1, then the master-slave synchronization with x as the synchronizing coordinate is stable.

From Theorem 3, we can then conclude that the partially coupled systems along the variable y

$$\begin{aligned} x_1' &= \sigma(y_1 - x_1) \\ y_1' &= rx_1 - y_1 - x_1 z_1 - k(y_1 - y_2) \\ z_1' &= x_1 y_1 - b z_1 \\ x_2' &= \sigma(y_2 - x_2) \\ y_2' &= rx_2 - y_2 - x_2 z_2 - k(y_2 - y_1) \\ z_2' &= x_2 y_2 - b z_2, \end{aligned}$$

and

$$\begin{aligned} x_1' &= \sigma(y_1 - x_1) \\ y_1' &= rx_1 - y_1 - x_1 z_1 + k y_2 \\ z_1' &= x_1 y_1 - b z_1 \\ x_2' &= \sigma(y_2 - x_2) \\ y_2' &= rx_2 - y_2 - x_2 z_2 + k y_1 \\ z_2' &= x_2 y_2 - b z_2, \end{aligned}$$

are synchronized for large k. Similarly, the systems coupled along the variable x are synchronized for large k.

Example 2. Another example is the laser equation considered in [RT][R][H2]

$$\begin{aligned} E' &= \tau_c^{-1}[(G - \alpha)E + i\omega E \\ G' &= \tau_f^{-1}(p - G - G|E|^2), \end{aligned}$$

where $E = \mathcal{E} e^{i\phi}$ is the complex electric field and G the gain of a single transverse and longitudinal mode class B laser. τ_c is the cavity round trip time, τ_f the fluorescence time of the upper lasing level of the crystal, p the pump coefficients, α the cavity loss coefficients, ω the detuning of the laser from a common cavity mode.

In terms of $\mathcal{E}$ and ϕ, the equation is written as

$$\begin{aligned} \mathcal{E}' &= \tau_c^{-1}(G - \alpha)\mathcal{E} \\ \phi' &= \omega \\ G' &= \tau_f^{-1}(p - G - G\mathcal{E}^2). \end{aligned} \tag{17}$$

Treating ϕ as time and rescaling, we can write (19) as

$$\begin{aligned} \mathcal{E}' &= \tau_c^{-1}(G - \alpha)\mathcal{E} \\ G' &= \tau_f^{-1}(p - G - G\mathcal{E}^2). \end{aligned}$$

The master-slave synchronization with $\mathcal{E}$ as the driven signal and G as the response one is obvious. Hence, by Theorem 3,

$$\begin{aligned}
\mathcal{E}_1' &= \tau_c^{-1}(G_1-\alpha)\mathcal{E}_1 + k(\mathcal{E}_2-\mathcal{E}_1)\\
G_1' &= \tau_f^{-1}(p-G_1-G_1\mathcal{E}_1^2)\\
\mathcal{E}_2' &= \tau_c^{-1}(G_2-\alpha)\mathcal{E}_2 + k(\mathcal{E}_1-\mathcal{E}_2)\\
G_2' &= \tau_f^{-1}(p-G_2-G_2\mathcal{E}_2^2),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_1' &= \tau_c^{-1}(G_1-\alpha)\mathcal{E}_1 + k\mathcal{E}_2\\
G_1' &= \tau_f^{-1}(p-G_1-G_1\mathcal{E}_1^2)\\
\mathcal{E}_2' &= \tau_c^{-1}(G_2-\alpha)\mathcal{E}_2 + k\mathcal{E}_1\\
G_2' &= \tau_f^{-1}(p-G_2-G_2\mathcal{E}_2^2),
\end{aligned}$$

are synchronized for large k.

The exact model of the coupling system of two lasers used in [RT] is

$$\begin{aligned}
E_1' &= \tau_c^{-1}[(G_1-\alpha_1)E_1 + kE_2] + i\omega_1 E_1\\
G_1' &= \tau_f^{-1}(p_1-G_1-G_1|E_1|^2)\\
E_2' &= \tau_c^{-1}[(G_2-\alpha_2)E_2 + kE_1] + i\omega_2 E_2\\
G_2' &= \tau_f^{-1}(p_2-G_2-G_2|E_2|^2)
\end{aligned}$$

In the case of coupling of identical lasers one has, in terms of $\mathcal{E}_j$ and ϕ_j, $j=1,2$,

$$\begin{aligned}
\mathcal{E}_1' &= \tau_c^{-1}[(G_1-\alpha)\mathcal{E}_1 + k\cos(\phi_2-\phi_1)\mathcal{E}_2]\\
\phi_1' &= \tau_c^{-1}k\sin(\phi_2-\phi_1)\mathcal{E}_2\mathcal{E}_1^{-1}+\omega\\
G_1' &= \tau_f^{-1}(p-G_1-G_1\mathcal{E}_1^2)\\
\mathcal{E}_2' &= \tau_c^{-1}[(G_2-\alpha)\mathcal{E}_2 + k\cos(\phi_2-\phi_1)\mathcal{E}_1]\\
\phi_2' &= \tau_c^{-1}k\sin(\phi_1-\phi_2)\mathcal{E}_1\mathcal{E}_2^{-1}+\omega\\
G_2' &= \tau_f^{-1}(p-G_2-G_2\mathcal{E}_2^2).
\end{aligned}$$

