

## On Independent Sets

O.T. Alas

This note contains some results presented in my talk 'Three Applications of Combinatorics to General Topology' at the *Semana de Combinatória* (São Paulo, November 1994), dedicated to Paul Erdős. Generalizations in ZFC and under Martin's axiom of former results by Erdős and Bagemihl are shown.

First let us recall some definitions and results. Let  $X$  be a non-empty set. A function  $f: X \rightarrow p(X)$  from  $X$  into the power set  $p(X)$  of  $X$  is called a *set-map* if  $x \notin f(x)$  for all  $x \in X$ . A subset  $S$  of  $X$  is said to be an independent set for  $f$ , if for all  $x, y \in X$ , the condition  $x \neq y$  implies  $x \notin f(y)$  and  $y \notin f(x)$ .

Let  $(X, \tau)$  be a topological space. A subset  $S$  of  $X$  is *first category* if it is contained in a countable union of closed nowhere dense subsets (i.e., closed subsets with empty interior); if  $S$  is not first category it is said to be *second category*. A subset  $M$  of  $X$  is everywhere second category if  $M \cap U$  is second category for every non-empty open set  $U$ .

In 1953 Erdős proved that if  $f: \mathbf{R} \rightarrow p(\mathbf{R})$  is a set-map and  $\overline{f(x)} = \emptyset$  for all  $x \in \mathbf{R}$ , then there is an infinite *countable* independent set for  $f$ . In 1973 Bagemihl generalized the previous result so as to obtain a *dense countable* independent set. We shall present a generalized version of these results for a Baire Hausdorff topological space.

In what follows, we let  $(X, \tau)$  be a non-empty Baire Hausdorff space (hence, every non-empty open subset is second category in  $X$ ) with a countable  $\pi$ -base ([3], page 9). Let us fix  $\kappa$ , an infinite cardinal number such that  $\aleph_0 < \kappa < 2^{\aleph_0}$ ; in the next theorem we shall work under Martin's axiom for cardinal  $\kappa$  ([4], definition 2.5, page 54).

**Theorem 1** (MA( $\kappa$ )) *If  $(A_\alpha)_{\alpha \in \kappa}$  is a family of first category subsets of  $X$ , then  $\bigcup_{\alpha \in \kappa} A_\alpha$  is also of first category.*

*Proof.* The proof follows the same pattern of that of theorem 2.20 in [4] with minor modifications. It is sufficient to prove that if  $(U_\alpha)_{\alpha \in \kappa}$  is a family of dense open subsets of  $X$ , then  $\bigcap_{\alpha \in \kappa} U_\alpha$  is dense too. Fix  $\mathcal{B} = \{B_0, B_1, \dots\}$  a  $\pi$ -base of  $X$  such that if  $B \in \mathcal{B}$ , then either  $B$  is a unitary set, or  $B$  contains a non-isolated point. Since if the set  $D$  of isolated points is dense in  $X$  the result is trivial (because  $D \subset U_\alpha$  for all  $\alpha \in \kappa$ ) we shall assume that  $D$  is not dense. Hence the set  $J$  of all  $j \in \omega$  such that the intersection of  $B_j$  with every open dense subset is infinite. For each  $j \in J$  define

$$c_j = \{i \in \omega \mid B_i \subset B_j\}$$

and for each  $\alpha \in \kappa$  let

$$a_\alpha = \{i \in \omega \mid B_i \not\subset U_\alpha\}.$$

Put  $\mathcal{C} = \{c_j \mid j \in J\}$  and  $\mathcal{A} = \{a_\alpha \mid \alpha \in \kappa\}$ . To apply theorem 2.15 of [4] it is enough to prove that if  $F$  is a non-empty finite subset of  $\kappa$  and  $j \in J$ , then  $c_j \setminus \bigcup_{\alpha \in F} a_\alpha$  is an infinite set. Indeed,

$$c_j \setminus \bigcup_{\alpha \in F} a_\alpha = \left\{ i \in \omega \mid B_i \subset B_j \cap \bigcap_{\alpha \in F} U_\alpha \right\},$$

which is infinite, since  $\bigcap_{\alpha \in F} U_\alpha$  is dense and open. By theorem 2.15 there is  $d \subset \omega$  such that, for all  $x \in \mathcal{A}$ ,  $|x \cap d| < \omega$  and for all  $y \in \mathcal{C}$ , we have  $|d \cap y| = \omega$ .

Now define, for each  $n \in \omega$ ,

$$V_n = \bigcup \{B_i \mid i \in d, i > n\} \cup \text{set of all isolated points};$$

$V_n$  is open and dense in  $X$ . Since  $|d \cap c_j| = \omega$  for  $j \in J$ ; for each  $n \in \omega$ , there is  $i \in d \cap c_j$ ,  $i > n$ , hence  $B_i \subset B_j$  and  $V_n \cap B_j \neq \emptyset$  and it follows that  $V_n$  is dense in  $X$ . Finally  $\bigcap_{n \in \omega} V_n \subset \bigcap_{\alpha \in \kappa} U_\alpha$ , which shows that  $\bigcap_{\alpha \in \kappa} U_\alpha$  is dense in  $X$ . Indeed,  $|d \cap a_\alpha| < \omega$  implies that there is  $n \in \omega$  so that  $d \cap c_\alpha \subset n$ , hence, if  $i > n$ ,  $i \in d$ , then  $i \notin a_\alpha$  and  $B_i \subset U_\alpha$  and  $V_n \subset U_\alpha$ .  $\square$

Let  $f: X \rightarrow p(X)$  be a set-map such that  $\overline{f(x)} = \emptyset$  for all  $x \in X$ .

