Delay-Induced Transient Oscillations in a Two-Neuron Network

K. Pakdaman, C. Grotta-Ragazzo, C.P. Malta and J.-F. Viber

Abstract: Finite transmission times between neurons, referred to as delays, may appear in hardware implementation of neural networks. We analyze the dynamics of a two-neuron network in which the delay modifies the transient and not the long-term behavior of the network. We show that the delay causes some trajectories to oscillate transiently before reaching stationary behavior and the duration of these transients increases exponentially with the delay. Such a phenomenon deteriorates network performance.

Key words: Continuous time neural network, Nonlinear graded response neuron, Transient regime, Delay Oscillation.

1 Introduction

In many neural network applications, "information" is stored as stable equilibria of a convergent or almost convergent system (Hopfield, 1984; Hirsch, 1989). Thus, a given information is retrieved by initializing the network at a point within the basin of attraction of the corresponding equilibrium point and letting the system reach its steady state.

Finite inter-unit transmission times, referred to as delays, present in hardware implementation of neural networks, can interfere with information retrieval in three ways. *i)* Delays may cause a stable equilibrium point to become unstable, thus rendering the retrieval of the stored information impossible. *ii)* The network with delay may have attractors that are not present in the system without delay (Marcus *et al.*, 1991; Gilli, 1993; 1995). For initial conditions (ICs) within the basin of these attractors, the network activity displays sustained oscillations and no information is retrieved. *iii)* The basin of attraction of the stable equilibria and, consequently, the classification of ICs performed by the network, can be altered by the delay (Pakdaman *et al.*, 1995a; 1995b).

These can be avoided if the following three respective properties hold: (P1) *local stability of all stable equilibria is preserved in presence of delays* (Marcus & Westervelt, 1989; Bélair, 1993; Ye *et al.*, 1994a; 1994b), (P2) *the network with delay is convergent or almost convergent* (Bélair, 1993; Ye *et al.*, 1994a; 1994b; Burton, 1991; 1993; Civalleri *et al.*, 1993; Gopalsamy and He, 1994a; 1994b; Roska & Chua, 1992; Roşka *et al.*, 1992; 1993), (P3) *for constant initial functions, the basins of attractions of stable equilibria are independent of the delay.*

The dynamics of a two-neuron network satisfying (P1), (P2) and (P3), is studied. It is shown that even when the steady state is unaffected by the delay, information retrieval may deteriorate due to considerable lengthening of the transient regime duration.

2 The model

The dynamics of two identical nonlinear graded response neurons (NGRNs) (Hopfield, 1984) connected to each other by symmetric positive weights $W > 0$ and delays $A > 0$, are determined by the following delayed differential equations (DDEs):

$$\begin{cases} \frac{dx}{dt}(t) = -x(t) + W\sigma_\alpha(y(t-A)) \\ \\ \frac{dy}{dt}(t) = -y(t) + W\sigma_\alpha(x(t-A)) \end{cases} \tag{1}$$

$$\text{where} \quad \sigma_\alpha(x) = \tanh(\alpha x) = \frac{e^{\alpha x} - e^{-\alpha x}}{e^{\alpha x} + e^{-\alpha x}} \quad \text{for } 0 < \alpha < \infty,$$
$$\text{and} \quad \sigma_\infty(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x \leq 0 \end{cases} \quad \text{for } \alpha = \infty$$

For $\alpha = \infty$, there are two locally asymptotically stable equilibria, $r_1 = (-W, -W)$ and $r_3 = (W, W)$ (Fig. 2a). The basins of attraction of r_1 and r_3 for constant initial conditions are $\{(u,v) \in \mathbb{R}^2,\ u < -v\} \bigcup \{(0,0)\}$ and $\{(u,v),\ u > -v\}$ respectively. Thus, (P1) and (P2) are satisfied, and (P3) holds for r_3.

