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# Closed Submodules of Centred Bimodules over Prime Rings, and Applications[1]

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**Abstract:** This paper is a survey on results of several papers by the author. In the first two sections we summarize the main results on closed submodules of centred bimodules over prime rings. In the rest of the paper we mention several applications of the results, mainly to study properties concerning prime ideals and radicals of centred extensions.

**Key words:** Centred bimodules, closed submodules, centred extensions, closed ideals, prime ideals, radicals.

**AMS classification:** 16D20, 16D25, 16D30, 16S20.

# Introduction

Prime ideals in ring extensions have been studied extensively in recent years. In particular, we wrote a series of papers in which we developed a method to study prime ideals of (not necessarily finite) centred extensions (see bibliography). It turns out that the method can be extended to study submodules of centred bimodules over prime rings, and this has been done in [6]. The purpose of this survey is to present some ideas and results of these papers.

Let $R$ be a ring and let $M$ be an $R$-bimodule. We say that $M$ is a centred bimodule over $R$ if there exists a generating set of $R$-centralizing elements; i.e, there exists $X = (x_i)_{i\in\Omega} \subseteq M$ such that $M$ is generated over $R$ by the set $X$ and $ax_i = x_ia$, for every $a \in R$, $i \in \Omega$. A ring extension $S \supseteq R$ is said to be a centred extension of $R$ if $S$ is a centred bimodule over $R$.

There are many natural examples of centred bimodules and centred extensions. Namely, a group or even a semigroup ring $RG$. In particular, a polynomial ring in any set of, either commuting or non-commuting, indeterminates. Also a matrix ring over R, and a tensor product $S = R\bigotimes_L K$, where $L$ is a field and $R$ and $K$ are $L$-algebras. These are all examples of centred extensions of a ring $R$, and, of course, they are centred bimodules over $R$. On the other hand, any module $M$ over a commutative ring $R$ is a centred bimodule, and if $S$ is an algebra over $R$, then $S\bigotimes_R M$ is a centred bimodule over $S$. Finally, a ring of infinite matrizes over $R$ provided that every matrix has a finite number of non-zero entries is also a centred bimodule over $R$.

In this survey we give the main results of the first two sections of [6]. Then we present several applications of these results. Some are applications to the theory of bimodules, some others to the study of prime ideals of centred extensions and questions on radicals. Thus we take a look to the main results in several papers

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on the subject. We refer the reader to the original papers for proofs. However, since [6] has been finished we obtained some new results which are not published yet. So we include here some of them, and in this case we include also a short proof. At the end we introduce some open questions.

Throughout this paper $R$ is a prime ring and $M$ is a centred bimodule over $R$ with $X = (x_i)_{i\in\Omega}$ as a set of $R$-centralizing generators, unless otherwise stated. Submodule of $M$ means sub-bimodule. Also, an ideal $I$ of $R$ is always a two-sided ideal, and this is denoted by $I \lhd R$. The notations $\subset$ and $\supset$ mean strict inclusions.

# 1 Closed Submodules

Assume that $N \subseteq P$ are submodules of $M$. We define the closure of $N$ in $P$ by

$$[N]_P = [N] = \{x \in P : \text{there exists } 0 \neq H \lhd R \text{ such that } xH \subseteq N\}.$$

We will omit the subscript $P$ when there is no possibility of misunderstanding.

It is clear that the closure $[N]_P$ of $N$ in $P$ is a submodule of $M$ with $N \subseteq [N]_P \subseteq P$. A submodule $N$ of $P$ is said to be closed in $P$ if $[N]_P = N$.

One of the crucial points of this method is to have a good characterization of $[N]$. It is necessary to consider first the free case.

Assume that $L$ is free over $R$ with the centralizing basis $E = (e_i)_{i\in\Omega}$. Any $x \in L$ can be uniquely written as a finite sum $x = \sum_{i\in\Omega} a_i e_i$, where $a_i \in R$. The $e$-coefficient of $x$ is denoted by $x(e)$; i.e., for $x$ given above $x(e_i) = a_i$, $i \in \Omega$. The support of $x$ is defined as usual by $supp(x) = \{e \in E : x(e) \neq 0\}$.

Let $N$ be a submodule of $L$. A non-zero element $x \in N$ is said to be of minimal support in $N$ if for every $y \in N$ with $supp(y) \subset supp(x)$ we have $y = 0$. We denote by $m(N)$ the set of all the elements of minimal support in $N$. Now we are ready to give the very useful notion of minimality in the free case. The minimality of $N$ is defined by $Min(N) = \{supp(x) : x \in m(N)\}$.

Before considering the general case we need a lemma. The proof is straightforward.

**LEMMA 1.1** *Let $M$ and $M'$ be two centred bimodules and $\phi: M \to M'$ an epimorphism of $R$-bimodules. If $N' \subseteq P'$ are submodules of $M'$, $N = \phi^{-1}(N')$, and $P = \phi^{-1}(P')$, we have $[N]_P = \phi^{-1}([N']_{P'})$. In particular, $N$ is closed in $P$ if and only if $N'$ is closed in $P'$.*

Under the situation of Lemma 1.1 we know that there exists a one-to-one correspondence between the set of all the submodules of $M'$ which are contained in $P'$ and the set of all the submodules of $M$ which contains $Ker\phi$ and are contained in $P$. The lemma shows that this correspondence preserves closed submodules. This fact allows us to give a definition of $Min(N)$ in the general case.

