Resenhas IME-USP 1997, Vol. 3, No. 1, 55 - 72.

Attracting Manifolds for Evolutionary Equations

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1. Introduction. We are interested in the dependence of compact global attractors upon the parameters in evolutionary equations described by ODE, PDE and even FDE. We formulate the problem in a general way and concentrate primarily on PDE. Spatial discretization of the PDE will lead to ODE which have similar properties. Many applications contain delays in the vector fields describing the dynamics and the ideas apply equally well to these.

Let Λ be a subset of a locally compact Banach space, let X be a Banach space and, for any $\lambda \in \Lambda$, let $T_\lambda(t) : X \to X, t \geq 0$, be a C^0-semigroup of transformations which is dissipative and has a compact global attractor $\mathcal{A}_\lambda$; that is, $\mathcal{A}_\lambda$ is a compact invariant set which is invariant ($T_\lambda(t)\mathcal{A}_\lambda = \mathcal{A}_\lambda, t \geq 0$) and, for any bounded set $B \subset X$, dist $(T_\lambda(t)B, \mathcal{A}_\lambda) \to 0$ as $t \to \infty$. Let $M \subset X$ be a C^1-manifold and let $S \subset \Lambda$. We say that M is an *attracting manifold for* $T_\epsilon(t)$ *at* S if, for any $\epsilon > 0$, there is an open neighborhood U of S such that, for any $\lambda \in U \cap \Lambda$, we have dist $(\mathcal{A}_\lambda, M) < \epsilon$.

If S is compact and M is an attracting manifold for $T_\lambda(t)$ at S and $\lambda_0 \in S$, then $\mathcal{A}_{\lambda_0} \subset M$. If, in addition, M is invariant for $T_{\lambda_0}(t)$; that is, $T_\lambda(t)M = M$, then M is called an *inertial manifold* for each $\lambda \in S$. We remark that the dimension of M is not assumed to be finite.

If it happens that $\mathcal{A}_\lambda \subset M$ for $\lambda \in S$, then we also will use the term that $T_\lambda(t)$ is *synchronized with respect to* M *for each* $\lambda \in S$.

If we suppose that M is invariant under $T_\lambda(t)$ for $t \geq 0, \lambda \in \Lambda$, then synchronization with respect to S is concerned only with the global asymptotic stability of M for $\lambda \in S$. For systems for which the semigroup can be respresented in a natural way as a linear part $T^L_\lambda(t)$ plus a nonlinear part and M is invariant under $T^L_\lambda(t)$, the verification of this stability property in many applications proceeds by the verification of the following steps:

(1) the introduction of new coordinates $X = M \oplus M^\perp$, where $M^\perp$ also is invariant under the linear part of $T^L_\lambda(t)$,
(2) obtaining uniform bounds on $\mathcal{A}_\lambda$ for $\lambda \in U \cap \Lambda$, where U is an open neighborhood of S,
(3) showing that there exist β, $\alpha_\lambda > 0$ such that

$$\|T^L_\lambda(t)|M^\perp\| \leq \beta e^{-\alpha_\lambda t}, \quad t \geq 0,$$

and $\alpha_\lambda, \lambda \in S$, can be made large by an appropriate choice of the set S.

If $T_\lambda(t)x$ is continuous with respect to λ uniformly with respect to t, x in bounded sets, there is an open set $U_0 \supset S$ such that $\{\mathcal{A}_\lambda, \lambda \in U_0 \cap \Lambda\}$ is bounded and M is an attracting manifold for $T_\lambda(t)$ at S, then, for any $\lambda_0 \in S$, there is a neighborhood V of λ_0 such that $\{\mathcal{A}_\lambda, \lambda \in V \cap S\}$ is upper semicontinuous at λ_0. If S is compact, this implies that, for any $\epsilon > 0$, there is a neighborhood U of S

such that

$$\operatorname{dist}(\{\mathcal{A}_\lambda, \lambda \in U \cap \Lambda\}, \{\mathcal{A}_\lambda, \lambda \in S\}) < \epsilon.$$

If $T_\lambda(t)x$ is continuous with respect to λ uniformly with respect to t, x in bounded sets, there is an open set $U_0 \supset S$ such that $\{\mathcal{A}_\lambda, \lambda \in U_0 \cap \Lambda\}$ is bounded, $S = \{\lambda_0\}$ and M is an attracting manifold for $T_\lambda(t)$ at λ_0, then $\{\mathcal{A}_\lambda, \lambda \in U_0 \cap \Lambda\}$ is upper semicontinuous at λ_0 and $\mathcal{A}_{\lambda_0} \subset M$. Conversely, if $\mathcal{A}_{\lambda_0} \subset M$ and $\{\mathcal{A}_\lambda, \lambda \in U_0 \cap \Lambda\}$ is upper semicontinuous at λ_0, then M is an attracting manifold for $T_\lambda(t)$ at λ_0.

Therefore, if $S = \{\lambda_0\}$, then a natural way to prove that M is an attracting manifold for $T_\lambda(t)$ at λ_0 is to prove the upper semicontinuity of the attractors at λ_0.

In some of the applications that we will mention below, the difficulties involved are the identification of the semigroup λ_0 and proving the above type of continuity with respect to λ at λ_0.

It often is the case that the semigroup at λ_0 is obtained from a semigroup defined on another space because there is some type of degeneracy that occurs at λ_0. There is an abstract setup for our examples. It is necessary first to identify the limit problem on the other space. Suppose that this has been done. More specifically, let $\lambda_0 \in \Lambda$ be fixed and suppose that there is a C^0-semigroup $\tilde{T}_{\lambda_0}(t) : \tilde{X} \to \tilde{X}$, where $\tilde{X}$ is a Banach space, and suppose that $\tilde{A}_{\lambda_0}$ is the compact global attractor. It is then necessary to imbed the flow defined by this semigroup into the space X. In general, the space $\tilde{X}$ is not as big as X, but it is possible to show that there is a C^1-manifold M in X and and a homeomorphism $h : \tilde{X} \to M$ with the additional property that this induces a C^0-semigroup $T_{\lambda_0}(t)$ on M and $hT_{\lambda_0}(t) = \tilde{T}_{\lambda_0}(t)h$. We then have a satisfactory arrangement for checking if M is an attracting manifold for $T_\lambda(t)$ at λ_0.