For $0 < \alpha < \infty$, the positive feedback condition ($W > 0$) (Smith, 1987; Roska *et al.*, 1992; Pakdaman *et al.*, 1995a) and the invariance of DDEs (1) under the transformations $x \to -x$, $y \to -y$ and $x \to y$, $y \to x$, imply the following. For $0 < \alpha W < 1$, $r_0 = (0,0)$ is a globally asymptotically stable equilibrium point. For $1 < \alpha W < \infty$, there are one unstable ($r_2 = (0,0)$) and two locally asymptotically stable ($r_1 = (-a,-a)$, $r_3 = (a,a)$) equilibrium points (a is the strictly positive solution of: $-x + W\tanh(\alpha x) = 0$). The basins of attraction of r_1 and r_3 for constant initial conditions are $\{(u,v),\ u < -v\}$ and $\{(u,v),\ u > -v\}$ respectively. Thus, for $0 < \alpha < \infty$, (P1), (P2) and (P3) are satisfied.

3 Transient regime

In this section, the transient regime for constant ICs $r = (u,v)$ with $v > -u \geq 0$ is studied. The transient regime refers to the dynamics before the system stabilizes to its steady state. Practically, the transient regime ends when the state of the system cannot be distinguished from the equilibrium point with some given precision η. We denote by $T(r,A)$ *the transient regime duration* (TRD) of a solution $z(t,r) = (x(t,r), y(t,r))$ of DDEs (1) with IC r. $z(t,r) = (x(t,r), y(t,r))$ has a zero at time t if $x(t,r) \times y(t,r) = 0$, and we denote by $N(r,A)$ the number of zeros of $z(t,r)$.

Case of $\alpha = \infty$. For $\alpha = \infty$, solutions are characterized by iterates of a one-dimensional map (appendix A), which yields the following result.

There is a sequence $v_1(A) > v_2(A) > \ldots > v_k(A) > \ldots > 0$, tending to zero as $k \to \infty$, such that for an integer p:

$$N(r,A) = \begin{cases} 1 & v > v_1 - (1+\frac{v_1}{W})u \\ 2p\ (p \geq 1) \text{ and } T(r,A) \geq pA \text{ for} & v = v_p - (1+\frac{v_p}{W})u \\ 2p+1\ (p \geq 1) & v_{p+1} - (1+\frac{v_{p+1}}{W})u < v < v_p - (1+\frac{v_p}{W})u \end{cases}$$

The temporal evolutions and trajectory of a solution with 29 zeros are represented in Figs.1a,b, respectively, showing the oscillatory transient prior to stabilization at r_3. In Fig. 2a, dotted lines correspond to ICs r with even $N(r,A)$ indicated on the line, and regions between two consecutive lines correspond to ICs r with the odd $N(r,A)$ indicated. From the description of the trajectories it can be derived that the TRD increases with the number of zeros. This is illustrated in Fig. 2b showing the TRD for ICs $(-10^{-3}, v)$. Each "hump" (for $A = 2$ and $A = 3$) corresponds to ICs that have the same number of zeros. For example, the humps indicated by the arrows correspond to the TRD of solutions with three zeros.
Furthermore, for a fixed IC $r = (u,v)$ $(v > -u \geq 0)$, there is an unbounded sequence of delays $0 < A_1 < A_2 < \ldots < A_k < \ldots$, such that $z(t,r)$ has exactly 1, $2p$ or $2p+1$ zero(s) for $A < A_1$, $A = A_p$ or $A_p < A < A_{p+1}$, respectively. Thus, for large enough delays, $N(r,A)$, and consequently $T(r,A)$ are increasing functions of A. The expression of $v_n(A)$ (appendix A), indicates that the rate of increase is exponential. This is in accord with numerical results as exemplified by the dotted line in Fig. 3a.

