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Auto-Oscillations in Continuous Systems with Impulsive Self-Support

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Abstract: The surway of impulsive-continuous autonomous systems of various types is represented, especially those of in which the impulsive self-suport (ISS) generates discontinuous auto-oscillations. The main objects are: discontinuous dynamical systems, linear oscillator with one degree of freedom and ISS, scalar functional differential equations with ISS, heat conductions and vibration of the string with energy dissipation and ISS.

Key words: impulsive differential equation, impulsive-continuous system, auto-oscillations, discontinuous dybamical system.

1 Introduction

Impulsive-continuous evolution of a system consists of alternation of stages of continuous variation of its state and very short-term stages of its essential change. Similar processes arise when the studying system is subjected to impulsive outer action (e.g. under the controlling) or when the fast changes of the state happen immanently as the system parameters have reached its critical values ("transition from quantity to quality").

The impulsive-differential equations (IDEs) are used as mathematical instrument of mentioned subject. These equations include differential equations to describe stages of continuons variation of the stage, and "finite" equation to describe discontinuities of the 1st kind of the solution in instants of impulses. The IDEs theory began about 40 years ago and hundreds of papers and a lot of books [1 - 7] have been published up to now.

The IDEs theory with given moments t_i of impulses is mostly developed. This assumption excludes autonomous equations but allows linear IDEs, the theory of which is similar in many respects to the usual linear differential equations theory. (It is quite natural, because the linear IDE is equivalent to the linear differential equation with addens of delta-function type in coefficients and inhomogeneous term.) Basing on these equations, the theory of nonlinear IDEs with principal linear part can be constructed to a marked degree also. Some obtained results can be extended to the case $t_i = t_i(x)$, i.e. when the instants of impulses depend on the state of the system.

The autonomous IDEs theory, where the moments of impulses are defined by reaching the critical states of a system, has been studied much weaker. At the same time these systems, nonlinear in principle, are quite natural both in theoretical and in applied aspects, so their detailed study is highly desirable.

The theory of discontinuous oscillations is the main (but not the unique) field of applications of autonomous IDEs. The example of such oscillations was already considered by E.Friedlaender in 1926 [8]. In 1930, realizing L.I.Mandelshtam and N.D.Papaleksi idea, A.A.Andronov and A.A.Witt [9] interpreted discontinuous oscillations as discontinuous trajectories on phase plane; the development of this subject see in [10].

The simplest model of watch was the first mathematical model of the mechanical discontinuous oscillations; it was considered independently in 1937 by A.A.Andronov and S.E.Khaikin [11] with method of pointwise maps, and by N.M.Krylov and N.N.Bogolyubov [12] with asymptotic methods of nonlinear mechanics. More realistic model of watch including two degrees of freedom was studied by A.A.Andronov and Yu.I.Neimark [13].

Autonomous systems with impulsive-continuouos evolution are connected immediately with the theory of discontinuous dynamical systems, which has appeared quite recently and has not been developed sufficiently. (The difference may exist only in approaches: we begin from precise indication of phase space for the letter, but we start from IDE as mathematical model, whereas the phase space is fitted, and moreover sometimes it is fitted not uniquely for the former systems.) Therefore we give in Sec.2 one of the possible definitions of discontinue dynamic systems and indicate some of their properties. Different examples of differential equations (the equation of linear oscillator with one degree of freedom, the scalar equation with retarded argument, the partial differential equations) are considered in further sections, with impulsive self-support, which may generate discontinuous auto-oscillations, describing asymptotically stable periodic impulsive-continuous processes.

2 Discontinuous dynamical systems

The general notion of discontinuous dynamical system both for arbitrary metric space [14] and for $\mathbb{R}^n$ [15] seems to be given first by Th.Pavlidis in 1966. The scholar investigated some properties of stability for such systems and indicated their possible applications to the theory of neuron nets [16, 17]. S.Kaul gave another general approach to the notion of discontinuous dynamical system in metric space (see [4], 4.7). Beginning from 1969 V.F.Rozhko extended the main concepts of topological dynamics (Poisson's stability, minimal sets etc.) on these systems in his series of papers (see e.g. [18 - 20]). K.S.Sibirskii and I.A.Chirkova introdused (see [21]) more general notion of discontinuous set-valued dynamical system in 1971; P.I. Morozanu made this [22] for discontinuous dynamical system with aftereffect. More concrete discontinuous dynamical systems in $\mathbb{R}^n$ (specifically in $\mathbb{R}^2$) was considered in [23 - 26] a.o.; unfortunately, several papers in this field are difficult to get.