We illustrate how these concepts are related to results that have been obtained in the literature for several different types of problems. The hope is that a rephrasing with the above terminology will clarify, unify and lead to the formulation of other problems of a similar nature. We do not attempt to give a complete bibliography and the reader should refer to the references in the quoted papers.

2. Examples of synchronization. In this section, we give various examples illustrating the concept of synchronization and the manner in which dissipation leads to synchronization. In some cases, all of the necessary dissipation is due to diffusive coupling and, in other cases, it arises as a combination of diffusive coupling and the internal damping in the systems themselves. All of the examples are chosen from PDE defined on a bounded domain Ω with a smooth boundary. If the reader does not like PDE, then it is sufficient to think of the Laplacian as the discretized Laplacian and the unknown functions to be vector functions in a finite dimensional space. In this case, everything is an ODE.

2.1. Well mixing in reaction diffusion equations. Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded set with a smooth boundary, Δ is the Laplacian operator and $f : \mathbb{R} \to \mathbb{R}$ is a C^2-function. We consider the reaction diffusion equation with

Neumann boundary conditions:

$$\begin{aligned} \partial_t u &= \alpha^{-1}\Delta u + f(u) \quad \text{in } \Omega, \\ \partial_n u &= 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{2.1}$$

We suppose that f is *dissipative* (with respect to the Neumann boundary conditions); that is, there is a constant $\delta > 0$ such that

$$\limsup_{|u|\to\infty} \frac{f(u)}{|u|} \le -\delta < 0. \tag{2.2}$$

If we let $X = H^1(\Omega)$, then it is well known that (2.1) is a gradient system, there is a compact global attractor $\mathcal{A}_\alpha$ and the set $\mathcal{A}_\alpha, \alpha > 0$, is uniformly bounded in X (see, for example, Hale (1988)).

If we let

$$M = \{\varphi \in X : \varphi \text{ is independent of } x\},$$

then M is invariant under the flow defined by (2.1).

We extend the parameter range to $[0,\infty)$ by defining the system for $\alpha = 0$ to be the ODE

$$\dot{\xi} = f(\xi),$$

which has a compact global attractor $\mathcal{A}_0$. It is easy to prove the following result of Conway, Hoff and Smoller (1978), Hale (1986).

Theorem 2.1. *There is a $\alpha_0 > 0$ such that, for $0 \le \alpha \le \alpha_0$, we have $\mathcal{A}_\alpha = \mathcal{A}_0 \subset M$; that is, (2.1) is synchronized with respect to M for each $\alpha \in [0, \alpha_0]$.*

Proof. If we define

$$M^\perp = \{\varphi \in X : \int_\Omega \varphi = 0\},$$

then $X = M \oplus M^\perp$ and the linear subspaces M and $M^\perp$ are invariant under the flow defined by the linear part $\partial_t u - \alpha^{-1}\Delta u = 0$ of (2.1). If we designate Δ_N the Laplacian with Neumann boundary conditions and if we let $\sigma(\Delta_N)$ denote the set of eigenvalues of Δ_N, then

$$\begin{aligned} \sigma(\Delta_N) &= \{\mu_0 = 0 > -\mu_1 \ge -\mu_2 \ge \ldots \to -\infty\} \\ \sigma(\alpha^{-1}\Delta_N|M^\perp) &= \{-\alpha^{-1}\mu_j, j = 1, 2, \ldots \to -\infty \text{ as } \alpha \to 0\}. \end{aligned} \tag{2.3}$$

If, in (2.1), we make the change of variables

$$u = v + w, \quad v \equiv m(u) \equiv |\Omega|^{-1}\int_\Omega u \in M, \quad w \in M^\perp$$

then

$$\begin{aligned}\partial_t w &= \alpha^{-1}\Delta w + f(v+w) - m(f(v+w)) \\ &= \alpha^{-1}\Delta w + h(v,w)w\end{aligned}$$

and the function $h(v,w)$ is uniformly bounded in neighborhood of $\cup_{\alpha\geq 0}\mathcal{A}_\alpha$. From (2.3), this implies that there is an α_0 such that, for $\alpha \in [0,\alpha_0]$, the function $w \to 0$ as $t \to \infty$ if $v + w \in \mathcal{A}_\alpha$. Since $\mathcal{A}_\alpha$ is invariant, it follows that $w = 0$ on $\mathcal{A}_\alpha$ and, thus, $\mathcal{A}_\alpha = \mathcal{A}_0 \subset M$. This proves the result.

2.2. Partial diffusion. In this subsection, we give an example where there is diffusion in only some of the components and yet it is possible to conclude that the system is synchronized with respect to a manifold M as a result of other dissipative terms in the system. More specifically, we consider the Fitzhugh-Nagumo equations

$$\begin{aligned}\partial_t u - \alpha^{-1}\Delta u &= f(u) - v \\ \partial_t v &= -\gamma(v - \delta u) \text{ in } \Omega\end{aligned} \tag{2.4}$$

with Neumann boundary conditions, where $\gamma > 0, \delta > 0$ are constants and f is dissipative.

It is known (see, for example, Marion (1989)) that there is a compact global attractor $\mathcal{A}_\alpha \subset X = H^1(\Omega) \times L^2(\Omega)$ and the set $\mathcal{A}_\alpha, \alpha \in (0,\alpha_0]$ is uniformly bounded in X.