Let $M$ be a centred bimodule over $R$ with $X = (x_i)_{i\in\Omega}$ as a set of $R$-centralizing generators. Take a free centred bimodule $L$ over $R$ with a centralizing basis $E = (e_i)_{i\in\Omega}$. We fix $L$ and the basis $E$. Consider the canonical epimorphisms $\pi: L \to M$ given by $\pi(e_i) = x_i$, for every $i \in \Omega$. Now, given a submodule $N$ of

$M$ we define the $E$-minimality of $N$ as being $Min(\pi^{-1}(N))$. We call it simply minimality of $N$, since $L$ and $E$ are fixed.

Now we are ready to give a description of $[N]_P$ directly in the general case. We have

**THEOREM 1.2** *Let $M$ be a centred bimodule over $R$ and $N \subseteq P$ submodules of $M$. Then $[N]_P$ is the largest submodule $K$ of $P$ which contains $N$ and satisfies $Min(K) = Min(N)$. Also, $[N]$ is closed and, moreover, it is the smallest closed submodule of $P$ which contains $N$. In particular, $[N]$ is the unique closed submodule of $P$ which contains $N$ and satisfies $Min([N]) = Min(N)$.*

The proof of the theorem has two different parts. First we have to prove the free case. This has been done in [6] (Lemma 1.1 and Theorem 1.2). The proof of the general case in just canonical, and for this reason is not included here. We point out that this proof is also not contained in [6]. In that paper we follow a different approach for the general case, and so the description we give here is just Remark 1.15 of [6].

Using Theorem 1.2 we can obtain some interesting consequences. First, we can define $[N]$ in a dual way. In fact, we can use the condition $Hx \subseteq N$ instead of the condition $xH \subseteq N$. Then we can proof a similar result as Theorem 1.2. Since the notion of minimality does not depend on left or right side, it follows that the two definitions coincide. So we have the following

**COROLLARY 1.3** *For submodules $N \subseteq P$ of $M$ we have*

$$[N]_P = \{x \in P : \text{there exists}\, 0 \neq H \lhd R\, \text{such that}\ Hx \subseteq N\} =$$

$$\{x \in P : \text{there are ideals}\, A \neq 0\, \text{and}\, B \neq 0\, \text{of}\, R\, \text{with}\, AxB \subseteq N\}.$$

An element $x \in M$ is said to be a torsion element if there exists an ideal $H$ of $R$ with $xH = 0$. Thus the torsion elements of $M$ are just the elements of the submodule $[0]_M$ of $M$. A submodule $P$ of $M$ is said to be torsion-free (resp. torsion), if $[0]_P = 0$ (resp. $[0]_P = P$). It is clear that if $N \subseteq P$, then $N$ is closed in $P$ if and only if the factor module $P/N$ is torsion-free.

On the other hand, recall that a right $R$-module $A$ is said to be prime if for $x \in A$ and $r \in R$ we have that $xRr = 0$ implies either $x = 0$ or $Ar = 0$ [3]. The submodule $B$ of $A$ is prime in $A$ if the factor module $A/B$ is a prime module. According to this definition for $N \subseteq P \subseteq M$, $N$ is prime in $P$ if and only if for $x \in P$ and $0 \neq r \in R$ we have that $xRr \subseteq N$ implies that either $x \in N$ or $Pr \subseteq N$.

There is a close relation between the closed submodules we are studying in our papers and the prime submodules. We was not aware of this relation when we write [6]. So we use this opportunity to give the following.

**PROPOSITION 1.4** *Assume that $N \subset P$ are submodules of $M$. Then $N$ is closed in $P$ if and only if $N$ is a prime submodule of $P$ and the factor module $P/N$ is not a torsion module.*

Proof. If $N$ is closed in $P$, then the factor module $P/N$ is torsion-free, so it is not torsion. Also, if $x \in P$, $0 \neq r \in R$, and $xRr \subseteq N$, then $xRrR \subseteq N$ and since $RrR$ is a non-zero ideal of $R$ we have $x \in [N] = N$. Thus $N$ is prime in $P$.

Conversely, assume that $P/N$ is not torsion and $N$ is prime in $P$. If $xH \subseteq N$, for $x \in P$ and $0 \neq H \lhd R$, take any $0 \neq a \in H$. We have $xRa \subseteq N$. It follows that either $x \in N$ or $Pa \subseteq N$. The argument shows that either $x \in N$ or $PH \subseteq N$. The proof is complete because the last possibility gives a contradiction.

We point out that if $S \supset R$ is a centred extension of a prime ring $R$, the case we studied in several papers, and $I$ is an $R$-disjoint ideal $S$, then the factor ring $S/I$ is never torsion because the identity element of $S$ is not a torsion element in the factor ring. So the lattice of closed ideals is just the lattice of $R$-disjoint ideals which are prime $R$-submodules of $S$.

If every generator $x_i$ of $M$, $i \in \Omega$, is a torsion element, then $M$ is a torsion bimodule. Thus $[0]_P = P$, for every submodule $P$ of $M$. It follows that $P$ is the unique closed submodule of $P$. So the lattice of closed submodules of $P$ is trivial in this case. Thus it is natural to assume that there exist generators of $M$ which are not torsion elements. It is easy to see that any such a generator is an element of $M$ which is free over $R$.

Hereafter, we assume that $M$ is not a torsion bimodule. Consequently, by Zorn's lemma there exists a subset $E$ of $X$ which is a maximal $R$-independent subset of $X$. Denote by $L$ the (free) submodule of $M$ which has $E$ as a centralizing basis. There is a nice relation between $M$ and $L$.