The solution of Cauchy problem is built in the direction of increasing t in all these papers, so we really consider semidynamical systems.

Here we give one of the possible definitions of the discontinuous dynamical system in $\mathbb{R}^n$. Namely, we consider IDE of the form

$$\dot{x}(t) = f(x(t)) \qquad (x(t) \notin M), \tag{1}$$

$$(x(t) \in M) \Rightarrow x(t^+) = F(x(t)). \tag{2}$$

Here $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $M \subset \mathbb{R}^n$ is a closed set, $F \in C(M, \mathbb{R}^n)$. The solution $x = \varphi(t; a)$ of IDE (1), (2) under given initial condition

$$x(0) = a \tag{3}$$

is built in the following way. If $a \notin M$, so $\varphi(t; a)$ coincides with solution of problem (1), (3) on $[0, t_1]$, where $t_1 (= t_1(a))$ is the lowest of values $t > 0$, for which $\varphi(t; a) \in M$ ($\varphi(t; a)$ coincides with the mentioned solution for $t > 0$ on all interval of its existence, if there are no such values t). Then we set $\varphi(t_1^+; a) = F(\varphi(t_1^+; a))$ and take $\varphi(t; a)$ on $(t_1, t_2]$ as the solution of equation (1) under initial condition $x(t_1) = \varphi(t_1^+; a)$, where t_2 is the lowest of values $t > t_1$, for which $\varphi(t; a) \in M$ (if such t exist), etc.

But if $a \in M$, we take $t_1 = 0$, and after that we continue the described procedure.

The following conditions are demanded to hold, in order the mentioned definition to be correct (namely, the definition of the value t_2 for $\varphi(t_1^+; a) \in M$ has the proper meaning):

C1. If $a \in M \cap F(M)$ and $f(a) \neq 0$, then the solution of problem (1),(3) has no common points with M for all sufficiently small $t > 0$.

C2. If $a \in M \cap F(M)$ and $f(a) = 0$, then $F(a) = a$;

in the latter case we accept $\varphi(t; a) \equiv a$.

The solution of IDE (1),(3) under initial condition $x(t_0) = a$ is defined analogously; this solution is equal to $\varphi(t - t_0; a)$.

The solution $\varphi(\cdot; a)$ of (1)-(3), defined above, can have either finite or infinite number of discontinuity points t_i, and all of them are of the 1st kind. It coincides with one of the solutions of equation (1) in the first case after the last discontinuity point, and therefore it is defined on some maximal interval $[0, T_a)$, and moreover if $T_a < \infty$, then $|\varphi(t; a)| \to \infty$ as $t \to T_a^-$. It is $T_a = \infty$ if solutions of (1) do not blow up (e.g. if $|f(x)| = O(|x|)$ as $|x| \to \infty$).

Now we are considering several simple examples in which the second case takes place. We accept $n = 2$, $f(x) = (-1, 0)$ in all these examples.

Example 1. Let $M = \{(x_1, x_2) : x_1 = 0\}, F(0, x_2) = (1, x_2)$. Then the solution $\varphi(\cdot; a)$ has discontinuity points at $t = a_1, a_1 + 1, ...$ and it is periodic as $t \geq a_1$, if $a_1 \geq 0$.

Example 2. Let $M = \{(x_1, x_2) : x_1 > 0, x_1 x_2 = 1\}, F(x_1, x_2) = (x_1, x_2 + 1)$. Then the solution $\varphi(\cdot; a)$ has discontinuity points at $t = a_1 - a_2^{-1}, a_1 - (a_2 + 1)^{-1}, a_1 - (a_2 + 2)^{-1}, ...$ and tends to ∞ as $t \to a_1^-$, if $a_1 > 0, a_1 a_2 \geq 1$.

