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Torsion-Free Metabelian Groups With Finite Commutator Quotients [1]

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The existence of a torsion-free metabelian group G with a prescribed torsion commutator quotient G/G' is a fundamental problem which requires attention. Our recent paper [7] with Narain Gupta is a study of this problem in the case where G/G' has prime exponent. An elementary argument using the transfer map from G into G' shows that G/G' cannot be finite cyclic and so the answer is not uniformly positive.

The problem under consideration happens to fall in the realm of the theory of Bieberbach groups, of geometric origins. We recall that a group G is called Bieberbach provided it is torsion-free and has a maximal abelian normal subgroup V of finite rank, called the translation subgroup, and finite quotient G/V, called the point-group, or holonomy group. Fundamental groups of compact flat Riemannian manifolds are Bieberbach groups and they characterize completely such manifolds ; see Charlap [3] for these facts.

Since we are interested in having G/G' finite, and since G' is abelian, the translation subgroup V must contain G'. A classical example that comes to mind is the Hantzsche-Wendt 3-dimensional manifold which has the fundamental group defined by

$$K(2) = < a, b \mid (a^2)^b = a^{-2}, (b^2)^a = b^{-2} > .$$

The commutator quotient of this group is isomorphic to $C_4 \times C_4$. Furthermore, the translation subgroup V is generated by $a^2, b^2, [a, b]$, and G/V is isomorphic to $C_2 \times C_2$.

But what about the existence of torsion-free metabelian groups having commutator quotients of prime exponent? If we consider the infinite 'dihedral' group

$$D = < a, b \mid a^b = a^{-1} >$$

we note that $[a, b] = a^{-2}$ and that D/D' is isomorphic to $C_2 \times C$. Though the group D group fails to have an elementary abelian quotient, still it suggests to build a group where b would be inverted by another generator and so on indefinitely. Indeed, the following infinitely generated group is built accodingly

$$S(2) = < x_i \ (1 \leq i \leq \infty) \mid x_i^{x_{i+1}} = x_i^{-1} \ (\forall i), \ [x_i, x_j] = 1 (\forall j \neq i+1 >$$

and it can be verified that this is indeed an example of a torsion-free metabelian group with commutator quotient of exponent 2.

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One of Bieberbach's theorems tells us that given a natural number k and a finite group T, there exist a finite number of torsion-free groups with translation subgroup V of rank k and point-group $T = G/V$. It is a theorem due to Auslander-Kuranishi that there exists at least one Bieberbach group with a prescribed point-group T. It follows that the number of isomorphism classes of n-generated torsion-free groups G with prescribed finite abelian group $T = G/G'$ of order m is finite. This is so since the rank of G' is bounded by a function of n, m, and therefore the possibilities for the rank of V are also finite.

Hiller and Sah consider in [8] Bieberbach groups G having finite commutator quotient G/G'. Here an application of the transfer map from G into G' proves that the center of G is trivial. In this work they characterize finite groups H which are realizable as point-groups of such Bieberbach groups, as those groups H where no cyclic p-Sylow subgroup has a normal complement ; see also, Cliff-Weiss [4]. In particular, any non-cyclic finite abelian p-group is realizable as a point-group. We note that such examples have been constructed by cohomological methods which do not appear to be amenable to the question of the realization of a prescribed T as G/G'.

In the rest of this exposition we will concentrate mainly on the question of the realization of an elementary abelian p-group T of arbitrary rank k, finite or infinite, as the commutator quotient G/G' of a torsion-free metabelian group G.

1. 2-Generated Groups

Early on in our study we found it exceedingly difficult to produce examples of 2-generated torsion-free metabelian groups with elementary abelian commutator quotient. Actually, there are no such groups. This surprisingly negative result is a corollary to a positive one about Bieberbach groups with point-groups isomorphic to $C_p \times C_p$.