Set $\phi=\phi_2-\phi_1$, then

$$\phi' = -\tau_c^{-1}k\sin(\phi)(\mathcal{E}_2\mathcal{E}_1^{-1}+\mathcal{E}_1\mathcal{E}_2^{-1}), \tag{18}$$

and clearly $\phi=0$ is a locally stable equilibrium of (20) for $k>0$. If we restrict $\phi\in(-\frac{\pi}{2},\frac{\pi}{2})$, then the attractor lies in $\{\phi=0\}$ and on it one has

$$\begin{aligned}
\mathcal{E}_1' &= \tau_c^{-1}[(G_1-\alpha)\mathcal{E}_1 + k\mathcal{E}_2]\\
G_1' &= \tau_f^{-1}(p-G_1-G_1\mathcal{E}_1^2)\\
\mathcal{E}_2' &= \tau_c^{-1}[(G_2-\alpha)\mathcal{E}_2 + k\mathcal{E}_1]\\
G_2' &= \tau_f^{-1}(p-G_2-G_2\mathcal{E}_2^2),
\end{aligned}$$

and for k large, the system is synchronized by Theorem 3.

It is believable that the phenomena of synchronization are more general than the existence of stable foliation structure in chaotic systems. The study of the phenomena could lead more understanding toward the chaotic behavior.

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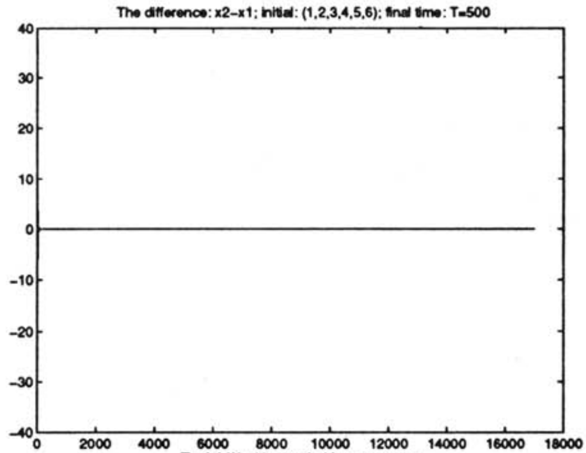
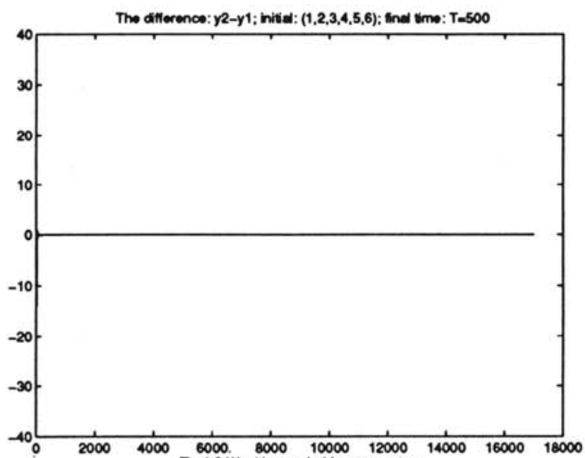


Fig.1.1 Weakly coupled Lorenz system



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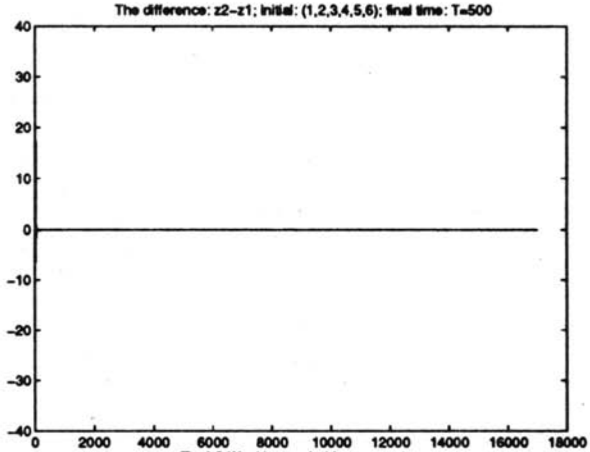