Example 3. Let $M = \{(x_1, x_2) : x_1 = x_2\}, F(x_1, x_2) = (x_1, x_2/2)$. Then the solution $\varphi(\cdot; a)$ has discontinuity points at $t = a_1 - a_2, a_1 - a_2/2, a_1 - a_2/2^2, ...$ and tends to 0^+ as $t \to a_1^-$, if $a_1 \geq a_2 > 0$.

Example 4. Let $M = \{(x_1, x_2) : |x_2| = 1 + x_1\}, F(x_1, x_2) = (x_1, -(1 + x_1/2)\text{sgn}x_2)$. Then the solution $\varphi(\cdot; a)$ has discontinuity points at $t = a_1 - (|a_2| - 1), a_1 - (|a_2| - 1)/2, a_1 - (|a_2| - 1)/2^2, ...$, if $a_1 > 0, 1 < |a_2| \leq 1 + a$; this solution is picewise constant, whereas its signes alternate on intervals of constancy and its values tend to ± 1 as $t \to a_1^-$ having no unique limit.

The solution $\varphi(\cdot; a)$ is defined at all $t \geq 0$ in example 1, whereas it blows up in example 2 on account to "accumulation of jumps", i.e. the question on further continuation of solution does not arise in these examples. The solution is defined on $[0, a_1)$ in example 3; but it is natural to continue this solution for $t \geq a_1$ as the solutition of equation (1) under limit initial condition $x(a_1) = 0$, because of its limite behavior as $t \to a_1^-$. At last, the natural continuation of the solution for $t \geq a_1$ as single-valued function does not exist apparently in example 4.

Generalizing the example 3, let us agree in all cases, when the solution $\varphi(\cdot; a)$ of (1),(3) is built on the interval $[0, \bar{t}), 0 < \bar{t} < \infty$ and the limit $\varphi(\bar{t}^-; a)$ exists, to continue this solution for $t \geq \bar{t}$ as the solution of IDE (1),(2) under the initial condition $x(\bar{t}) = \varphi(\bar{t}^-; a)$. We note that $\varphi(\bar{t}; a) \in M$ and $F(\varphi(\bar{t}; a)) = \varphi(\bar{t}; a)$ if $\bar{t}$ is the limit point for discontinuity points of the solution in this situation.

The construction of the solution of (1),(3) mentioned above, can be considered as its constructive definition. There is the equivalent descriptive definition.

The function $\varphi(\cdot; a) : [0, T_a) \to \mathbb{R}^n$ $(0 < T_a \leq \infty)$ is called the *solution* of problem (1)-(3), when the following conditions hold:

1. It is continuous from the left, with $\varphi(0; a) = a$.

2. It is continuously differentiable and it satisfies equation (1) on some open interval with the left end t , for any $t \in [0, T_a)$.

3. $\forall t \in [0, T_a) \Rightarrow$ if $\varphi(t; a) \notin M$, then $\varphi(t^+; a) = \varphi(t; a)$, but if $\varphi(t; a) \in M$, then $\varphi(t^+; a) = F(\varphi(t; a))$.

We call the solution of (1)-(3) the *noncontinuable* one, if it satisfies, in addition, the condition 4:

4. If $T_a < \infty$, than $\varphi(t; a)$ has no (finite) limit as $t \to T_a^-$.

Theorem 1. Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, the set $F \in C(M, \mathbb{R}^n)$ be closed, and C1, C2 be satisfied. Then any two solutions of the problem (1),(3) coincide on common interval of their definition, and the noncontinuable solution of this problem exists.

Proof. The first assertion of theorem 1 can be proved by contradiction, if we consider the infimum of disagreement points of two solutions. With regard to this, the second assertion is obtained with the help of join of all the intervals of existence of the solutions of (1),(3). 2

We call the noncontinuable solution $\varphi(\cdot; a) : [0, T_a) \to \mathbb{R}^n$ of (1)-(3) the *regular* one, if $T_a = \infty$, or $T_a < \infty$ and $|\varphi(t; a)| \to \infty$ as $t \to T_a^-$; otherwise, we call this solution the *singular* one.

Theorem 2. In addition to conditions of theorem 1 let the following conditions be satisfied:

C3. $F(M)$ is a closed set.

C4. $x \in M \cap F(M) \Rightarrow F(x) = x$.

C5. The set $\{x : f(x) = x\}$ is totally disconnected (i.e. it contains only the single-point connected components).

Then there are no irregular solutions.