We extend the parameter range to $[0,\infty)$ by defining the system for $\alpha = 0$ to be the ODE

$$\begin{aligned}\dot{\xi} &= f(\xi) - \eta \\ \dot{\eta} &= -\gamma(\eta - \delta\xi),\end{aligned}$$

which has a compact global attractor $\mathcal{A}_0$.

Again, if we let $M = \{\text{constant functions}\}$, then we can prove the following result.

Theorem 2.2. *There is an $\alpha_0 > 0$ such that (2.4) is synchronized with respect to M for $\alpha \in [0,\alpha_0]$.*

Proof. The proof of this result is a little more difficult than that of Theorem 2.1, but the ideas are the same. If we let $u = \xi + w$, $\xi = m(u)$, $\int_\Omega w = 0$, $v = \eta + z$, $\eta = m(v)$, $\int_\Omega z = 0$, then we can rewrite the equations as

$$\begin{aligned}\dot{\xi} &= m[f(\xi + w)] - \eta \\ \dot{\eta} &= -\gamma(\eta - \delta\xi) \\ \partial_t w &= \alpha^{-1}\Delta w + (I - m)(f(\xi + w)) - z \\ &= \alpha^{-1}\Delta w + h(\xi, w)w - z \\ \partial_t z &= -\gamma(z - \delta w)\end{aligned} \tag{2.5}$$

where the function $h(\xi, w)$ is uniformly bounded on the attractors.

We can now apply invariant manifold theory to (2.5) and show that there is an $\alpha_0 > 0$ and positive constants c_1, c_2 such that, for $\alpha \in [0, \alpha_0]$, there is an invariant manifold in a neighborhood of the attractors described by the function $w = p(\alpha, \xi, \eta, z)$ and the function p satisfies the estimate

$$|p(\alpha, \xi, \eta, z)| \le \frac{c_1}{\alpha^{-1} - c_2}|z|. \tag{2.6}$$

The equation for z on this manifold is given by

$$\partial_t z = -\gamma(z + p(\alpha, \xi, \eta, z)).$$

If we further restict α_0, then we note that $z \to 0$ as $t \to \infty$. Since the attractor is invariant, this implies that $z = 0$ on $\mathcal{A}_\alpha$. From (2.6), this implies that $w = 0$ on $\mathcal{A}_\alpha$. Therefore, we have completed the proof.

2.3. Synchronized oscillators. In this subsection, we discuss the possibility of synchronization of identical oscillators described by second order ODE which are coupled by diffusion. The ODE could correspond for example to a linearly damped, periodically forced Duffing equation which, as is well known, can exhibit chaotic behavior. The ODE also could correspond to the autonomous van der Pol equation for which the dynamics is simple and corresponds to a self excited oscillation. In order to achieve synchronization, we diffusively couple both the position and velocity. At the end of the subsection, we make a few remarks about the use of only position coupling.

For positive constants k, c, we consider the strongly damped nonlinear wave equation

$$\begin{aligned} &\partial_t^2 u = k^{-1}\Delta u + c^{-1}\Delta\partial_t u + f(t, u, \partial_t u) \quad \text{in } \Omega \\ &\partial_n u = 0 \quad \text{in } \partial\Omega \end{aligned} \tag{2.7}$$

where $f(t, u, v)$ is a smooth function which is either periodic in t or autonomous (independent of t).

We take the space of initial data for (2.7) to be $X = H^1(\Omega) \times L^2(\Omega)$. The basic existence theory of (2.7) is contained in Fitzgibbon (1981) and Massatt (1983). Massatt (1983) also has shown that, if f is independent of t, then the semigroup $T_{k,c}(t)$ generated by (2.7) is point dissipative and has a compact global attractor $\mathcal{A}_{k,c}$. If f is periodic in t, then the Poincaré map has a compact global attractor $\mathcal{A}_{k,c}$. We suppose that there are positive constants $\bar{k}$, $\bar{c}$ such that $\{\mathcal{A}_{k,c}, k \in (0, \bar{k}], c \in (0, \bar{c}]\}$ is uniformly bounded in X.

We extend the parameter range to include $(k, c) = (0, 0)$ by defining the system at this point to be the ODE

$$\ddot{\xi} = f(t, \xi, \dot{\xi}), \tag{2.8}$$

which has a compact global attractor $\mathcal{A}_{0,0}$.

As in the previous examples, we define

$$M = \{\text{constant functions in } X\}$$

and let $M^\perp = \{(\varphi, \psi) \in X : m(\varphi) = 0, m(\psi) = 0\}$. We then have $X = M \oplus M^\perp$.

The following result is due to Hale (1996).

Theorem 2.3. *There are constants $c_0 > 0$, $K > 0$, such that (2.7) is synchronized with respect to M in the sector $S = \{(k, c) : c \in (0, c_0], 0 < k \le Kc\}$. Furthermore, the attractor $\mathcal{A}_{k,c}$ is the attractor $\mathcal{A}_{0,0}$ for the ODE (2.8).*

Proof. The proof will be very similar to the proof in Section 2.1 for well mixing in the reaction diffusion equation. If we let $u = v + w$, $v = m(u)$, $w \in M^\perp$, then w is a solution of the equation

$$\begin{aligned} &\partial_t^2 w = k^{-1}\Delta w + c^{-1}\Delta\partial_t w + h(t, v, w, \partial_t v, \partial_t w)w \quad \text{in } \Omega \\ &\partial_n w = 0 \quad \text{in } \partial\Omega \end{aligned} \tag{2.9}$$

where $h(t, v, w, y, z)$ is bounded on a neighborhood of the set $\{\mathcal{A}_{k,c}, k \in (0, \bar{k}], c \in (0, \bar{c}]\}$.