**Lemma 1** *Assume that  $X$  has countable hereditary density. Let  $M \subset X$  be everywhere of second category in  $X$ . Then there exists  $S \subset M$  such that*

- 1)  $S \cap T \neq \emptyset$  for every second category subset  $T$  of  $X$ ;
- 2) for every  $y \in S$ , the set  $M \setminus \{x \in X \mid y \in f(x)\}$  is everywhere of second category in  $X$ .

*Proof.* Assume to the contrary that such an  $S$  does not exist. It follows that there exists a second category set  $T \subset X$  such that for all  $y \in T$ , the set  $M \setminus \{x \in X \mid y \in f(x)\}$  is not everywhere second category<sup>(1)</sup>. Indeed, if for every second category set  $T \subset X$  there is a  $y_T \in T$  such that

$$M \setminus \{x \in X \mid y_T \in f(x)\} \text{ is everywhere of second category,}$$

then the set  $S$  of all these points  $y_T$  verifies the required conditions.

Fix a second category set  $T$  as in (1) and fix  $\mathcal{B}$  a countable  $\pi$ -base of  $X$  (assume  $\emptyset \notin \mathcal{B}$ ). For each  $y \in T$  fix  $B_y \in \mathcal{B}$  such that

$$(M \setminus \{x \in X \mid y \in f(x)\}) \cap B_y \text{ is of first category.}$$

Since  $T = \bigcup_{B \in \mathcal{B}} \{y \in T \mid B_y = B\}$  and  $T$  is of second category fix  $B \in \mathcal{B}$  such that  $T_0 = \{y \in T \mid B_y = B\}$  is of second category. Choose a dense countable subset  $T_1$  of  $T_0$ ; then

$$B \cap \bigcup_{y \in T_1} (M \setminus \{x \in X \mid y \in f(x)\}) \text{ is of first category;}$$

since  $B \cap M$  is of second category, there is  $z \in B \cap M$  and  $y \in T_1$  implies  $y \in f(z)$ . So  $T_1 \subset f(z)$ ; but

$$\begin{array}{ccc} T_0 \subset \overline{T_1} \subset \overline{f(z)} & & \\ \downarrow & & \text{(contradiction!)} \\ \text{second category} & & \end{array}$$

The lemma is proved.  $\square$

**Lemma 2** ( $\text{MA}(\kappa)$ ) Assume  $X$  has hereditary density  $\leq \kappa$ . Let  $M \subset X$  be everywhere of second category in  $X$ . Then there exists  $S \subset M$  such that

- 1)  $S \cap T \neq \emptyset$  for every second category subset  $T$  of  $X$ ;
- 2) for every  $y \in S$ , the set  $M \setminus \{x \in X \mid y \in f(x)\}$  is everywhere of second category in  $X$ .

*Proof.* The proof is analogous to the preceding one. Using the same notation, choose a dense subset  $T_1$  of  $T_0$  with  $|T_1| \leq \kappa$ . Then

$$B \cap \bigcup_{y \in T_1} (M \setminus \{x \in X \mid y \in f(x)\})$$

is of first category by virtue of theorem 1.  $\square$

**Theorem 2** Assume  $f: X \rightarrow p(X)$  is a set-map such that  $\overset{o}{f(x)} = \emptyset$  for all  $x \in X$ . Then the following results hold:

- 1) if  $(X, \tau)$  has countable hereditary density, then there is a set  $S \subset X$  dense in  $X$  and independent for  $f$ ;
- 2) ( $\text{MA}(\kappa)$ ) if  $(X, \tau)$  has hereditary density  $\leq \kappa$ , then there is a set  $S \subset X$  dense in  $X$  and independent for  $f$ .

*Proof.* Let  $\mathcal{B} = \{B_0, B_1, \dots\}$  be a  $\pi$ -base for  $X$  such that each  $B_n$  is non-empty. Since  $X$  is everywhere of second category, applying lemma 1 (respectively, lemma 2) with  $M_0 = X$ , there is  $S_0 \subset M_0$  such that

- 1)  $S_0 \cap T \neq \emptyset$  for every second category set  $T$ ;
- 2) for every  $y \in S_0$ , the set  $M_0 \setminus \{x \in X \mid y \in f(x)\}$  is everywhere of second category in  $X$ .

$B_0$  is second category, so  $B_0 \cap S_0 \neq \emptyset$  and fix  $y_0 \in B_0 \cap S_0$ ; then

$$M_1 = X \setminus (\{x \in X \mid y_0 \in f(x)\} \cup f(y_0))$$

is everywhere of second category. For every  $y \in M_1$ ,  $y$  and  $y_0$  are independent for  $f$ . Apply lemma 1 (respectively, lemma 2) once again to  $M_1$  and obtain  $S_1$  such that

- 3)  $S_1 \cap T \neq \emptyset$  for every second category subset of  $X$ ;
- 4) for all  $y \in S_1$ , the set  $M_1 \setminus \{x \in X \mid y \in f(x)\}$  is everywhere of second category in  $X$ .

Choose  $y_1 \in S_1 \cap B_1$  and proceed as before with

$$M_2 = M_1 \setminus (\{x \in X \mid y_1 \in f(x)\} \cup f(y_0) \cup f(y_1)),$$

which is everywhere of second category in  $X$ . Notice that  $y, y_0, y_1$  are independent for  $f$  for all  $y \in M_2$ . Following this same idea we find a set  $\{y_0, y_1, \dots, y_n, \dots\}$  which is independent for  $f$  and dense in  $X$ .  $\square$

## References

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**O.T. Alas**

Instituto de Matemática e Estatística

Universidade de São Paulo

Caixa Postal 66281

Agência Cidade de São Paulo

05389-970 São Paulo, SP

**Brazil**

*E-mail address:* alas@ime.usp.br