Case of $1 < \alpha W < \infty$. ICs $r = (u,v)$ with $u = -v$ are on the boundary separating the basins of attraction of r_1 and r_3. Solutions of these ICs satisfy a scalar delay differential equation with negative feedback:

$$\begin{cases} \frac{dx}{dt}(t) = -x(t) - W\tanh(\alpha x(t-A))) \\ y(t) = -x(t) \end{cases} \tag{2}$$

Thus, for large enough delays ($A > \frac{1}{\sqrt{\alpha^2 W^2 - 1}}\arccos(\frac{-1}{\alpha W})$) solutions of constant initial conditions $r = (u,v)$ with $u = -v \neq 0$, tend to periodic oscillations (Walther, 1995). The continuous dependence of solutions on ICs for finite α implies that, for large enough delays, solutions of (1) close to the boundary, display transient oscillations before converging. The closer the IC is to the boundary, the longer the duration of the transient oscillations. Figure 3b represents the TRD for $\alpha = 2.5$ and for three delay values ($A = 0.1$, 2 and 3), for ICs $(-10^{-3}, v)$, with v ranging from 10^{-3} to 100. Solutions displaying transient oscillations correspond to ICs to the left of the sudden change of slope in the curves (around $v = 3$ for $A = 2$ and $v = 12$ for $A = 3$, indicated by arrows).

Figure 3a shows the TRD for a given IC $(u,v) = (-10^{-3}, 5)$ as function of the delay A, for three values of α (0.5, 1 and ∞). As the delay increases, the TRD undergoes an exponential increase in all three curves. For larger α, the increase in the TRD is faster. However, for a given α, the rapid increase in the TRD does not take place at the same delay value for all ICs.

In summary. For $1/W < \alpha \leq \infty$, we have shown that: *i)* For a fixed delay, there are ICs in the neighborhood of the boundary that display delay-induced oscillatory transients, the size of this neighborhood increases with the delay so that *ii)* for any IC (u, v) such that $v > -u > 0$ the TRD increases exponentially with the delay, due to the onset of transient oscillations.

4 Discussion and conclusion

We showed that due to the onset of delay-induced transient oscillations, the TRD of some ICs underwent rapid increase with the delay, even though the presence of delay did not alter the asymptotic behavior of the system.

An important issue is to determine whether a given network architecture is prone to delay-induced transient oscillations similar to those described in this paper. The lengthening of the TRD which results from the onset of oscillations can be detrimental to the network performance. This preliminary investigation suggests that delay-induced transient oscillations can be related to the behavior of the discrete-time system obtained at the singular limit $A \to \infty$ (Sharkovsky *et al.*, 1993). For Eq. (1) the discrete time system is given by:

$$\begin{cases} x(t+1) = W\sigma_\alpha(y(t)) \\ y(t+1) = W\sigma_\alpha(x(t)) \end{cases} \tag{3}$$

Systems (1) and (3) have the same stable and unstable equilibria. However, the latter has also a stable period-two cycle formed by the succession of $(a, -a)$ and $(-a, a)$, which attracts trajectories of ICs (u, v) with $u.v < 0$.

The transient oscillations observed in the continuous time system reflect this change of behavior at the singular limit. Therefore, stable limit cycles in the discrete-time system associated with an almost convergent delayed system, indicate delay-induced transient oscillations for ICs within the basin of attraction of the limit cycle.

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5 Reference

Bélair, J. (1993). Stability in a model of a delayed neural network. *Journal* *of Dynamics and Differential Equations*, **5**, 607–623.

Burton, T.A. (1991). Neural networks with memory. *Journal of Applied Mathematics and Stochastic Analysis*, **4**, 313–332.

Burton, T.A. (1993). Averaged neural networks. *Neural Networks*, **6**, 677–680.

Civalleri, P.P., Gilli, M., & Pandolfi, L. (1993). On stability of cellular neural networks with delay. *IEEE Transactions on Circuits and Systems_I: Fundamental Theory and Applications*, **40**, 157–165.