Proof. Let $\varphi(\cdot; a)$ be an irregular solution of (1)-(3). Then the sequence $\tau_1 < \tau_2 < \ldots \to T_a^-$ exists such that $\sup_i |\varphi(\tau_i; a)| < \infty$; we can assume without loss of generality that

$$\varphi(\tau_i; a) \to \bar{x} \quad (i \to \infty). \tag{4}$$

It follows from irregularity of the solution that

$$\exists p > 0 : \overline{\lim_{t \to T_a^-}} |\varphi(t; a) - \bar{x}| > p,$$

and we can suppose without loss of generality that all $|\varphi(\tau_i; a) - \bar{x}| < p/2$.

The function $\varphi(\cdot; a)$ has discontinuity points in any proximity to T_a, as f is locally bounded; it follows hence and from (4) at once that $\bar{x} \in M$. We consider two following cases.

1. Let $h > 0$ exist such that it can be found in any proximity to T_a discontinuity points θ of the function $\varphi(\cdot; a)$, for which $|\varphi(\theta; a) - \bar{x}| < p$, $|\varphi(\theta^+; a) - \varphi(\theta; a)| > h$. If $\theta_1 < \theta_2 < \ldots \to T_a$ is the sequence of these points, then we can assume, after passage to subsequence, that $\varphi(\theta_i; a) \to b$, $\varphi(\theta_i^+; a) \to c$ as $i \to \infty$. But then it is $b \in M$, $F(b) = c$ and $|b - c| \geq h$. It follows from continuity of the solution from the left, that it is $\inf |\varphi(\xi; a) - \varphi(\theta_i; a)| \to 0$ as $i \to \infty$, where inf is taken over all discontinuity points $\xi < \theta_i$. It follows from here and from C3, that $b \in F(M)$, and we come to contradiction with C4.

2. Let the mentioned h do not exist. Then $F(\bar{x}) = \bar{x}$, as otherwise we can take any value from $(0, |F(\bar{x}) - \bar{x}|)$ as h. Taking t sufficiently near to T_A, we obtain from absence of h and from boundedness of $|f(x)|$ for $|x - \bar{x}| \leq p$, that the (discontinuous) arc of considering trajectory, which starts in $\{x : |x - \bar{x}| < \varepsilon\}$, ends in $\{x : |x - \bar{x}| > p - \varepsilon\}$, and is such that diameters of all its continuous parts and sizes of all its jumps are less then ε exists for any $\varepsilon > 0$.

Let us choose the sequence $\varepsilon_1 > \varepsilon_2 > \ldots \to 0$ and denote by J_i the arc of trajectory, which corresponds to ε_i; let us denote the set, which is obtained from J_i by means of supplement the segment of the line with the same ends to any jump with ends $\varphi(\theta; a)$, $\varphi(\theta^+; a)$, and of closing of the result by J_i'. The set J_i' is connected, and the points $x \in M \cap J_i'$ in which $|F(x) - x| < \varepsilon_i$, form $2\varepsilon_i$-net on J_i'. The relation

$$J' := \overline{\mathrm{lt}} J_i' \subseteq \{x : F(x) = x\}$$

follows from that statement. But $\underline{\mathrm{lt}} J_i'$ is not an empty set, because it contains $\bar{x}$. Therefore, the set J' is connected, by virtue of Zoretti's theorem (see, e.g., [26]). But $\bar{x} \in J'$ and, moreover, J' contains at least one common point with $\{x : |x - \bar{x}| = p\}$, what contradicts to C5.

Theorem 2 is proved. 2

Corollary. If the set M is bounded in conditions of theorem 2, and solutions of (1) do not blow up, then $\forall a \in \mathbb{R}^n \Rightarrow T_a = \infty$.

In conclusion we note that it is not difficult to alter the definition of IDE so that its trajectories are placed not in all $\mathbb{R}^n$, but in the fixed domain. This situation arises while investigating the evolution of mechanical systems, which include impacts on immovable obstacles (see, e.g., [28, 29]). The mathematical theory of billiards is the special case; being begun in 1927 by G.Birkhoff (see [30], Sec. VI. 6-9), it gave the impulse to many issues with very deep results. Although this very theory is covered formally by general theory of IDEs, it is quite specific and so we shall not touch it here.