**LEMMA 1.5** *For every $x \in M$ there exists a non-zero ideal $H$ of $R$ such that $xH \subseteq L$.*

Let $N \subseteq P$ be submodules of $M$. We say that $N$ is dense in $P$ if $[N]_P = P$. Equivalently, the factor module $P/N$ is a torsion module. Lemma 1.5 says that for every centred bimodule $M$ over $R$ there exists a dense submodule $L$ which is free over $R$. We will refer to it as a free dense submodule of $M$.

We have the following

**THEOREM 1.6** *Assume that $K \subseteq P$ are submodules of $M$ such that $K$ is dense in $P$. Then there is a one-to-one correspondence between the set of all the closed submodules of $P$ and the set of all the closed submodules of $K$. Moreover, this correspondence associates the closed submodule $N$ of $P$ with the closed submodule $I$ of $K$ if $N \cap K = I$ (equivalently $N = [I]_P$).*

The above theorem is very interesting for several reasons. One of these is because when applied to a free dense submodule $L$ of $M$ reduces the study of closed submodules to the free case. This reduction can also be done using Lemma 1.1, as we have already seen. However the last way is much more convenient, since we are having a free module contained in $M$. Thus the reduction works inside the given bimodule $M$, while in the first case we have to define a new free bimodule such that $M$ is a factor module of it. We have

**COROLLARY 1.7** *Assume that $P$ is a submodule of $M$ and $L$ is a free dense submodule of $M$. Then there is a one-to-one correspondence via contraction between the set of all the closed submodules of $P$ and the set of all the closed submodules of $P \cap L$. In particular, there is a one-to-one correspondence between the set of all the closed submodules of $M$ and the set of all the closed submodules of $L$.*

Using this corollary we can give a new characterization of $[N]_P$ in terms of another notion of minimality. In fact, we define the minimality of $N$ as the minimality of $N \cap L$ as a submodule of the free module $L$. Then we can prove a theorem corresponding to Theorem 1.2 using this new concept of minimality. In [6] we follow this approach.

We want to explain why we change the presentation of the subject in this paper comparing with [6]. First, we wanted to make more evident the new approach. Second, there is also a natural advantage using this way. We can give the characterization of the closure of an ideal just from the begining. With the other approach we have to develop a lot of machinery before we are in position to give this characterization, as we can see in [6].

# 2 Enlarging and Contracting Closed Submodules

Let $Q$ be either the maximal (complete) or the Martindale right quotient ring of $R$ ([18], Chap. IX; [13], Section 4.3; [16]). The extended centroid of $R$ is the center of $Q$, and we denote it here by $C$. Recall the following basic properties.

**LEMMA 2.1** (i) $R \subseteq Q$.

(ii) *If $J$ is a dense right ideal (resp. non-zero ideal) of $R$ and $f: J \to R$ is a homomorphism of right $R$-modules, then there exists $q \in Q$ such that $f(r) = qr$, for all $r \in J$.*

(iii) *For any $q_1, ..., q_n$ in $Q$ there exists a dense right ideal (resp. non-zero ideal) $J$ of $R$ such that $q_i J \subseteq R$, for $i = 1, ..., n$.*

(iv) *If $qJ = 0$ for some $q \in Q$ and dense right ideal (resp. non-zero ideal) $J$ of $R$, then $q = 0$.*

(v) *$Q$ is also a prime ring and $C$ is a field.*

(vi) *$q \in C$ if and only if there exists a non-zero ideal $I$ of $R$ and an $R$-bimodule homomorphism $f: I \to R$ such that $f(r) = qr$, for every $r \in I$.*

The purpose of this section is to extend the bimodule $M$ to a $Q$-bimodule $M^*$ and then to contract $M^*$ to a vector space $V$ over $C$. We show that there exists a one-to-one correspondence between the closed submodules of $M$, $M^*$, and the subspaces of $V$. First we have to consider the free case.

Let $L$ be a free centred bimodule with the centralizing basis $E = (e_i)_{i\in\Omega}$. Denote by $L^*$ the free $Q$-bimodule $\sum_{i\in\Omega} \oplus Qe_i$, where $E$ is a centralizing basis of $L^*$. Put $V = \sum_{i\in\Omega} Ce_i$, a vector space over $C$ with the same basis $E$.

Assume that $N$ is a submodule of $L$ and $\Gamma = \{e_1, ..., e_n\} \in Min(N)$. Take any $e \in \Gamma$, say $e = e_1$, and consider the non-zero ideal $H$ of $R$ defined as the set of all $a \in R$ such that there exists $x = ae_1 + a_2e_2 + ... + a_ne_n \in N$. Then such an $x$ is unique and the map $f_i: H \to R$ defined by $f_i(a) = a_i$ is a (well-defined) $R$-bimodule map, where $a_1 = a$. Hence there exist $c_i \in C$ with $c_ia = a_i$, for $i = 1, ..., n$, where $c_1 = 1$. We write $m_{\Gamma,e} = e_1 + c_2e_2 + ... + c_ne_n$. We can easily see that the element $m_{\Gamma,e}$ is the unique element of $V$ such that for every element $x \in N$ with $supp(x) = \Gamma$ we have $x = m_{\Gamma,e}x(e) = x(e)m_{\Gamma,e}$. Moreover, $supp(m_{\Gamma,e}) = \Gamma$ and $m_{\Gamma,e}(e) = 1$.