It is convenient to introduce the following notation. Given a polynomial $f(x) = n_0 + n_1 x + .. + n_m x^m$, with integer coefficients, and given a, b elements of a group G, then define

$$a^{f(b)} = (a^{n_0})(a^{n_1})^b ... (a^{n_m})^{b^m}.$$

Also, define the polynomial $t(n;x) = (x^n - 1)/(x-1)$ for any positive integer n. Thus the dihedral group relation $a^b = a^{-1}$ may be expressed as $a^{t(2;b)} = e$ in this notation. Now we define the generalized form of the group $K(2)$: let p be a prime number and define in the variety of metabelian groups

$$K(p) = < a, b \mid [a, b, a^p],\ [a, b, b^p],\ (a^p)^{t(p;b)}, (b^p)^{t(p;a)}, \ \text{metabelian} > .$$

The commutator quotient of $K(p)$ is isomorphic to $C_{p^2} \times C_{p^2}$.

Theorem 1. Let G be a torsion-free metabelian group having a finite commutator quotient G/G'. Let V be a maximal abelian normal subgroup of G and suppose that the point-group is $G/V \cong C_p \times C_p$. Furthermore, let a, $b \in G$ generate G

modulo V, and let $K = < a, b >$. Then K is isomorphic to $K(p)$. Also, $K(p)$ is a torsion-free group.

We have not specified in the theorem the number of generators of the group G. We note that Cobb constructed in [5] an infinite family of finitely generated torsion-free groups having point-groups isomorphic to $C_2 \times C_2$; these groups have commutator quotients of exponent 4.

Suppose in the above theorem that G is 2-generated then necessarily G is isomorphic to $K(p)$, and so G/G' is isomorphic to $C_{p^2} \times C_{p^2}$. Therefore, the commutator quotient cannot be isomorphic to $C_p \times C_p$.

Corollary. Let G be a 2-generated torsion-free metabelian group. Then, G/G' cannot be of exponent p, for any prime number p.

Sketch of proof of Theorem 1.

The proof is developed in a series of steps. First, we define $d(x) = x - 1$, $t(p; x) = (x^p - 1)/(x - 1)$. Also, as $t(p; 1) = p$, we define

$$l(p; x) = (t(p; x) - p)/(x - 1) = x^{p-2} + ... + i.x^{p-i-1} + ... + (p - 1),$$

and note that $l(p; 1) = p(p - 1)/2$. We simplify the notation $t(p; x)$ and $l(p; x)$ to $t(x)$ and $l(x)$, respectively, since the prime p is fixed.

(1) As was commented earlier, since G is torsion-free, by elementary transfer results, it follows that the center of G is trivial, and that G does not contain an abelian normal subgroup N with G/N cyclic.

(2) The subgroup V is a G/V-module and, so is G'. Let a, b induce respectively the automorphisms A, B on V. Denote $[a, b]$ by c. We have

$$[a^p, b] = [a, b]^{t(a)}$$

which can be written additively as

$$a^p.d(B) = c.t(A).$$

Similarly, we have

$$b^p.d(A) = -c.t(B).$$

(3) Extend the integer coefficients of G' to the rational numbers and thus define the $\boldsymbol{Q}[G/V]$-module $M = \boldsymbol{Q} \otimes G'$. Then as $t(x)$ and $x - 1$ are relatively prime over $\boldsymbol{Q}$, the module M decomposes as a direct sum of G/V-submodules

$$M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$$

where

$$\begin{aligned} M_1 &= M.d(A).d(B)\ ,\ \ M_2 = M.d(A).t(B), \\ M_3 &= M.t(A).d(B)\ ,\ \ M_4 = M.t(A).t(B). \end{aligned}$$

Furthermore, since M_4 is centralized by G/V, it is the trivial module.

(4) Using the module decomposition it is possible to show that

$$a^p = c.(-\frac{1}{p}l(B).t(A)) \ , \ \ b^p = c.(\frac{1}{p}l(A).t(B)).$$

and

$$a^{p^2} = [a,b]^{-t(a).l(b)} \ , \ \ b^{p^2} = [a,b]^{l(a).t(b)},$$

and that these equations are equivalent to

$$(a^p)^{t(b)} = e \ , \ \ (b^p)^{t(a)} = e.$$

(5) In this step it shown that G may be assumed to be K which is 2-generated, and the problem becomes that of deciding the possible ranks for V. It is easy to see that this rank is at most p^2-1. But by a theorem from [9], we see that the rank is exactly p^2-1.

(6) So far G has been shown to satisfy the relations of $K(p)$. and the main work ahead is to show that other relations which are not consequences of the relations of $K(p)$ will introduce torsion.