Fig.1.3 Weakly coupled Lorenz system

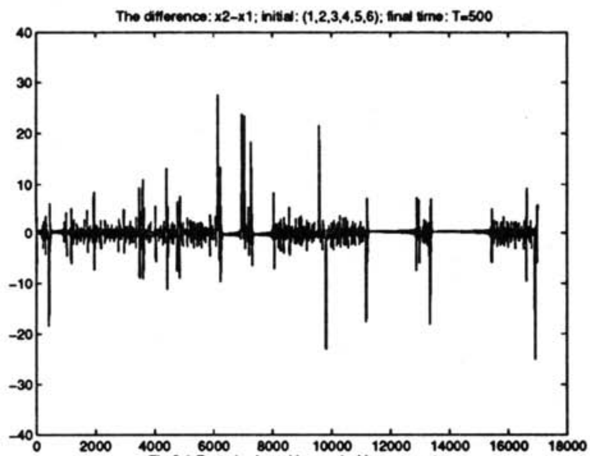
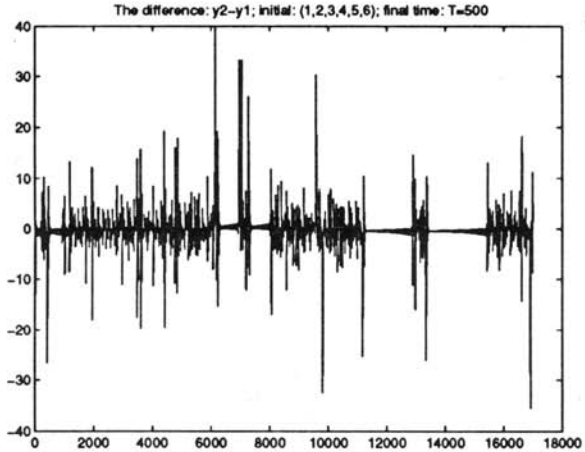


Fig.2.1 Perturbed weakly coupled Lorenz system



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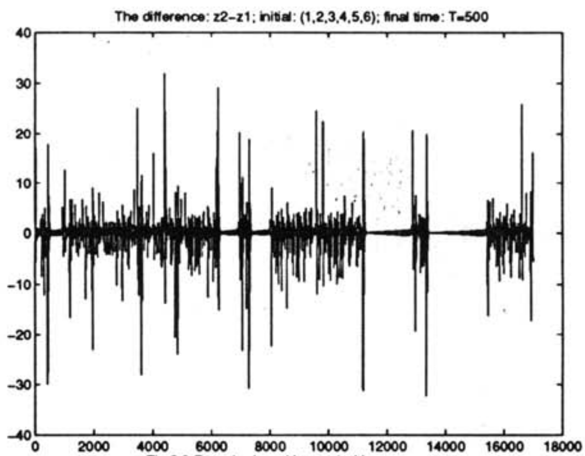
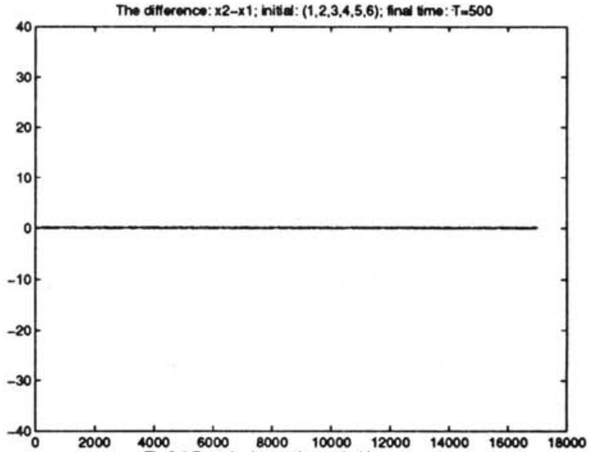


Fig.2.3 Perturbed weakly coupled Lorenz system



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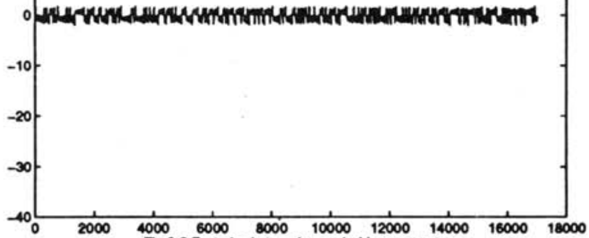

Fig.3.2 Perturbed strongly coupled Lorenz system

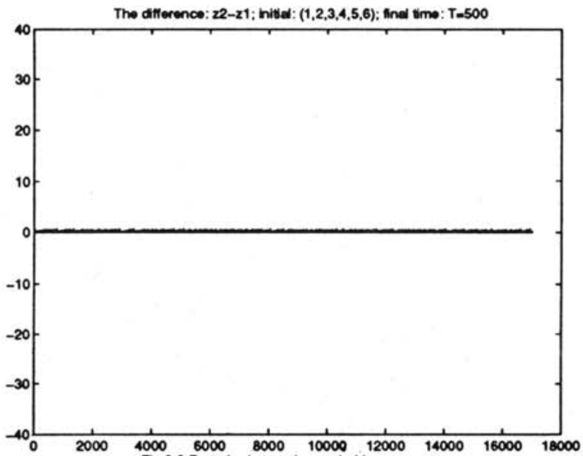

Fig.3.3 Perturbed strongly coupled Lorenz system