3 Linear oscillator with one degree of freedom and impulsive support

The following example is considered in Sec. 5 of the book [3]. Here IDE (1),(2) has the form

$$\ddot{a} + 2\lambda\dot{x} + \omega^2 x = 0, \tag{5}$$

$$(x(t) = 0, \dot{x}(t^-) \geq 0) \Rightarrow (x(t^+) = x(t), \dot{x}(t^+) = \dot{x}(t^-) + g(\dot{x}(t^-))) \tag{6}$$

(we use the same notation as in [3]); here $0 < \lambda < \omega$, and $g(\cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a given function.

The trajectories of IDE (5),(6) in phase plane $x, \dot{x}$ consist of twisting arcs of affine transformed logarithmic spirales alternating with jumps along the positive semiaxis $\dot{x}$. It is not difficult to verify that a *simple* (i.e. with one jump on minimal period) cycle has a jump with a top point $(0, y)$, iff

$$y = py + g(py),\ p := \exp(-2\pi\lambda/(\omega^2 - \lambda^2)^{1/2}).$$

The right-hande side of the equation for y is the Poincaré's first return function for this autonomous system. Any trajectory is either a simple cycle, or it asymptotically approaches to some simple cycle, or to origin of coordinates, or to infinity, if this function is nondecreasing. Not-simple cycles, and an asymptotically approach to such cycle, trajectories with more complicated asymptotic behavior may exist also, if the Poincaré's function is not monotone. If $g \in C^1$, then a simple cycle with a top point $(0, y)$ is asymptotically stable (unstable), when $p|1 + g'(py)| < 1$ (> 1).

These assertions can be applied easily to the partical cases $g(y) = \Delta v = \text{const}$ and $g(y) = (J^2 + y^2)^{1/2} - y$ (i.e. $\dot{x}^2(t^+) = \dot{x}^2(t^-) + J^2, J = \text{const}$).

More complicate IDE is considered in [31], where impulses for equation (5) are defined by relation

$$([m\dot{x}^2(t^-) + kx^2(t)]/2 = E_0) \Rightarrow x(t^+) = x(t),\ \dot{x}(t^+) = \dot{x}(t^-) + \Delta v;$$

here $E_0 > 0$ is a given critical value of total energy, $\Delta v > 0$ is the constant added speed, and the meaning of the other notations is obvious. After introduction of dimensionless variables and parameters IDE for the system in consideration takes the form ($\xi = \xi(\tau)$)

$$\ddot{\xi} + 2c\dot{\xi} + \xi = 0 \ (\dot{\xi}^2(\tau^-) + \xi^2(\tau) \neq 1), \tag{7}$$

$$(\dot{\xi}^2(\tau^-) + \xi^2(\tau) = 1) \Rightarrow \xi(\tau^+) = \xi(\tau), \ \dot{\xi}(\tau^+) = \dot{\xi}(\tau^-) + w. \tag{8}$$

The dependence of polar angle of the first point of contact of trajectory with the critical circle, on polar angle of its starting point (the lowest point of the jump) on this circle is taken as the first return function. The investigation of this function results in assertions:

Theorem 3. $\forall w \in (0, 2)$, IDE (7),(8) has the only simple positive (i.e. with $\xi(\tau) > 0$) cycle.

Theorem 4. $\forall c > 0, \exists w_0(c) > 0 : \forall w \in (0, w_0(c)) \Rightarrow$ this cycle is (orbitally) unstable.

Theorem 5. $\forall w > 0, \exists c_0(w) > 0 : \forall c > c_0(w) \Rightarrow$ IDE (7),(8) has the only simple cycle and this cycle is positive and asymptotically stable.

Note that conclusions on stability in theorems 4 and 5 are obtained on "applied" level, since its proofs are based on an analysis of coefficients of asymptotical representations, the validity of which is assumed apriori.