To complete the proof of the theorem, we analyze the eigenvalues of the linear system

$$\partial_t^2 u - k^{-1}\Delta u - c^{-1}\Delta\partial_t u = 0 \tag{2.10}$$

with Neumann boundary conditions. If the eigenvalues of Δ_N consists of the set $\{\mu_j\}$, then the eigenvalues λ of (2.10) satisfy

$$\lambda^2 + c^{-1}\lambda\mu_m + k^{-1}\mu_m = 0.$$

The eigenvalue $\mu_0 = 0$ leads to a double eigenvalue $\lambda = 0$ which are the eigenvalues of the ODE $\partial_t^2 u = 0$. This is natural since the subspace M is invariant for (2.10) as well as (2.7).

If $\mu_m \neq 0$, then

$$2\lambda = -c^{-1}\mu_m\{1 \pm [1 - \frac{4c^2}{\mu_m k}]^{1/2}\}.$$

It is now easy to verify that for any $p > 0$ there exists constants $c_0 = c_0(p) > 0, K = K(p) > 0$ such that

$$\begin{aligned} &(+\text{ sign}) \quad \operatorname{Re}\lambda \le -p, \quad 0 < c \le c_0, k > 0, \\ &(-\text{ sign}) \quad \operatorname{Re}\lambda \le -p, \quad 0 < k \le Kc. \end{aligned}$$

With these estimates on the eigenvalues, we can proceed exactly as before to complete the proof of the theorem.

Remark 2.1. In making the above estimates on the eigenvalues, we have been obligated to take both k and c sufficiently small. The analysis does not work if either $c = 0$ (in this case, the linear equation (2.10) is conservative) or $k = 0$ (this constant was used to control the eigenvalues corresponding to the $-$ sign).

Remark 2.2. For the periodically forced Duffing equation with linear damping, it is possible to show that (2.7) can be synchronized with $c = 0$ provided that k exceeds a threshold value; that is, synchronization can be achieved by using only the wave equation. Such a result can be obtained because of the linear damping in the Duffing equation (Hale (1996)). This positive linear damping in the Duffing equation is necessary because otherwise the system with $c = 0$ has a first integral.

For the forced Duffing equation with zero damping, it seems as if synchronization should occur in a sector in k, c space as a consequence of the boundedness of the solutions of the ODE from KAM theory.

If f depends upon x; that is, the coupled oscillators are not identical, then synchronization in general cannot be achieved without using the constant c (see, for example, Hale (1996)).

Remark 2.3. For f corresponding to the van der Pol equation, we obtain synchronization by using both k and c. It is not known what happens when $c = 0$.

2.4. Oscillators described by transmission lines. Lossless transmission lines of finite length with circuitry across the lines is modeled by the telegraph equation with the boundary conditions reflected by Kirchhoff's laws. It has been known for some time (see, for example, Hale and Verduyn-Lunel (1993) for derivation and references) that the voltage (also the current) at one end of line must satisfy a neutral delay differential equation of the form

$$\frac{d}{dt}Dv_t = f(t, v_t) \tag{2.11}$$

where

$$\begin{aligned} v_t(\theta) &= v(t+\theta), -r \le \theta \le 0, \\ Dv_t &= v(t) - \int_{-r}^{0} d\mu(\theta)v(t+\theta), \end{aligned} \tag{2.12}$$

$r \ge 0$ is constant, the function $f : C([-r, 0] \to \mathbb{R}$ is smooth and the matrix function μ is of bounded variation and nonatomic at zero.

For simple circuitry corresponding to constant voltage input, a resistor, a capacitor and a diode, the functional D has the form

$$Dv_t = v(t) - qv(t-r), \tag{2.14}$$

where $|q| < 1$.

For a continuium of resistively coupled transmission lines, the corresponding model is a partail neutral functional differential equation (PNFDE) of the form

$$\frac{d}{dt}Du_t - k^{-1}\Delta Du_t = f(t, u_t) \quad \text{in } \Omega \tag{2.15}$$

with some type of boundary conditions and Du_t is defined by

$$(Du_t)(x) = u(t,x) - \int_{-r}^{0} d\mu(\theta)u(t+\theta, x), \quad x \in \Omega. \tag{2.16}$$

To be specific, let us assume Neumann boundary conditions. A natural space for the initial data for (2.15) is $X = C([-r,0], H^1(\Omega))$. It is shown in Hale (1994) that, for any $\varphi \in X$, there is a unique solution $u(t,\varphi)$ with $u(0,\varphi)(\theta) = \varphi(\theta)$ for $\theta \in [-r, 0]$. If we let $T(t)\varphi \in X$ be defined by $(T(t)\varphi)(\theta) = u(t+\theta, \varphi), \theta \in [-r, 0]$ and assume that all solutions are defined for $t \geq 0$, then $T(t) : X \to X$ is a C^0-semigroup with $T(t)\varphi$ being C^p in φ if f is C^p.

In the applications to transmission lines, the operator D is always *stable*; that is, there are positive constants β, α such that

$$Dw_t = 0 \Rightarrow \|w_t\|_{C([-r,0],\mathbf{R})} \leq \beta e^{-\alpha t}, \ t \geq 0. \tag{2.17}$$

If

$$M = \{\varphi \in X : \varphi \text{ is independent of } x\},$$

then the following result is proved in Hale (1996).

Theorem 2.4. *If we assume that D is stable and there is a compact global attractor $\mathcal{A}_k$ for (2.17) for each $k > 0$ and there is a $k_0 > 0$ such that the set $\mathcal{A}_k, 0 < k \leq k_0$, is bounded in X, then there is a $k_1 > 0$ such that (2.17) is synchronized with respect to M for $k \in (0, k_1]$. Furthermore, the attractor $\mathcal{A}_k$ is the attractor for the spatially independent equation (2.11).*

The proof follows along the same lines as in the previous proofs. The space X is first decomposed as $X \oplus X^\perp$ where $X^\perp = \{\varphi \in X : \int_\Omega \varphi(\cdot, x)dx = 0\}$. For the linear system

$$\frac{d}{dt}Du_t - k^{-1}\Delta Du_t = 0 \quad \text{in } \Omega \tag{2.18}$$

with Neumann boundary conditions, it is then shown that

$$\|T(t)|M^\perp\| \leq ke^{-\gamma(k)(t-s)}, t \geq s,$$

where $\gamma(k) \to \infty$ as $k \to 0$. The remainder of the proof is as before. See Hale (1996) for details.