Denote by $M_C(N)$ the set of all the elements $m_{\Gamma,e}$ constructed above, where $\Gamma \in Min(N)$ and $e \in \Gamma$. So $M_C(N) \subseteq V$ and for every $m \in M_C(N)$ there exists a non-zero ideal $H$ of $R$ with $mH = Hm \subseteq N$.

Now, given a $Q$-submodule $P$ of $L^*$ denote by $P_0 = P \cap L$ and by $P_V = CM_C(P_0)$, the subspace of V generated by $M_C(P_0)$.

**THEOREM 2.2** *Let $L$ be a free centred bimodule over $R$ and suppose that $P$ is a submodule of $L^*$. Then there is a one-to-one correspondence between the following:*

(i) *The set of all the submodules of $L$ which are closed submodules of $P_0$.*

(ii) *The set of all the submodules of $L^*$ which are $Q$-closed submodules of $P$.*

(iii) *The set of all the $C$-subspaces of $P_V$.*

*Moreover, this correspondence associates the closed submodule $N$ of $P_0$ with the $Q$-closed submodule $N^*$ of $P$ and the subspace $K$ of $P_V$ if $N^* \cap L = N$ and $N^* = QK \cap P$.*

The proof of this theorem is really hard, and to obtain it is necessary to prove several lemmas. One of these lemmas shows that for submodules $N \subseteq K$ of $L$, $N$ is closed in $K$ if and only if $N = QM_C(N) \cap K$.

Now we are in position to define the canonical torsion-free extension. Let $M$ be a centred bimodule over $R$ with $X = (x_i)_{i\in\Omega}$ as a set of $R$-centralizing generators. When $L$ is free with $E$ as a centralizing basis, the canonical torsion-free extension of $L$ is defined as $L^* = \sum_{i\in\Omega} \oplus Qe_i$. In general, take a free $R$-bimodule

$L$ with the centralizing basis $E$, as above, and an epimorphism $\pi: L \to M$ given by $\pi(e_i) = x_i$. By Lemma 1.1 the submodule $I = \pi^{-1}([0]_M)$ is a closed submodule of $L$ and so there exists a closed submodule $I^*$ of $L^*$ such that $I^* \cap L = I$, by Theorem 2.2. Put $M^* = L^*/I^*$ and denote by $\phi: L^* \to M^*$ the canonical projection. Thus $M^*$ is a centred bimodule over $Q$ with $(\phi(e_i))_{i\in\Omega}$ as a generating set of centralizing elements and $\phi$ is a $Q$-bimodule homomorphism. Also, since $\phi^{-1}(0) = I^*$ is closed we have that $M^*$ is torsion-free as a $Q$-module. We can easily see that $j(x_i) = \phi(e_i)$, $i \in \Omega$, induces a well-defined $R$-bimodule homomorphism $j: M \to M^*$.

**DEFINITION 2.3** The pair $(M^*, j)$ is said to be the canonical torsion-free extension of $M$.

Let $P$ be a right $R$-module. We say that $P$ is torsion-free if the following condition holds: $x \in P$ and $xJ = 0$, for a dense right ideal $J$ of $R$, imply $x = 0$. It is not hard to see that this definition agrees with the one given in Section 1 for centred bimodules. The canonical torsion-free extension has the following universal property.

**PROPOSITION 2.4** *Under the above assumptions , $M^*$ is a centred bimodule over $Q$ which is torsion-free as a right $R$-module and $j: M \to M^*$ is an $R$-bimodule homomorphism. Moreover, for every right $Q$-module $P$ which is torsion-free as right $R$-module and every homomorphism of right $R$-modules $f: M \to P$ there exists a unique homomorphism of right $Q$-modules $f^*: M^* \to P$ such that $f^* \circ j = f$.*

By Proposition 2.4 it is clear that the canonical extension $(M^*, j)$ of $M$ is unique up to isomorphism. Some other facts which are convenient to remark are the following. The bimodule $M^*$ is a torsion-free centred bimodule over $Q$ with $(j(x_i))_{i\in\Omega}$ as a set of $Q$-centralizing generators. Also, $Ker j = [0]_M$. So we may consider $M \subseteq M^*$ if and only if $M$ is torsion-free over $R$.

Now we can give one of the main results of this paper.

**THEOREM 2.5** *For every centred bimodule $M$ over $R$, the canonical torsion-free extension $M^*$ of $M$ is free over $Q$. Moreover, if $L$ is a free dense submodule of $M$ with the basis $E$, then $M^* = L^*$ is free over $Q$ with the centralizing basis $(j(e))_{e\in E}$.*

The above theorem says, loosely speaking, that every torsion-free centred bimodule over a prime ring $R$ is always free when considered as a $Q$-bimodule.

As we said above, $M^*$ is a free bimodule which is uniquely determined by $M$. Note that $V$ is the set of all $x \in M^*$ such that $rx = xr$, for every $r \in R$. Then $V$ is also uniquely determined by $M$.

Now we can give the corresponding of Theorem 2.2 for the general case. We denote again by $L$ a free dense submodule of $M$.

**THEOREM 2.6** *Let $M$ be a centred bimodule over $R$, $(M^*, j)$ the canonical torsion-free extension of $M$, and $P$ a submodule of $M^*$. Then there is a one-to-one correspondence between the following:*

(i) *The set of all the submodules of $M$ which are closed in $j^{-1}(P)$.*

(ii) *The set of all the submodules of $M^*$ which are closed in $P$.*

(iii) *The set of all the $C$-subspaces of $CM_C(P \cap L)$.*

*Moreover, the correspondence associates the closed submodule $N$ of $j^{-1}(P)$ with the closed submodule $N^*$ of $P$ and the subspace $K$ of $CM_C(P \cap L)$ if $j^{-1}(N^*) = N$ and $N^* = QK \cap P$.*

We point out that the proof of Theorem 2.6 follows directly, via $L$, from Theorems 1.6 and 2.2.