(7) The main extra possible relation is of the form $a^p = c^{m(a,b)}$, where $c=[a,b]$ and $m(x,y)$ is a polynomial with integer coefficients. This last relation when translated to the representation of G on G' produces conditions on the coefficients of the polynomial $m(x,y)$ which imply the existence of torsion in G'.

Observation. If we extend our discussion to torsion-free center-by-metabelian groups then G/G' can be isomorphic to $C_p \times C_p$. An example is

$$G = < a, b \ | \ a^3 = b^3 = (ab)^3 >$$

which is the fundamental group of the Seifert 3-manifold $C_{28}(76)$ in [10]. The commutator series quotients of this group is

$$G/G'' = \mathbb{Z}_3 \times \mathbb{Z}_3, G'/G'' = \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}, G'' = \mathbb{Z}.$$

2. k-Generated Groups, $k \geq 3$

For $k \geq 3$, we are able to construct by generators and relations diverse classes of examples of torsion-free metabelian groups. The following class is quite uniform and serves effectively our purpose. Let

$$\begin{aligned} G(p,k) = \ & < x_i \ \ (1 \leq i \leq k) \ \mid \ x_i.(x_i)^{x_{i+1}}..(x_i)^{(x_{i+1})^{p-1}}, \\ & [x_i, x_j, x_l] \ \ (\text{if } i,j,l \text{ are distinct}), \quad \text{metabelian} >, \end{aligned}$$

where p is any prime number, $3 \leq k \leq \infty$, and the indices are computed modulo k. The presentation is written relative to the variety of k-generated metabelian groups. We prove

Theorem 2. The group $G = G(p, k)$ has an elementary abelian quotient group of rank k. Furthermore, G is torsion-free if and only if $k \geq 4$, or p is odd and $k = 3$.

Open Cases. Let G be a k-generated torsion-free metabelian group with $k \geq 3$. Can G/G' be isomorphic to $C_p \times C_p$ for some prime p ? Can G/G' be isomorphic to $C_2 \times C_2 \times C_2$?

The proof of Theorem 2 requires taking a close look at the algorithmic theory of metabelian groups given by a 'canonical' metabelian presentation, and having a torsion commutator quotient. In this respect, we note the recent article by Baumslag et.al. on the algorithmic theory for finitely generated metabelian groups [1]. The problem of deciding torsion-freeness divides naturally into two steps. The first consists in finding a presentation for G', finding a basis for G', and then verifying that G' is torsion-free. We have followed the strategy of ambiguity resolution of Bergman [2] in finding a basis for G'. Once it is shown that G' is torsion-free then the second step consists in checking the non-existence of torsion elements outside G'. In our examples, we have found it convenient to work in the p-group G/N where $N = (G')^p \gamma_{p+1}(G)$, and $\gamma_{p+1}(G)$ is the $p+1$-th term of the descending central series of G. It turns out that for G to be torsion-free, it is sufficient to verify that the elements having order p in G/N are all in G'/N.

3. Computation of examples in GAP

The group theory software GAP [6] can be applied to prove that the group

$$H(2,4) = < x_i (1 \leq i \leq 4) \mid (x_i)^{t(2;x_{i+1})} \quad \text{(indices mod(4)), metabelian} >$$

is torsion-free. Note that the relation $x_i^{x_{i+1}} = (x_i)^{-1}$ is equivalent to $[x_i, x_{i+1}] = (x_i)^2$. We include below the routine used in this calculation.

(i) Define the generators and their commutators

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x1:=AbstractGenerator("x1"); x2:=AbstractGenerator("x2");
x3:=AbstractGenerator("x3"); x4:=AbstractGenerator("x4");
x21:=Comm(x2,x1); x31:=Comm(x3,x1); x32:=Comm(x3,x2);
x41:=Comm(x4,x1); x42:=Comm(x4,x2); x43:=Comm(x4,x3);
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(ii) Define the group h24, and its subgroup sgp generated by the simple commutators.