Gilli, M. (1993). Strange attractors in delayed cellular neural networks. *IEEE Transactions on Circuits and Systems_I: Fundamental Theory and Applications*, **40**, 849–853.

Gilli, M. (1995). A spectral approach for chaos prediction in delayed cellular neural networks. *International Journal of Bifurcation and Chaos*, **5**, 869–875.

Gopalsamy, K., & He, X.-Z. (1994a). Stability in asymmetric Hopfield nets with transmission delays. *Physica D*, **76**, 344–358.

Gopalsamy, K., & He, X.-Z. (1994b). Delay-independent stability in bidirectional associative memory networks. *IEEE Transactions on Neural Networks*, **5**, 998–1002.

an der Heiden, U., & Mackey, M.C. (1982). The dynamics of production and destruction: analytic insight into complex behavior. *Journal of Mathematical Biology*, **16**, 75–101.

Hirsch, M.W. (1989). Convergent activation dynamics in continuous time networks. *Neural Networks*, **2**, 331–350.

Hopfield, J.J. (1984). Neurons with graded response have collective computational properties like those of two-state neurons, *Proceedings National Academy of Science USA*, **81**, 3088–3092.

Marcus, C.M., Waugh, F.R., & Westervelt, R.M. (1991). Nonlinear dynamics and stability of analog neural networks. *Physica D*, **51**, 234–247.

Marcus, C.M., & Westervelt, R.M. (1989). Stability of analog neural networks with delay. *Physical Review A*, **39**, 347–359.

Pakdaman, K., Grotta-Ragazzo, C., Malta, C.P., & Vibert, J.-F. (1995a). Effect of delay on the boundary of the basin of attraction in a system of two neurons. *Technical report IFUSP/P–1169, Instituto de Física, Universidade de São Paulo, Brasil.*

Pakdaman, K., Malta, C.P., Grotta-Ragazzo, C., & Vibert, J.-F. (1995b). Effect of delay on the boundary of the basin of attraction in a self-excited single neuron. *Technical report IFUSP/P-1164, Instituto de Física, Universidade de São Paulo, Brasil.*

Roska, T., & Chua, L.O. (1992). Cellular neural networks with non-linear and delay-type template elements and non-uniform grids. *International Journal of Circuit Theory and Applications*, **20**, 469–481.

Roska, T., Wu, C.F., Balsi, M., & Chua, L.O. (1992). Stability and dynamics of delay-type general and cellular neural networks. *IEEE Transactions on Circuits and Systems_I: Fundamental Theory and Applications*, **39**, 487–490.

Roska, T., Wu, C.F., & Chua, L.O. (1993). Stability of cellular neural networks with dominant nonlinear and delay-type templates. *IEEE Transactions on Circuits and Systems_I: Fundamental Theory and Applications*, **40**, 270–272.

Sharkovsky, A.N., Maistrenko, Yu.L., Romanenko, E.Yu. (1993). Difference Equations and Their Applications. Dordrecht: Kluwer Academic Publishers.

Smith, H. (1987). Monotone semiflows generated by functional differential equations. *Journal of Differential Equations*, **66**, 420–442.

Walther, H.-O. (1995). The 2-dimensional attractor of $x'(t) = -\mu x(t) + f(x(t-1))$. *Memoirs of the American Mathematical Society*, **113**, n 544.

Ye, H., Michel, A.N., & Wang, K. (1994a). Global stability and local stability of Hopfield neural networks with delays. *Physical Review E*, **50**, 4206–4213.

Ye, H., Michel, A.N., & Wang, K. (1994b). Qualitative analysis of Cohen-Grossberg neural networks with multiple delays. *Physical Review E*, **51**, 2611–2618.

A Analysis for $\alpha = \infty$

The behavior of some non-constant initial conditions for a system modeling a single self-exciting neuron with a step transfer function, or a smooth transfer function close to a step function have been studied through the iterations of a map (Heiden & Mackey, 1982; Sharkovsky *et al.*, 1993). For the two neuron system we have the following result.