# 3 Applications

Throughout this section we will consider several applications of the results of the former ones.

### 3.1 Non-singular Submodules

Recall that the singular submodule $Z(P)$ of a right $R$-module $P$ is defined as the set of all the elements $x \in P$ such that the right annihilator $r(x)$ of $x$ in $R$ is an essential right ideal of $R$. The submodule $P$ is said to be non-singular if $Z(P) = 0$. We say that a submodule $N$ of $P$ is non-singular in $P$ if $Z(P/N) = 0$ ([12], pp. 30-36).

When $M$ is a bimodule over $R$ and $P$ is a submodule of $M$, we consider $P$ as a right $R$-module. So $Z(P)$ is the right singular submodule of $P$ and is, in fact, a sub-bimodule of $M$. We will say simply singular submodule and non-singular, omitting right. Also, $r(x)$ denotes the right annihilator of $x$ in $R$.

For a ring $R$, the singular ideal of $R$ is the ideal $Z(R)$, which is the singular submodule of $R$ when considered as right $R$-module. We say that $R$ is non-singular if $Z(R) = 0$.

It is easy to see that if $P$ is a submodule of a centred bimodule $M$ and $N$ is a non-singular submodule of $P$, then $N$ is closed in $P$. The purpose of the first part of this section is to consider the converse of this fact. We have

**LEMMA 3.1** *Assume that $R$ is a prime non-singular ring and $N \subseteq P$ are submodules of $M$. Then $Z(P/N) = [N]_P/N$.*

Lemma 3.1 shows that when $R$ is a prime non-singular ring, then the singular submodule of a submodule of $M$ can be determined using closed submodules. The following is clear.

**COROLLARY 3.2** *Assume that $R$ is a prime non-singular ring and $N \subseteq P$ are submodules of $M$. Then $N$ is closed in $P$ if and only if $N$ is non-singular in $P$. In particular, $P$ is torsion-free if and only if $P$ is non-singular.*

The main result concerning non-singular rings and submodules is the following ([6], Theorem 4.6)

**THEOREM 3.3** *Let $M$ be a centred bimodule over the prime ring $R$ and $P$ a submodule of $M$ which is not a torsion submodule. Then the following conditions are equivalent:*

(i) *$R$ is a non-singular ring.*

(ii) $Z(P/[0]_P) = 0$.

(iii) *Every closed submodule of $P$ is non-singular in $P$.*

(iv) $Z(P/N) = [N]_P/N$, *for every submodule $N$ of $P$.*

### 3.2 Strongly Closed Submodules

Now we turn our attention to strongly prime rings and strongly closed submodules. Recall that a ring $R$ is said to be (right) strongly prime if every non-zero ideal $I$ of $R$ contains an insulator; i.e., there exists a finite set $F \subseteq I$ such that $Fa = 0$, $a \in R$, implies $a = 0$. An ideal $P$ of $R$ is said to be (right) strongly prime if $R/P$ is a strongly prime ring.

Let $P$ be a submodule of a centred bimodule $M$. A submodule $N$ of $P$ is said to be (right) strongly closed in $P$ if for any submodule $K$ of $M$ with $N \subset K \subseteq P$ there exists a finite set $F \subseteq K$ such that $Fa \subseteq N$, $a \in R$, implies $a = 0$. The submodule $P$ is said to be strongly closed if the ideal (0) is strongly closed in $P$. Every strongly closed submodule of $P$ is closed in $P$. Moreover, it is not hard to prove that any such a submodule is also non-singular in $P$.

Denote by $s(P/N)$ the smallest strongly closed submodule of $P/N$ (it is easy to see that there exists such a submodule). Corresponding to the results of the first part of this section we can obtain the following.

**LEMMA 3.4** *Assume that $R$ is a strongly prime ring. Then for submodules $N \subseteq P$ of $M$ we have $s(P/N) = [N]_P/N$.*

**COROLLARY 3.5** *Assume that $R$ is a strongly prime ring and $N \subseteq P$ are submodules of $M$. Then $N$ is strongly closed in $P$ if and only if $N$ is closed in $P$.*

The second main result of this section can be proved using Corollary 3.5 ([6], Theorem 4.13).

**THEOREM 3.6** *Let $M$ be a centred bimodule over a prime ring $R$ and $P$ a submodule of $M$. Then the following conditions are equivalent:*

(i) *$R$ is strongly prime.*

(ii) *$[0]_P$ is a strongly closed submodule of $P$.*

(iii) *Every closed submodule of $P$ is strongly closed in $P$.*

(iv) *$s(P/N) = [N]_P/N$, for every submodule $N$ of $P$.*

### 3.3 The Torsion-Free Rank of a Submodule

Now we study the torsion-free rank of a submodule. Let $N$ be a submodule of a centred bimodule $M$ over a prime ring $R$. The torsion-free rank of $N$ is defined as the length of the longest possible direct sum of non-zero torsion-free sub-bimodules of $N$, if such a bound exists, or infinite in the contrary case. We denote the torsion-free rank of $N$ by $rank(N)$ ([17], Definition 1.5).

We can give an equivalent definition of $rank(N)$ using the results of the former section. As a consequence this notion becomes more tractable, and we are able to prove results on the torsion-free rank using well-known properties of vector spaces.