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h24:=Group(x1,x2,x3,x4);
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h24.relators:=[$x1^2/x21, x2^2/x32, x3^2/x43, x4^2/x41^-1$];
sgp:=Subgroup(h24, [x21, x31, x32, x41, x42, x43]);

Find the presentation of the commutator subgroup of h24 which is the normal closure of sgp in h24, using the Reidemeister-Schreier algorithm, and its simplification by Tietze transformations.
P:=PresentationNormalClosure(h24,sgp);
Transform the above presentation P into a group.
K:=FpGroupPresentation(P);
Compute the commutator quotient of K.
Kf:=CommutatorFactorGroup(K);
# Find the abelian invariants of Kf, using the Normal Smith form.
v:=AbelianInvariants(Kf);
The result is : (h24)' is torsion free of rank 10.

(iii) Apply the nilpotent quotient algorithm to h24 to compute its 2-central series for prime=2, and level=2.
Q2:=pQuotient(h24,2,2);
The output is that the successive quotients of the series are elementary abelian 2-groups of ranks 4 , 6 , respectively. A power- commutator AG-presentation is also given.
Transform Q2 into a group
aq:=FpGroup(AgGroup(Q2));;
List the elements of aq as permutations with respect to the trivial subgroup.
oq:=OperationCosetsFpGroup(aq,TrivialSubgroup(aq));;
List the elements of exponent 2 in oq.
E:=Elements(oq);;
S:=[ ];
for x in E do if $x^2 = (\ )$ then Add(S,x);else;fi;od;
s:= Length(S);
Print s;
The result is $s = 2^6$; that is, the set S is precisely the second center in the above 2-central series. Hence, the group H(2,4) is torsion-free.

We remark here that by introducing the extra relation $[x3, x1] = e$, a quotient of $H(2,4)$ is obtained which continues to be torsion-free, and the rank of the commutator subgroup diminishes from 10 to 7.

4. m-powers.

Once it has been shown that G' is torsion-free and its rank is decided, in the next stage it becomes necessary to have formulae for $(ab)^p$ and $a^{t(p;b)}$ in a metabelian group G , especially modulo $N = (G')^p\gamma_{p+1}(G)$. On denoting the commutator

$[a,b]$ by c, we prove that in G modulo N, the following formulae hold:

(i) $(ab)^p = a^p.b^p.c^{u'(a-1,b-1)}$,

where u' is a polynomial defined by

$$u'(x,y) = x^{p-2} + (-1)^{\frac{p-1}{2}} x^{\frac{p-3}{2}} y^{\frac{p-1}{2}} + \sum_{j=1}^{(p-3)/2} (-1)^j (x^{p-2j-1} + y^{p-2j-1}) x^{j-1} y^j ;$$

(ii) $a^{t(p;b)} = a^p c^{-v'(a-1,b-1)}$,

where v' is a polynomial defined by

$$v'(x,y) = \sum_{j=0}^{p-2} x^j y^{p-2-j}.$$

5. More examples

Two other classes of torsion-free metabelian groups which arose during our investigation may be of interest. The first

$$\begin{aligned} Q(p,4) = \ & < x_1, x_2, x_3, x_4 \mid \\ & x_1^p = x_2^p = [x_4, x_3] , \ \ x_3^p = x_4^p = [x_2, x_1] , \ \ [x_4, x_1] = [x_3, x_2] , \\ & \text{metabelian} > \end{aligned}$$

gives us an example of a torsion-free group having G/G' of exponent p, for $p = 2$, and also for p odd provided we add the relations

$$[x_2, x_1, x_2] = [x_2, x_1, x_1] , \ \ [x_4, x_3, x_4] = [x_4, x_3, x_3].$$

The second class is

$$\begin{aligned} S(p,k) = \ & < x_i \ \ (1 \le i \le k) \mid x_i^{t(p;x_{i+1})} \ (\forall i) , \ \ [x_{i+1}^{-l} x_i x_{i+1}^{l}, x_j] \\ & (\forall j \ne i+1, \ \ 0 \le l < p) >, \end{aligned}$$

where $3 \le k \le \infty$, and the indices are computed modulo k. The groups in this class are absolutely metabelian and torsion-free, but the only case for which the commutator quotient is torsion is $k = \infty$. We note that for $p = 2$, these groups have the simpler and more attractive form

$$S(2,k) =< x_i \ \ (1 \le i \le k) \mid x_i^{x_{i+1}} = x_i^{-1} \ \ (\forall i), \ \ [x_i, x_j] = e \ \ (\forall j \ne i+1) > .$$

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