Computing experiment, which was carried out by V.A.Larin for $w = 3/2^{1/2}$, shows that asymptotically stable cycle of rank 2 (i.e. having two jumps on minimal period) arises instead of simple positive cycle when c decreases and passes over value 1.018. If c decreases further and passes over value 0.528 it turns into stable cycle of rank 4. The picture becomes difficult to visual analyse, when c decreases further, but the stable cycle of rank 3 arises on "penultimate" stage. The elements of known A.N.Sharkovskii's sequence [32] are possible to be seen here.

$$\ddot{x} + a\dot{x} + f(x) = 0 \ (\dot{x}(t^-) \neq 0),$$

$$(\dot{x}(t^-) = 0) \Rightarrow x(t^+) = x(t), \ \dot{x}(t^+) = L \ (a, L = \text{const} > 0)$$

was investigated by B.S.Kalitin in his series of papers [33-35].

4 Scalar functional IDE of retarded type

Scalar functional-differential equation

$$\dot{x}(t) = f(x(t+\theta_1), ..., x(t+\theta_m), x_t) \ (x_t(\theta) := x(t+\theta), -h \leq \theta \leq 0) \tag{9}$$

with self-supporting condition

$$(x(t) = 0) \Rightarrow x(t^+) = a \ (= \text{const} > 0) \tag{10}$$

was considered in [36]. Here $0 < h < \infty$, all $\theta_j \in [-h, 0], f : \mathbb{R}^m \times K[-h, 0] \to \mathbb{R}$, where $K[-h, 0]$ is the set of function $[-h, 0] \to \mathbb{R}$ which are continuous from the left and have no more then finite number of discontinuity points, of the 1st kind only. The function f satisfies the conditions:

$$|f(u_1, ..., u_m, \varphi) - f(v_1, ..., v_m, \psi)| \le \sum_{j=1}^{m} M_j |u_j - v_j| + M_0 \int_{-h}^{0} |\varphi(\theta) - \psi(\theta)| d\theta,$$

$$-\delta := \sup\{f(u_1, ..., u_m, \psi) : \forall u_j \in [0, a],\ 0 \le \psi(\theta) \le a\} < 0.$$

The map of the set $\{\varphi : \varphi \in K[-h, 0], \varphi(0) = 0\}$ in itself, defined by formula $\varphi \mapsto (x(\cdot, \varphi))_\tau$, is taken as the first return function, where $x(\cdot, \varphi)$ is the solution of IDE (9),(10) under initial condition $x_0 = \varphi$, and τ is the first positive zero of this solution. We denote

$$f_0 := |f(0, ..., 0, 0)|,\ M := M_1 + ... + M_m + hM_0.$$

The following theorem is proved in [35]:

Theorem 6. Let the inequalities

$$2aM \le (f_0^2 + 2f_0\delta + 5\delta^2)^{1/2} - f_0 - \delta,\ h \le a(f_0 + aM)^{-1}$$

hold. Then IDE (9),(10) has the only periodic solution in the strip $0 \le x(t) \le a$ (up to arbitrary translation in t). This solution is simple and globally asymptotic stable (in natural sence) in mentioned strip.

The smallness of retardation was used in proof essentially, because the first return function was considered in uniform metric. The condition of this smallness is remuved in [37] where the equation

$$\dot{x}(t) = f(x(t + \theta_1), x(t + \theta_2), ..., x_t) \tag{11}$$
$$(0 \ge \theta_1 > \theta_2 > ... \to -\infty, \quad x_t(\theta) := x(t + \theta), \quad -\infty < \theta \le 0)$$

(with the same self-support condition (11) is considered.

Changing the definition from [37] quite a bit, we denote by K a set of functions $(-\infty, 0] \to [0, a]$, which are continuous from the left and have a set of discontinuity points without limit points. The values $f(u_1, u_2, ..., \psi)$ must be defined, if $\forall u_j \in [0, a], \psi \in K$; the inequalities

$$-\Delta \le f(u_1, u_2, ..., \psi) \le -\delta,\ 0 < \delta < \Delta < \infty$$

must be hold, as well as the Lipschitz condition in the following form

$$|f(u_1, u_2, ..., \varphi) - f(v_1, v_2, ..., \psi)| \le \sum_{s=1}^{\infty} M_s |u_s - v_s| + M_0 \int_{-\infty}^{0} e^{kt} |\varphi(t) - \psi(t)| dt,$$

$k > 0$; discrete retardations θ_s must satisfy the condition

$$M_0 := \sup_{s \ge 1}\{M_s \exp(k|\theta_s|)\} < \infty;$$

$\exists p > 0$: there are no more then p of values θ_s on any interval of length less then a/δ. The following theorem is proved in [37]:

Theorem 7. In supposition of previous paragraph, it exists $\eta > 0$ for any $\varepsilon > 0, E > \varepsilon$, such that if $k > \varepsilon, \delta > \varepsilon, \varepsilon < a < E, \Delta < E, p < E, M_0 < \eta, M^0 < \eta$, then the same assertion, as in theorem 6 for IDE (9),(10) is valid for IDE (11),(10).