The following is a key result.

**LEMMA 3.7** *Let $N$ be a submodule of $M$. Then we have $rank(N) = rank([N]) = rank([N]/[0])$, where $[N]/[0]$ is a submodule of $M/[0]_M$.*

The above lemma shows that to compute $rank(N)$ we may always assume that $M$ is torsion free and $N$ is closed in $M$. Denote again by $(M^*, j)$ the canonical torsion-free extension of $M$ and by $V$ the corresponding $C$-vector space. We have

**THEOREM 3.8** *Let $N$ be a closed submodule of $M$, $N^*$ the closed submodule of $M^*$ with $j^{-1}(N^*) = N$, and $K = N^* \cap V$. Then $rank(N) = dim_C(K)$, where $dim_C(K)$ denotes the dimension of $K$ as a $C$-vector space. In particular, $rank(N) = rank(N^*)$.*

Theorem 3.8 has several applications. As an example we give here one result which was proved in ([6], Corollary 3.5) and other which is new.

**COROLLARY 3.9** *Assume that $N \subseteq P$ are submodules of a centred bimodule $M$. Then $rank(P) = rank(N) + rank(P/N)$.*

Now we consider rank of submodules of a tensor product. Assume that $M_1$ and $M_2$ are centred bimodules with centralizing generators $(x_i)_{i\in\Omega}$ and $(y_j)_{j\in\Delta}$, respectively. It is easy to see that $M_1 \otimes_R M_2$ is a centred bimodule over $R$ with centralizing generators $(x_i \otimes y_j)_{i,j}$. Assume that $N_1$ and $N_2$ are submodules of $M_1$ and $M_2$, respectively.

**THEOREM 3.10** *$rank(N_1 \otimes_R N_2) = rank(N_1)rank(N_2)$, where $N_1$ and $N_2$ are as above.*

Sketch of the proof. First we have to show that the canonical torsion-free extension of $M_1 \otimes_R M_2$ is equal to $M_1^* \otimes_Q M_2^*$, where $(M_1^*, j_1)$ and $(M_2^*, j_2)$ are the canonical torsion-free extensions of $M_1$ and $M_2$, respectively. To see this we check that $(M_1^* \otimes_Q M_2^*, j_1 \otimes j_2)$ satisfies the universal property given in Proposition 2.4.

Next, if $L_1$ and $L_2$ are free dense submodules of $M_1$ and $M_2$, respectively, then $L_1^* = M_1^*$ and $L_2^* = M_2^*$. Hence $L_1^* \otimes_Q L_2^* = M_1^* \otimes_Q M_2^*$. From this easily follows that $L_1 \otimes_R L_2$ can be considered as a free dense submodule of $M_1 \otimes_R M_2$. Since $N_i^* = (N_i \cap L_i)^*$, where $N_i^*$ means the extension of the submodule $N_i$ of $M_i$, $1 = 1, 2$, we can reduce the problem to the free case.

It remains only to prove that if $K_i = N_i^* \cap V_i$ and $K = (N_1 \otimes_R N_2)^* \cap V$, then $K = K_1 \otimes_C K_2$, where $V_i$ and $V$ denotes the $C$-spaces associated with $M_i$, $i = 1, 2$, and $M_1 \otimes_R M_2$, respectively. Clearly we have that $K_1 \otimes_C K_2 \subseteq K$. Assume that K is strictly largest than $K_1 \otimes_C K_2$. Then there exists some element $v \in K$ which is $C$-independent of the elements of a basis of $K_1 \otimes_C K_2$. Also there exists a non-zero ideal $H$ of $R$ such that $vH \subseteq N_1 \otimes N_2$. It is now easy to obtain a contradiction.

We point out that more results on submodules of tensor products of centred bimodules over a prime ring will be contained in a forthcoming paper that I have in preparation.

### 3.4 The Goldie Dimension of a Submodule

Let $P$ be a right $R$-module. The Goldie dimension of $P$ is defined as the lenght of the longest possible direct sum of non-zero submodules of $P$ if such a bound exists, or infinite in the contrary case ([1], Chap. 1).

We consider here a submodule $N$ of a centred bimodule $M$ and denote by $Gdim(N)$ the Goldie dimension of $N$ as a sub-bimodule of $M$; i.e, considering direct sums of non-zero sub-bimodules of $M$. There is a nice relation between the Goldie dimension and the torsion-free rank of a submodule. First we have the following

**PROPOSITION 3.11** *Let $N$ be a submodule of $M$ and $L$ a free dense submodule of $M$. Then the sum $(N \cap L) + [0]_N$ is a direct sum and an essential submodule of $N$.*

Proof. If $x \in N \cap L \cap [0]_N$, then there exists a non-zero ideal $H$ of $R$ such that $xH = 0$, and since $x \in L$ we get $x = 0$. So the sum is a direct sum. Assume that $P$ is a submodule of $N$ and $x \in P$. Then there exists $0 \neq H \lhd R$ such that $xH \subseteq N \cap L$. Hence $P$ has non-zero intersection with $N \cap L$ in case that $xH \neq 0$ and has non-zero intersection with $[0]_N$ in case that $xH = 0$. The proof is complete.

The following evident corollary is well-known using some other notions of torsion submodules. It shows that the Goldie dimension of a submodule is determined by the Goldie dimension of a torsion submodule and the torsion-free rank of a submodule, which is the dimension of a vector space.