For $r = (u, v) \in \mathbb{R}^2$, such that $v > -u \geq 0$, let $V(r) = W(v+u)/(W-u)$ and n the integer such that $f^{n-1}(V(r)) < v_1 \leq f^n(V(r))$, where $v_1 = W(\mathrm{e}^A - 1)$, $f(v) = \frac{W(2+\mathrm{e}^{-A})v}{2W-(v+W)\mathrm{e}^{-A}}$ and f^n represents f iterated n times. Then, there is $T \geq nA$, and $\theta > 0$ such that for $t \geq T$ $z(t, r) = z(t-\theta, r_n)$, where $z(t, r)$ is the solution of (1) for the IC r, and r_n represents $(0, f^n(V(r)))$, for n even, and $(f^n(V(r)), 0)$ for n odd. An example of a trajectory with $n = 3$ is shown in Fig. 4.

Thus, the sequence v_n is defined by:
$v_n = f^{-n}(v_1) = \frac{2W(2-\mathrm{e}^{-A})^n(\mathrm{e}^A-1)}{2(2+\mathrm{e}^{-A})^n[(2+\mathrm{e}^{-A})^n-(2-\mathrm{e}^{-A})^n](\mathrm{e}^A-1)}$. $v_n \sim \frac{2W\mathrm{e}^A}{n+2}$ as $A \to \infty$.

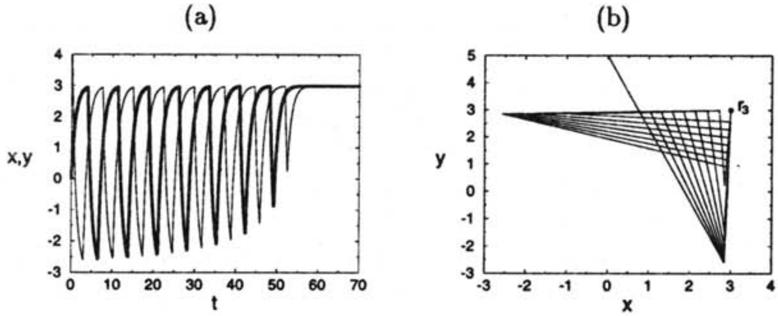

Figure 1: (a): Temporal evolution of $x(t)$ (thick line) and $y(t)$ (thin line). (b): Trajectory in (x, y)-plane. $\alpha = \infty$, delay $A = 3$, $W = 3$, and IC $(u, v) = (-10^{-3}, 5)$. Activation in a.u.; time in same a.u. as delay.

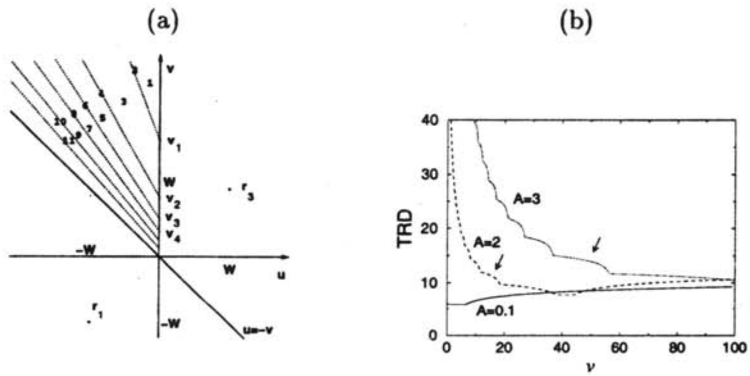

Figure 2: (a): Regions in the (u, v)-plane corresponding to different transients for $\alpha = \infty$. The line $u = -v$ is the boundary separating the basins of the two equilibria, r_1 and r_3. For $v > -u \geq 0$, solutions of ICs within a given area delimited by two consecutive dashed lines have the odd number of zeros indicated. Even numbers correspond to the number of zeros of solutions with ICs on the dashed lines. (b): TRD with precision $\eta = 10^{-2}$, for delays equal to $A = 0.1$ (solid line), $A = 2$ (dashed line) and $A = 3$ (dotted line) for ICs (u, v), with $u = -10^{-3}$, and v ranging from 10^{-3} to 100, for $\alpha = \infty$, $W = 3$. Abscissa: v in a.u.; ordinate: TRD in a.u..