Based on the proof it is possible to point out the explicit estimate η via ε and E which is apparently very far from the exact one.

The conditions of rise and stability of non-simple periodic solution for IDEs of retarded type have not studied yet neither on theoretical nor on computing levels. It is possible, that it should be start from simplest equations of such kind, i.e.

$$\dot{x}(t) = -1 - kx(t-h),\ x(t) = 0 \Rightarrow x(t^+) = 1\ (k, h = \text{const} > 0).$$

5 Heat conduction with impulsive support

The following partial IDE

$$u_t = au_{xx}\ (0 \le x \le l, 0 \le t < \infty, J(t) \ne J_0),\ u(0,t) = u(l,t) = 0 (0 \le t < \infty) \tag{12}$$

$$J(t) = J_0 \Rightarrow u(x,t^+) = u(x,t) + \alpha(x)\ (0 \le x \le l),\ J(t) := \int_0^l u(x,t)dx \tag{13}$$

is considered in [37]. Here $a, l, J_0 > 0$ are the given constants,

$$\alpha \in C([0,l], \mathbb{R}_+)\ (\alpha(x) \not\equiv 0)$$

is the given function. The value $J(t)$ is proportional to the amount of heat energy in a rod at an instant t.

Let us suppose that $J(0) > J_0$; then there is an infinite sequence $t = t_1(> 0) < t_2 < ...$ of discontinuity points of the solution. The main method of investigation in [37] is a bilateral estimate of the solution on intervals of its continuity, as well as the expansion of all functions under consideration in Fourier's series in x . Such rather rough estimations give enough reason to suppose that the requirements in following theorems can be weakened essentially.

Theorem 8. If $\alpha(x) = O(x)(x \to 0^+), \alpha(x) = O(l-x)(x \to l^-)$, then $t_k \to \infty$ as $k \to \infty$.

Theorem 9. If in addition $\alpha_{2j-1} \le 0\ (j = 2, 3, ...)$ and

$$\sum_{j=1}^{\infty} \alpha_{2j-1}/(2j-1) > 0$$

where

$$\alpha_j := 2l^{-1} \int_0^l \alpha(x) \sin(j\pi x/l)dx,$$

then $\inf_k(t_{k+1} - t_k) > 0$.

We suppose in addition that

$$\sum_{j=1}^{\infty} |\alpha_j| < \infty$$

in theorems 10 and 11.

Theorem 10. $\forall \alpha, J_0, \exists!$ (up to arbitrary translation in t) the simple periodic solution U of the IDE (12),(13). The similar statement holds if α and period of U are given before, but J_0 is fitted.

Theorem 11. $\forall \alpha \exists J^0(\alpha) > 0 : \forall J_0 < J^0(\alpha) \Rightarrow U$ from theorem 10 is stable asymptotically (in natural sence).

IDE of similar form that is more natural from physical point of view, is given in a [39]:

$$u_t = au_{xx} \ (x \in \mathbb{R},\ 0 \le t < \infty,\ u(0,t) \ne u_0), u(\pm\infty, t) = 0 \ (0 \le t < \infty), \quad (14)$$

$$u(0,t) = u_0 \Rightarrow u(x, t^+) = u(x,t) + b\delta(x) \ (x \in \mathbb{R}). \quad (15)$$

Here $a, u_0, b > 0$ are the given constants, δ is the delta-function. IDE (14),(15) has a solution

$$u(x,t) = [b/2(\pi a)^{1/2}] \sum_{j=0}^{k} (t - t_j)^{-1/2} \exp[x^2/4a(t - t_j)]$$

$$(x \in \mathbb{R},\ t_k < t \le t_{k+1}; k = 0, 1, ...; t_0 := 0)$$

under natural initial condition $u(x, 0^+) = b\delta(x)$, where $t_1 (> 0) < t_2 < ...$ are successive instants of impulses, which are defined from the equation $u(0,t) = u_0$.