2.5. Symmetric thin domains. Let us consider the system

$$\begin{aligned} &\partial_t u = \Delta u + f(u) \quad \text{in } \Omega_\epsilon = (0,1) \times (0,\epsilon) \\ &\partial_n u = 0 \quad \text{in } \partial\Omega_\epsilon \end{aligned} \tag{2.18}$$

where f is dissipative and $\epsilon > 0$ is a constant. If we consider (2.18) in the space $H^1(\Omega)$, then there is a compact global attractor $\mathcal{A}_\epsilon \subset X = H^1(\Omega_\epsilon)$.

We want to discuss the sets $\mathcal{A}_\epsilon$ as $\epsilon \to 0$. To do this, we scale Ω_ϵ to a fixed domain

$$\Omega_\epsilon \mapsto \Omega = (0,1) \times (0,1), \quad (x,y) \mapsto (x, \epsilon y)$$

to obtain the equation

$$\begin{aligned} &\partial_t u = L_\epsilon u + f(u) \quad \text{in } \Omega \\ &\partial_n u = 0 \quad \text{in } \partial\Omega \end{aligned} \tag{2.19}$$

where

$$L_\epsilon u = u_{xx} + \frac{1}{\epsilon^2} u_{yy}.$$

Equation (2.19) will be considered in the space X_ϵ which is the same as $H^1(\Omega)$ except we use the following norm which takes into account the scaling:

$$\|\varphi\|_{X_\epsilon} = (\|\varphi\|^2_{H^1(\Omega)} + \frac{1}{\epsilon^2}\|\varphi_y\|^2_{L^2(\Omega)})^{1/2}$$

It is not difficult to show that the set $\mathcal{A}_\epsilon, 0 < \epsilon \le \epsilon_0$, is uniformly bounded.

As $\epsilon \to 0$, the original domain approaches the line segment $(0,1)$ and there is a loss in dimension. It is thus very natural to compare $\mathcal{A}_\epsilon$ with the attractor $\tilde{\mathcal{A}}_0 \subset H^1(0,1)$ of the limit system

$$\begin{aligned} &\partial_t u = u_{xx} + f(u) \quad \text{in } (0,1), \\ &u_x = 0 \quad \text{at } x = 0,\, 1. \end{aligned} \tag{2.20}$$

There is a natural embedding of $\tilde{\mathcal{A}}_0 \subset H^1(0,1)$ into $\mathcal{A}_0 \subset X_\epsilon$. Hale and Raugel (1992) have proved the following result.

Theorem 2.5. *There is an $\epsilon_0 > 0$ such that, for $0 \le \epsilon \le \epsilon_0$, $\mathcal{A}_\epsilon = \mathcal{A}_0$; that is, for $\epsilon \in [0, \epsilon_0]$, system (2.18) is synchronized with respect to the linear manifold*

$$M = \{\varphi \in X_\epsilon : \varphi \text{ is independent of } y\}.$$

The proof is very similar to the previous ones. If we let

$$u = v + w, \quad v \equiv m(v) = \int_0^1 v(\cdot, y)dy, \; w \in M^\perp,$$

then

$$\begin{aligned} \partial_t v &= v_{xx} + m(f(v+w)) \\ \partial_t w &= L_\epsilon w + f(v+w) - m(f(v+w)) \\ &= L_\epsilon w + h(v,w)w \end{aligned}$$

and $h(v,w)$ is bounded on a neighborhood of $\cup_{\epsilon\in[0,\epsilon_0]}\mathcal{A}_\epsilon$. It is easy to check that

$$\sigma(L_\epsilon|M^\perp) = \frac{1}{\epsilon^2}\sigma(\Delta_N|M^\perp) \to -\infty \text{ as } \epsilon \to 0.$$

The remainder of the proof is the same as before.

3. Attracting manifolds. In this section, we discuss the diffusively coupling of systems which may not be identical. In this case, the functions in the PDE depend upon x. We do not expect to have synchronization and we turn to attracting manifolds. The spirit of the investigation is similar to the one used for synchronization. However, the tools from dynamical systems are different. We must combine the spectral theory with the theory of invariant manifolds in many of the cases. In some situations, we cannot use invariant manifold theory and can only show the upper semicontinuity of attractors. As remarked in the introduction, this is sufficient for our purposes.

3.1. Approximate well mixing in reaction diffusion equations. We consider the reaction diffusion system

$$\begin{aligned} \partial_t u &= \alpha^{-1}\Delta u + f(x,u) \quad \text{in } \Omega \\ \partial_n u &= 0 \quad \text{in } \partial\Omega \end{aligned} \tag{3.1}$$

where $\alpha > 0$ is constant, the function f is dissipative and, without loss of generality, we may suppose that $f(0) = 0$.

If $X = H^1(\Omega)$, then system (3.1) is gradient in X, defines a C^0-semigroup $T_\alpha(t)$ on X and there is a compact global attractor $\mathcal{A}_\alpha$ with the set $\mathcal{A}_\alpha, \alpha > 0$, being uniformly bounded.