**COROLLARY 3.12** *Under the same assumptions of Proposition 3.11 we have* $Gdim(N) = rank(N \cap L) + Gdim([0]_N)$.

### 3.5 Prime Ideals in Centred Extensions

A ring extension $S \supseteq R$ is said to be a centred extension if $S$ is a centred bimodule over $R$; i.e., it has a generating set $X$ of $R$-centralizing elements. An ideal $I$ of $S$ is said to be $R$-disjoint if $I \cap R = 0$.

Assume that $R$ is a prime ring. Then the closure of an $R$-disjoint ideal of $S$ is defined as in Section 1, and we can easily see that it is also an $R$-disjoint ideal. So all the results of the former sections can be applied to study closed ideals in $S$. It turns out that every $R$-disjoint prime ideal of $S$ is closed and so, in particular, our results can be applied to study prime ideals of centred extensions. Actually this study was our first purpose.

We studied $R$-disjoint prime ideals of free centred extensions in [5]. Then we considered the general case in ([6], Sections 5-8). When $S$ is a centred extension of $R$, the canonical torsion-free extension $S^*$ of $S$ is a ring which is a centred extension of $Q$, the canonical map $j$ is a ring homomorphism, and the corresponding vector space $V$ is an algebra over $C$. In this case everything goes smoothly and we have

**THEOREM 3.13** *The correspondence of Theorem 2.5 is a one-to-one correspondence between the following:*

(i) *The set of all the $R$-disjoint prime ideals of $S$.*

(ii) *The set of all the $Q$-disjoint prime ideals of $S^*$.*

(iii) *The set of all the prime ideals of $V$.*

This theorem is very useful to study special types of prime ideals and radical questions. Many results were obtained in [5] and [6] as applications of the theorem. As an example, we give the following.

**THEOREM 3.14** *Assume that $R$ is a strongly prime (resp. non-singular prime) ring, $S$ is a centred extension of $R$, and $P$ is an $R$-disjoint prime ideal of $S$. Then $P$ is also strongly prime (resp. non-singular) provided one of the following conditions is fulfilled:*

(i) *$P$ is maximal among the $R$-disjoint ideals of $S$.*

(ii) *The generating set $X$ is a commuting subset of $S$.*

The reader is refered to [5] and [6] for more results of this type. In particular, we point out that in the latest paper we also obtained results for prime ideals of intermediate extensions; i.e., rings $T$ with $R \subseteq T \subseteq S$.

### 3.6 Semisimplicity of Free Centred Extensions

An interesting application of the results in 3.5 is a theorem which reduces the question of whether a free centred extension is (Jacobson) semisimple to algebras over fields. This theorem was proved in [7] and states the following.

**THEOREM 3.15** *Let $R$ be a semisimple ring and $S = R[E]$ a free centred extension of $R$. Assume that $C[E]$ is semisimple for every field $C$ which is the extended centroid of a primitive factor of $R$. Then $S$ is also semisimple.*

This theorem has been extended in [11] to several other radicals, as we see now.

### 3.7 Radicals of Centred Extensions

Since the most popular radicals are intersections of prime ideals, is not surprising that we can obtain information concerning radicals of centred extensions from the information on radicals of algebras over $C$. This idea has been used in [11] to study first radicals of centred extensions and then to apply the results to tensor products of algebras over commutative rings.

In particular we proved, for example, that if $s$ denotes the strongly prime radical we have

**THEOREM 3.16** *Assume that $R$ is a strongly prime ring and $S$ is a torsion-free centred extension of $R$. Then $s(S) = Qs(V) \cap S$.*

Actually, in [11] we also obtained the corresponding of Theorem 3.16 for several other radicals, namely, the prime, locally nilpotent, nil, singular, Jacobson, and Brown-McCoy radicals. We proved also a theorem corresponding to Theorem 3.15 for all these radicals.

### 3.8 Prime Ideals and Radicals of Tensor Products

The most part of [11] was devoted to apply the results on prime ideals of centred extensions to study prime ideals and radicals of tensor products. If $A$ and $B$ are algebras over a commutative ring $F$, then $A \otimes_F B$ is a centred bimodule over $A$ (and $B$). So the former results can be applied.

If $I$ is an ideals of $A \otimes_F B$ we put $I_A = \{a \in A : a \otimes 1 \in I\}$ and $I_B = \{b \in B : 1 \otimes b \in I\}$. The ideal $I$ is said to be $A$-$B$-disjoint if $I_A = I_B = 0$.

In the next theorem we assume that $A$ and $B$ are prime algebras over a commutative ring $F$. We denote by $Q(A)$ and $Q(B)$ the Martindale right ring of quotients of $A$ and $B$ and by $C(A)$ and $C(B)$ its extended centroids, respectively. Also, $\phi: A \otimes B \rightarrow Q(A) \otimes Q(B)$ denotes the canonical map, and tensor products are always over $F$.

**THEOREM 3.17** *Let $A$ and $B$ be prime algebras over a commutative ring $F$. Then there is a one-to-one correspondence between the following:*

(i) *The set of all the $A$-$B$-disjoint prime ideals of $A \otimes B$.*

(ii) *The set of all the $Q(A)$-$Q(B)$-disjoint prime ideals of $Q(A) \otimes Q(B)$.*

(iii) *The set of all the prime ideals of $C(A) \otimes C(B)$.*

*Moreover, this correspondence associates a prime ideal $P$ of $A \otimes B$ with a prime ideals $P^*$ of $Q(A) \otimes Q(B)$ and a prime ideal $P_0$ of $C(A) \otimes C(B)$ if $P = \phi^{-1}(P^*)$ and $P^* = P_0(Q(A) \otimes Q(B))$.*

We obtained also a characterization for $C(A) \otimes C(B)$ to be a domain and to be a field. Finally, we proved results of the following type.