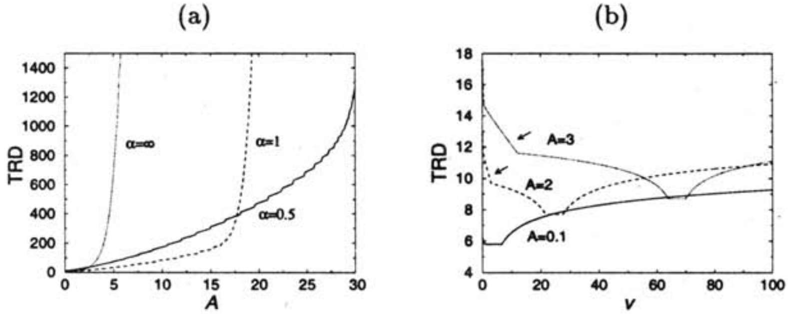

Figure 3: (a): Transient regime duration (TRD) with precision $\eta = 10^{-3}$, for a given IC $(u, v) = (-10^{-3}, 5)$ as a function of the delay for three different gains $\alpha = 0.5$ (solid line), $\alpha = 1$ (dashed line) and $\alpha = \infty$ (dotted line). Abscissa: delay in arbitrary units (a.u.) and ordinate: TRD same units as delay.(b): TRD with precision $\eta = 10^{-2}$, for delays equal to $A = 0.1$ (solid line), $A = 2$ (dashed line) and $A = 3$ (dotted line) for ICs (u, v), with $u = -10^{-3}$, and v ranging from 10^{-3} to 100, for $\alpha = 2.5$, $W = 3$. Abscissa: v in a.u.; ordinate: TRD in same units as delay.

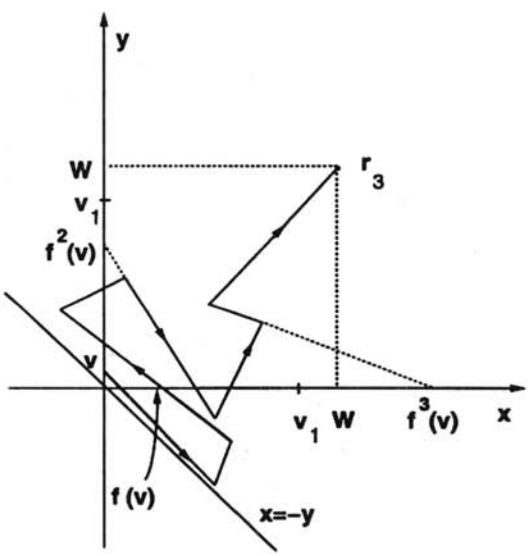

Figure 4: Trajectory of $r = (0, v)$ in the (x, y)-plane. $W = 3$.

K. Pakdaman
B3E, INSERM U 444, ISARS, Faculté de Médecine Saint-Antoine
27, rue Chaligny, 75571 Paris Cedex 12
France

C.P. Malta
Instituto de Física
Universidade de São Paulo
CP 66318, 05315-970 São Paulo
Brasil

J.-F. Viber
Department of Biophysical Engineering
Faculty of Engineering Science
Osaka University
Toyonaka 560 Osaka
Japan

C. Grotta-Ragazzo
Inst. de Mat. e Estatística
Universidade de São Paulo
CP 66281, 05315-970 São Paulo
Brasil
and
Mathematics Department
Princeton University
Fine Hall, Washington Road
Princeton, NJ 18540 **USA**