We are interested in the limiting behavior of $\mathcal{A}_\alpha$ as $\alpha \to 0$. If we let

$$M = \{\varphi \in X : \varphi = \text{independent of } x\}, \quad M^\perp = \{\varphi \in X : \int_\Omega \varphi = 0\},$$

and make the change of variables

$$u = v + w, \quad v \equiv m(u) \equiv |\Omega|^{-1}\int_\Omega u \in M, \quad w \in M^\perp,$$

then (3.1) becomes

$$
\begin{aligned}
\partial_t v &= m(f(\cdot, v+w)) \\
\partial_t w &= \alpha^{-1}\Delta w + f(x, v+w) - m(f(\cdot, v+w)).
\end{aligned}
\tag{3.2}
$$

The vector field in the second equation in (3.2) does not vanish when $w = 0$ if $f(x, u)$ depends upon x; that is, M is not invariant for (3.1). On the other hand, we know that

$$
\begin{aligned}
\sigma(\Delta_N) &= \{\mu_0 = 0 > -\mu_1 \geq -\mu_2 \geq \ldots \to -\infty\} \\
\sigma(\alpha^{-1}\Delta_N | M^{\perp}) &= \{-\alpha^{-1}\mu_j, j = 1, 2, \ldots \to -\infty \text{ as } \alpha \to 0\}.
\end{aligned}
$$

As a consequence of this fact, we know that there is an α_0 and positive constants c_1, c_2 such that, for $0 < \alpha \leq \alpha_0$, there is an invariant manifold $\mathcal{M}_\alpha \supset \mathcal{A}_\alpha$ defined by $w = p(v, \alpha)$ and

$$
|p(v, \alpha)| \leq \frac{c_1 |v|}{\alpha^{-1} - c_2}. \tag{3.3}
$$

The flow on M_α is given by the ODE

$$
\partial_t v = m(f(\cdot, v + p(v, \alpha)) \to m(f(\cdot, v)) \text{ as } \alpha \to 0
$$

Therefore, it is natural to take the limit system at $\alpha = 0$ to be the ODE

$$
\dot{\xi} = |\Omega|^{-1} \int_\Omega f(x, \xi) dx. \tag{3.4}
$$

Equation (3.4) defines a semigroup $T_0(t)$ on M with a compact global attractor $\mathcal{A}_0$. Extend this semigroup to all of X in any way whatsoever.

We then have the following result in Hale (1986).

Theorem 3.1. *There is an $\alpha_0 > 0$ such that M is an attracting manifold for $T_\alpha(t)$ at 0. Furthermore, the attractors $\mathcal{A}_\alpha, \alpha \geq 0$, are upper semicontinuous at $\alpha = 0$.*

The last statement of the theorem is a consequence of (3.4) and (3.3).

3.2. Attracting manifolds for diffusively coupled oscillators. For positive constants k, c, we consider the strongly damped nonlinear wave equation

$$
\begin{aligned}
&\partial_t^2 u = k^{-1}\Delta u + c^{-1}\Delta\partial_t u + f(t, x, u, \partial_t u) \quad \text{in } \Omega \\
&\partial_n u = 0 \quad \text{in } \partial\Omega
\end{aligned}
\tag{3.5}
$$

where $f(t, x, u, v)$ can depend explicitly upon x (the coupled oscillators may not be identical), $f(t, x, u, v)$ is a smooth function which is either periodic in t or autonomous. Again, we take the space of initial data for (3.5) to be $X = H^1(\Omega) \times L^2(\Omega)$.

If f is independent of t, then we assume that the semigroup $T_{k,c}(t)$ generated by (2.7) is point dissipative and thus there is a compact global attractor $\mathcal{A}_{k,c}$. If f is periodic in t, then the Poincaré map has a compact global attractor $\mathcal{A}_{k,c}$. We suppose that there are positive constants $\bar{k}$, $\bar{c}$ such that $\{\mathcal{A}_{k,c}, k \in (0, \bar{k}], c \in (0, \bar{c}]\}$ is uniformly bounded in X.

We can proceed exactly as in the previous section taking into account the spectral analysis in Section 3.3 to see that the subspace M should be

$$M = \{\varphi \in X : \varphi = \text{ independent of } x\}$$

and the appropriate limiting equation is the ODE

$$\ddot{\xi} = |\Omega|^{-1} \int_\Omega f(t, x, \xi, \dot{\xi}) dx, \tag{3.6}$$

which has a compact global attractor $\mathcal{A}_{0,0}$.

The following result is due to Hale (1996).

Theorem 3.2. *There is a sectorial region $S = \{(k, c) : c \in (0, c_0], 0 < k \le Kc\}$ such that M is an attracting manifold for the system (3.5) at $(0, 0)$ in the sector S. Furthermore, the attractors $\mathcal{A}_{k,c}, (k, c) \in S \cup \{(0, 0)\}$, are upper semicontinuous at $(0, 0)$.*

3.3. Nonsymmetric thin domains. Consider the system

$$\begin{aligned} &\partial_t u = \Delta u + f(u) \quad \text{in } \Omega_\epsilon = \{0 < y < \epsilon g(x), x \in (0, 1)\} \\ &\partial_n u = 0 \quad \text{in } \partial\Omega_\epsilon \end{aligned} \tag{3.7}$$

where $g \in C^3$, $g > 0$ and f is dissipative. This system is gradient with a compact global attractor $\mathcal{A}_\epsilon \subset X = H^1(\Omega_\epsilon)$.

To study the behavior of the attractors as $\epsilon \to 0$, we rescale to a common domain

$$\Omega_\epsilon \mapsto \Omega = (0, 1) \times (0, 1), \quad (x, y) \mapsto (x, \epsilon g(x) y)$$

to obtain an equivalent system

$$\partial_t u = L_\epsilon u + f(u) \quad \text{in } \Omega \tag{3.8}$$

with appropriate Neumann boundary conditions in the space $X_\epsilon \simeq H^1(\Omega)$, the same space as in Section 2.5 with the same norm involving ϵ and defining the L^2 inner product with the weight function g.

If, as before, $M = \{\varphi \in X_\epsilon : \varphi \text{ is independent of } y\}$, then M is not invariant if the function g is not constant.