**THEOREM 3.18** *Let $P$ be an $A$-$B$-disjoint prime ideal of $A \otimes B$. Then $P$ is strongly prime (resp. non-singular, locally nilpotent semisimple) if and only if $A$ and $B$ are strongly prime (resp. non-singular, locally nilpotent semisimple).*

Using the above results we can study prime ideals of tensor products which are not necessarily disjoint.

**COROLLARY 3.19** *Assume that $A$ and $B$ are algebras over a commutative ring $F$ and $P$ is a prime ideal of $A \otimes B$. Then $P$ is strongly prime (resp. non-singular, locally nilpotent semisimple) if and only if $P_A$ and $P_B$ are strongly prime (resp. non-singular, locally nilpotent semisimple).*

We refer the reader to [11] for many other results of the above type.

### 3.9 Prime Ideals in Polynomial Rings

To finish the section we consider polynomial rings. Actually, the case of a polynomial ring $R[x]$ in one indeterminate $x$ over a prime ring $R$ was the first considered by me [4], and this paper gave rise to the series of papers that I wrote on the subject. However, the case of polynomial rings in $n$-indeterminates was considered only recently, and the paper is not published yet [9]. There is another paper devoted to prime and maximal ideals [8] which deserves to be quoted. Finally, we point out that the method have also been used to study prime ideals in skew polynomial rings (see [2], [10], [14]).

The general results on closed and prime ideals are of course true for polynomial rings, but this case has some particularities that we should mention here. For example, we have

**THEOREM 3.20** *Assume that $R$ is a prime ring and $R[x]$ is a polynomial ring over $R$ in one indeterminate $x$. An $R$-disjoint ideal $I$ of $R[x]$ is closed if and only if $I = Q[x]f_0 \cap R[x]$, for some monic polynomial $f_0 \in C[x]$.*

Theorem 3.20 and some well-known results on polynomial rings immediately imply the following.

**COROLLARY 3.21** *An $R$-disjoint ideal $P$ of $R[x]$ is prime if and only if $P = Q[x]f_0 \cap R[x]$, for some monic irreducible polynomial $f_0 \in C[x]$.*

**COROLLARY 3.22** *Let $R$ be a prime ring. Then there exists a one-to-one correspondence between the following:*

(i) *The set of all the $R$-disjoint prime ideals of $R[x]$.*

(ii) *The set of all the $Q$-disjoint prime ideals of $Q[x]$.*

(iii) *The set of all the irreducible polynomials of $C[x]$.*

*Moreover, this correspondence associates the prime $P$ of $R[x]$ with the prime $P^*$ of $Q[x]$ and the irreducible polynomial $f_0$ of $C[x]$ if $P^* \cap R[x] = P$ and $P^* = Q[x]f_0$.*

The results on $R$-disjoint prime ideals have been improved and extended ([8], [9]). In particular, in [8] we got an intrinsic description of an $R$-disjoint prime ideal, namely, we can describe an $R$-disjoint prime ideal of $R[x]$ as an ideal which is closed and determined in some sense by a, so called, completely irreducible polynomial of $R[x]$.

Finally, in [9] we proved that any prime ideal $P$ of a polynomial ring $S = R[x_1, ..., x_n]$ is determined by its intersection with $R$ plus $n$ polynomials in $S$, where $R$ is here any (not necessarily prime) ring. These $n$ polynomials determine a sequence which is called a $(P \cap R)$-completely irreducible sequence. The converse is also true, i.e, given a prime ideal $P_0$ of $R$ and a $P_0$-completely irreducible sequence of $S$, there exists a prime ideal $P$ of $S$ which is determined by this sequence. It turns out that there exists a one-to-one corresponcence between prime ideals of

$S$ and equivalence classes of sequences of the type $(P_0, f_1, ..., f_n)$, where $P_0$ is a prime ideals of $R$ and $(f_1, ..., f_n)$ is a $P_0$-completely irreducible sequence. We refer the reader to [8] and [9] for more details.

# 4 Questions

We finish the paper with some open questions.

**QUESTION 4.1** I do not know whether the results in Sections 1 and 2 can be extended to study prime ideals of more general types of extensions of prime rings as, for example, normalizing extensions, strongly normalizing extensions, crossed products, etc. Some results in this direction are in [15].

**QUESTION 4.2** It should be very interesting to extend the method given in this paper to study closed submodules and ideals in the case when $R$ is a semiprime ring. By doing this we would be able to describe some radicals of the extension using algebras over commutative rings. I am trying to do this, and have some results in the case when $R$ is a semiprime noetherian ring.

**QUESTION 4.3** I think that probably Theorem 3.15 can be extended to more general kind of extensions as, for example, skew group rings.

**QUESTION 4.4** A ring $R$ is said to be a Jacobson ring if every prime ideal of $R$ is an intersection of primitive ideals. If $R$ is a Jacobson ring, then the polynomial ring $R[x]$ is a Jacobson ring [19]. It is clearly easier to prove the result when $R$ is a field instead of any Jacobson ring. So it would be very convenient to have a theorem of the following type: if $R$ is a Jacobson ring and $C[E]$ is a Jacobson ring for every field $C$ which is the extended centroid of a prime factor of $R$, then $R[E]$ is also a Jacobson ring. I was unable to prove such a result, even under some finiteness condition on the basis $E$ as, for example, $C[E]$ being a noetherian ring.

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