The behavior of differences $t_k - t_{k-1}$ as $k \to \infty$ plays an important role in determination of asymptotic behavior of solution as $t \to \infty$. We obtain designating

$$\tau_k := 4\pi a u_0^2 (t_k - t_{k-1})/b^2 \ (k = 1, 2, ...)$$

that $\tau_1, \tau_2, ... \in \mathbb{R}_+$ satisfy the recurrent relation

$$\sum_{j=1}^{k} (\sum_{s=j}^{k} \tau_s)^{-1/2} = 1 \ (k = 1, 2, ...).$$

The relation $u(x, t_k) \to u_0$ as $k \to \infty$ uniformly on every finite interval follows from the relation $\tau_k \to \infty$ as $k \to \infty$; but the last relation does not proved yet. It is easy to prove by contradiction that, $\forall A, p > 0$, every of inequalities $\tau_k < Ak^{1+p}$, $\tau_k > Ak^{1-p}$ $(k = 1, 2, ...)$ has infinite number of solutions. The direct calculation of the first 30 values of τ_k shows their monotone increasing under approximately linear law.

6 Vibrations of the string with energy dissipation and impulsive support

The following boundary value problem for partial IDE is considered in [40]:

$$u_{tt} = a^2 u_{xx} - 2cu_t \ (0 \le x \le l, 0 \le t < \infty, E(t) \ne E_0), \quad (16)$$
$$u(0,t) = u(l,t) = 0 \ (0 \le t < \infty),$$

$$E(t) = E_0 \Rightarrow (u(x,t^+) = u(x,t),\ u_t(x,t^+) = u_t(x,t^-) + \beta(x) \ (0 \le x \le l)). \quad (17)$$

Here $a, c, l, E_0 > 0$ are the given constants,

$$E(t) := \int_0^l (a^2 u_x^2 + u_t^2) dx/2,$$

and $\beta \in C^1[0,l]$ is the given function, for which

$$\beta(0) = \beta(l) = 0,\ \int_0^l \beta^2(x) dx > 4E_0.$$

Then, the solution of IDE (17),(17) has the infinite sequence of impulses at $t = t_1(>0) < t_2 < ...$, when the initial conditions for u ensure the existence of classical solution of the (17), as we shall suppose, and $E(0) > E_0$; the solution of (17),(17) is the classical solution of (17) on every rectangle $[0,l] \times [t_{k+1}, t_k]$ $(t_0 := 0)$. The method of investigation of solution of (17),(17) is similar to that in Sec.5 in connection with IDE (14),(15).

Theorem 12. If the initial conditions for u are given, and $E(0) > E_0$) then

$$\Delta_0 \ge (2c)^{-1} \ln[E(0)/E_0],\ \Delta_k \ge (2c)^{-1} \ln[(2E_0)^{-1} \int_0^l \beta^2(x) dx - 1] \ (k \ge 1),$$

$$\sup_k \Delta_k < \infty,\ (\Delta_k := t_{k+1} - t_k).$$

Theorem 13. Let $\beta' \in AC[0,l]$ and $\beta'' \in L^2[0,l]$. Then $\forall \Delta > 0, \exists!$ (up to arbitrary translation in t) the simple Δ-periodic solution of the IDE (17),(17). Moreover, E_0 is defined uniquely by the formula

$$E_0 = \frac{l}{2} e^{-2c\Delta} \sum_{j=1}^{\infty} \{[\frac{j\pi a\Delta}{l} \frac{\sin \omega_j \Delta}{\omega_j \Delta}]^2 + [\cos \omega_j \Delta - c\Delta \frac{\sin \omega_j \Delta}{\omega_j \Delta} - e^{-c\Delta}]^2\} (\frac{\beta_j}{D_j})^2,$$

where

$$\omega_j := [(j\pi a/l)^2 - c^2]^{1/2},\ D_j := 1 - 2e^{-c\Delta} \cos \omega_j \Delta + e^{-2c\Delta},$$

$$\beta_j := (2/l) \int_0^l \beta(x) \sin(j\pi x/l) dx,$$

and $(\sin z)/z$ is considered as analytic function for all $z \in \mathbb{C}$.

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