The first task is to find the correct limit system which will have a compact global attractor $\mathcal{A}_0$ that can be compared with the attractors $\mathcal{A}_\epsilon, \epsilon > 0$. Hale and Raugel (1992) have shown that the appropriate limit parabolic equation on the line is

$$\begin{aligned} \partial_t u &= \frac{1}{g}(g u_x)_x + f(u) \quad \text{in } (0,1) \\ u_x &= 0 \quad \text{at } x = 0, 1 \end{aligned} \tag{3.9}$$

Notice that the shape of the domain Ω_ϵ is reflected in the dispersive coefficient $(g'/g)u_x$.

If we let $\tilde{\mathcal{A}}_0 \subset H^1(0,1)$ be the compact global attractor for (3.9), then there is a natural map of $\tilde{\mathcal{A}}_0$ into a compact set $\mathcal{A}_0 \subset X_\epsilon$. It is not difficult to show that the sets $\mathcal{A}_\epsilon, 0 \le \epsilon \le \epsilon_0$ are uniformly bounded.

The following result has been proved by Hale and Raugel (1992).

Theorem 3.3. *The sets $\mathcal{A}_\epsilon, 0 \le \epsilon \le \epsilon_0$, are upper semicontinuous at $\epsilon = 0$ and, thus, the manifold M is attracting for (3.8) at $\epsilon = 0$.*

It does not seem to be possible to give a proof of this result using invariant manifold theory as in the previous examples. The difficulty arises because the limit system is a PDE. The linear part of the limit system has a fixed set of eigenvalues which approach $-\infty$. The original system has eigenvalues, some of which approach those of the limit system as $\epsilon \to 0$ and others that approach $-\infty$ because $\epsilon \to 0$. This seems to make it almost impossible to prove the existence of any invariant manifold of infinite dimension for which the flow is a parabolic PDE in one space dimension which approaches the limit system as $\epsilon \to 0$. It is for this reason that it is shown directly that the attractors are upper semicontinuous at $\epsilon = 0$.

3.4. Shadow systems. In this subsection, we consider systems of reaction diffusion equations for which some of the diffusion coefficients are very large and the others are fixed. It is natural to expect that the ones with large diffusion should behave as their average over the domain. We will make this precise.

For $d > 0$, $\alpha > 0$ constant, we consider the system

$$\begin{aligned} \partial_t u &= d\Delta u + f(u,v) \\ \partial_t v &= \alpha^{-1}\Delta v + g(u,v) \quad \text{in } \Omega \end{aligned} \tag{3.10}$$

with homogeneous Neumann boundary conditions. The functions f, g are assumed to be smooth. We consider d fixed and investigate some properties of the solutions as $\alpha \to 0$.

We assume that there is a compact global attractor $\mathcal{A}_\alpha$ for each $\alpha > 0$ and the set $\mathcal{A}_\alpha, \alpha > 0$, is uniformly bounded in $X = H^1(\Omega) \times H^1(\Omega)$. Let

$$M = \{(\varphi, \psi) \in X : \psi \text{ is independent of } x\}.$$

As $\alpha \to 0$, it is natural to expect that the limit system should be

$$
\begin{aligned}
\partial_t u &= d\Delta u + f(u,\xi) \quad \text{in } \Omega \\
\partial_t \xi &= m(g(u,\xi)),
\end{aligned}
\tag{3.11}
$$

a PDE coupled with an ODE, which in the terminology of Nishiura (1982) is referred to as the *shadow system*. The shadow system induces a flow on M which has a compact global attractor $\mathcal{A}_0$.

The following result is due to Hale and Sakamoto (1989).

Theorem 3.4. *The mainifold M is attracting for (3.10) at $\alpha = 0$ and the sets $\mathcal{A}_\alpha, \alpha \geq 0$ are upper semicontinuous at $\alpha = 0$.*

The proof is an application of invariant manifold theory.

3.5. Large diffusion in subdomains. In this subsection, we consider a reaction diffusion equation with variable diffusion with the diffusion coefficient being large on most of the region but dips to small values as one passes from one part of the region to another. Rather than give the most general result, we will discuss only a simple case and refer the reader to the original papers.

Consider the system

$$
\begin{aligned}
\partial_t u &= (a_\nu(x)u_x)_x + f(u) \quad \text{in } (0,1) \\
u_x &= 0 \quad \text{at } 0,1
\end{aligned}
\tag{3.12}
$$

where $\nu > 0$ is a real parameter, $a_\nu > 0$ is continuous and f is dissipative.

This system is gradient and there is a compact global attractor $\mathcal{A}_\nu \subset H^1(0,1)$.

To describe the function a_ν, fix $x_1 \in (0,1)$ (we could take any finite partition of $(0,1)$ and obtain similar results as below) and let

$$
\begin{aligned}
&S_\nu \subset (0,x_1) \cup (x_1,1) \text{ compact}, \quad \lim_{\nu \to 0} S_\nu = [0,1] \\
&\lim_{\nu \to 0} a_\nu | S_\nu = \infty, \quad \lim_{\nu \to 0} a_\nu | S_\nu^c = 0
\end{aligned}
$$

where S_ν consists of two open intervals and S_ν^c is the complement of S_ν. One must also impose some technical conditions on the the function a_ν. More precisely, suppose that b, ℓ, e are positive constants and let b', ℓ' be functions of ν which approach b, ℓ from above as $\nu \to 0$. The function a_ν is defined in the following way:

$$
\begin{aligned}
a_\nu(x) &\geq \frac{e}{\nu} & x &\in [0, x_1 - \nu\ell'] \cup [x_1 + \nu\ell', 1] \\
a_\nu(x) &\geq \nu b & x &\in [x_1 - \nu\ell', x_1 + \nu\ell'] \\
a_\nu(x) &\leq \nu b' & x &\in [x_1 - \nu\ell, x_1 + \nu\ell]
\end{aligned}
$$

For this problem, we expect that a solution on $\mathcal{A}_\nu$ will be approximately its average on $(0,x_1)$ and $(x_1,1)$ with some type of coupling between them.

To make this precise, let

$$(3.13)\qquad \begin{aligned} &v_1 \in H^1(0,x_1), \quad v_1(\cdot,x) = u(\cdot,x),\ x \in [0,x_1] \\ &v_2 \in H^1(x_1,1), \quad v_2(\cdot,x) = u(\cdot,x),\ x \in [x_1,1] \end{aligned}$$

to obtain a pair of equations

$$(3.14)\qquad \begin{aligned} \partial_t v_1 &= (a_\nu(x)v_{1x})_x + f(v_1) \quad \text{in } (0,x_1) \\ \partial_t v_2 &= (a_\nu(x)v_{2x})_x + f(v_2) \quad \text{in } (x_1,1) \end{aligned}$$

with the matching and boundary conditions

$$(3.15)\qquad \begin{aligned} &v_{1x}(0) = 0, \quad v_{2x}(0) = 0 \\ &v_1(x_1) = v_2(x_1), \quad v_{1x}(x_1) = v_{2x}(x_1). \end{aligned}$$

If we let

$$X \equiv H^1(0,x_1) \times H^1(x_1,1) \subset Y \equiv L^2(0,x_1) \times L^2(x_1,1)$$

then the set $\{\mathcal{A}_\nu \subset X \subset Y, \nu > 0\}$ is uniformly bounded. If we let

$$M = \{\text{constant functions}\},$$

then we can state the following result of Fusco (1987), Carvalho and Perreira (1994).

Theorem 3.5. *The manifold M is attracting in Y for (3.12) at $\nu = 0$.*

The proof uses the fact that there is a constant β such that the set of eigenvalues of the linear part contains $\lambda_1 = 0 > \lambda_2(\nu) \to \beta$ as $\nu \to 0$, $\lambda_j \to -\infty$ as $\nu \to 0$, $j \geq 3$. One can then apply invariant manifold theory to obtain an invariant two dimensional manifold containing the attractor. The ODE describing the flow on this two dimensional manifold (in an appropriate coordinate system) can be shown to converge to the limit system

$$(3.16)\qquad \begin{aligned} \dot{z}_1 &= f(z_1) + \mu_1(z_2 - z_1) \\ \dot{z}_2 &= f(z_2) - \mu_2(z_2 - z_1) \end{aligned}$$

and μ_1, μ_2 are positive constants defined explicitly in terms of the parameters defining the diffusion coefficient a_ν as

$$\mu_1 = \frac{b}{2\ell x_1}, \quad \mu_2 = \frac{b}{2\ell(1-x_1)}.$$

By varying the parameters defining a_ν, the constants μ_1, μ_2 can take on any positive values.

Also, z_1, z_2, respectively, are approximately the average of v_1, v_2 if $(v_1, v_2) \in \mathcal{A}_\nu$ and ν is small.

The following result is also true.

Theorem 3.6. *The limit system has a compact global attractor $\mathcal{A}_0 \subset Y$ and $\{\mathcal{A}_\nu\}$ is upper semicontinuous at $\nu = 0$ in the topology of Y.*

Carvalho and Cunimato (1994) have an extension of these results to the case where $\Omega \subset \mathbb{R}^N$ is divided into a finite number of subdomains with the diffusion coefficient being large on most of the subdomains and small at the boundaries.

3.6. Singularly perturbed hyperbolic equation. Consider the equation

$$\begin{aligned} \epsilon^2 \partial_t^2 u + \partial_t u &= \Delta u + f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{aligned} \tag{3.17}$$

where $\epsilon > 0$ is a parameter and f is a dissipative C^2 function satisfying

$$\limsup_{|u|\to\infty} \frac{f(u)}{|u|} < \mu_1$$

where μ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Let

$$X = H_0^1(\Omega) \times L^2(\Omega), \quad Y = (H^2(\Omega) \cap H_0^1(\Omega)) \times H^1(\Omega).$$

If we assume that the function f satisfies growth conditions at infinity which will ensure that $f : H_0^1(\Omega) \to L^2(\Omega)$ is compact, then system (3.17) is gradient in X, there exists a compact global attractor $\mathcal{A}_\epsilon \subset X$ and, furthermore, the set $\mathcal{A}_\epsilon \subset Y$ and the sets $\{\mathcal{A}_\epsilon, 0 < \epsilon \le \epsilon_0\}$ are uniformly bounded in Y. For a proof of these results, see, for example, Hale and Raugel (1990).

The limit system for (3.17) is the parabolic equation

$$\begin{aligned} \partial_t u &= \Delta u + f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{aligned} \tag{3.18}$$

in $H_0^1(\Omega)$. For (3.18), there is a compact global attractor $\tilde{\mathcal{A}}_0 \subset H^1(\Omega)$ and there is a natural map of $\tilde{\mathcal{A}}_0$ into a set $\mathcal{A}_0 \subset M \subset X$ where

$$M = \{(\varphi, \psi) \in X : \psi = \Delta\varphi + f(\varphi), \varphi \in H_0^1(\Omega)\}$$

The set M is a C^1 manifold in X (in general, not linear). The following result is due to Hale and Raugel (1990).

Theorem 3.7. *$\{\mathcal{A}_\epsilon, 0 \le \epsilon \le \epsilon_0\}$ is upper semicontinuous at $\epsilon = 0$. Thus, the manifold M is attracting for equation (3.17) at $\epsilon = 0$.*

For the same reasons as described in Section 3.3 on nonsymmetric thin domains, it does not seem to be possible to use invariant manifold theory to obtain a submanifold with the flow described by a parabolic equation which converges to (3.18) as $\epsilon \to 0$.

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