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New Results on the Canonical Structure of Classical Non-Linear Sigma Models [1,2]

Michael Forger

Abstract: The material presented here is based on recent work of the author (done in collaboration with colleagues from the University of Freiburg and from the University of São Paulo) which has produced new insight into the algebraic structure of classical non-linear sigma models as (infinite-dimensional) Hamiltonian systems [1-5]. After some introductory remarks intended to place this line of research into its appropriate context, I proceed to discuss the two main results obtained so far. The first, valid for general sigma models (defined on arbitrary Riemannian manifolds), is the explicit calculation of their extended current algebra, i.e., the algebra generated, under Poisson brackets, by the components of the Noether current referring to a given internal symmetry and by the components of the energy-momentum tensor; this calculation can be carried out in closed form by introducing a single new composite scalar field. The second, valid for integrable sigma models (defined on Riemannian symmetric spaces), is the identification of a new algebra which, in this class of models, should be regarded as the substitute for the classical Yang-Baxter algebra. To put this result into its proper perspective, a brief summary of basic definitions from the theory of two-dimensional integrable field theories is given, including that of ultralócal vs. non-ultralocal models.

Key words: Sigma models, current algebras, integrable systems.

Much of the remarkable progress that has been made in the understanding of two-dimensional quantum field theories over the last ten years is due to the identification of new algebras generated by certain composite fields and the systematic application of techniques from representation theory to analyze the algebraic structure of concrete models and unravel general features common to whole classes of models. A typical example is provided by the class of conformal quantum field theories, which can be defined as the quantum field theories possessing a traceless energy-momentum tensor. In two dimensions, these theories turn out to be chiral[3] and can roughly be classified according to their chiral algebras, generated by the chiral fields.

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[3] By definition, chiral quantum field theories are those two-dimensional quantum field theories in which all local fields $A(x)$ can be decomposed into sums of products $A_+(x^+)A_-(x^-)$ of mutually commuting chiral fields, i.e., of mutually commuting fields $A_+(x^+)$ and $A_-(x^-)$ depending on only one of the two light-cone variables x^+ and x^-, respectively.

The standard chiral algebra that turns up in any two-dimensional conformal quantum field theory is the Virasoro algebra, because the usual axioms of quantum field theory [6, 7, 8] imply [9, 10] that, in two space-time dimensions, a traceless energy-momentum tensor necessarily generates a chiral Virasoro algebra, that is, its chiral (= light-cone) components generate two commuting copies of the Virasoro algebra. However, the relevant chiral algebra may be larger. For example, it can similarly be shown that a chiral current, i.e., a current which is simultaneously conserved and curl-free, generates a chiral affine Kac-Moody algebra, that is, its chiral (= light-cone) components generate two commuting copies of an affine Kac-Moody algebra. Thus in some classes of conformal quantum field theories, the relevant chiral algebra is a suitable extension of the Virasoro algebra – normally its semidirect sum either with an affine Kac-Moody algebra, as in the case of the WZNW (Wess-Zumino-Novikov-Witten) models, or with a W-algebra, as in the case of the Toda field theories.

The main role of this chiral algebra is to organize all fields appearing in the theory into families headed by a so-called primary field and labelled by certain quantum numbers which are determined from the representation theory of the chiral algebra; mathematically, this amounts to decomposing the full field algebra into irreducible modules under the action of the chiral (sub)algebra and labeling these irreducible modules by the eigenvalues of the appropriate Casimir operators.

A more general class of two-dimensional quantum field theories in which algebraic methods have recently proved to be fruitful are the so-called integrable models. Here, an outstanding role is played by several new types of algebras, most notably the so-called Yang-Baxter algebras and certain families of Hopf algebras which – somewhat misleadingly – have come to be known as quantum groups. In relativistic quantum field theory, the Yang-Baxter equation – which can be viewed as the structure equation for Yang-Baxter algebras in exactly the same way as, for instance, the Jacobi identity can be viewed as the structure equation for Lie algebras – first appeared in some special two-dimensional models such as the Thirring model [11], the $O(N)$-invariant non-linear sigma model and the $O(N)$-invariant Gross-Neveu model [12], where it arises as the factorization equation for the two-body scattering matrix. (For a brief review, see [13].)

Although all the aforementioned algebras occur most naturally in quantum field theory, they or their classical precursors also appear in classical field theory and may find useful applications there. In conformal field theory, for example, we encounter the Witt algebra, i.e., the Virasoro algebra with vanishing central charge, as the classical precursor of the full Virasoro algebra and, when chiral currents are present, the relevant affine Kac-Moody algebra, as its own classical precursor. Similarly but somewhat less trivially, the classical Yang-Baxter equation, which plays an important role in the analysis of the Hamiltonian structure of integrable models in classical field theory [14], can be regarded as the classical precursor of the quantum Yang-Baxter equation and has in fact originally been derived from it by performing a semiclassical expansion (expansion in powers of $\hbar$) [15]. However, there is evidence that this derivation (see also [16]) must in some sense be incomplete, or that it perhaps contains some additional assumptions which have not been clearly specified, because the class of integrable models where the classical Yang-Baxter equation finds its natural place is only a subclass, namely that of the so-called ultralocal models[4] [14]. This of course raises the question as to what algebra will replace the classical Yang-Baxter algebra for other integrable models and by what kind of semiclassical approximation procedure that other algebra can be derived from the quantum Yang-Baxter equation.

As it is a priori unclear whether a concise answer to this problem exists at all, at least when the question is posed in such generality, it is certainly worthwhile to investigate it for a specific class of non-ultralocal models such as the integrable sigma models defined on Riemannian symmetric spaces. In fact, the essence of the puzzle becomes apparent when one compares the simplest (non-ultralocal) sigma model, namely the $O(N)$-invariant sigma model defined on the sphere S^{N-1}, with a simple (ultralocal) fermionic model, namely the $O(N)$-invariant Gross-Neveu model. Classical integrability of the latter involves a classical Yang-Baxter algebra (the standard r-matrix solution of the classical Yang-Baxter equation), but classical integrability of the former involves an entirely different algebra, whose defining relations have so far not been properly identified – despite the fact that quantum integrability of both models is governed by the quantum Yang-Baxter algebra and in fact by almost the same R-matrix solution of the quantum Yang-Baxter equation: the two R-matrices just differ by a simple pole factor reflecting the absence (sigma model) resp. presence (Gross-Neveu model) of bound states [12]. Obviously, the fact that a pole factor can produce such drastically different classical limits indicates that the problem of what should be the proper classical limit of the quantum Yang-Baxter algebra is a delicate one and is still far from being fully understood.

This unsatisfactory situation provided the motivation for our recent study [2] of classical integrable sigma models defined on Riemannian symmetric spaces from the viewpoint of (infinite-dimensional) Hamiltonian systems. This investigation, carried out completely in the framework of classical field theory, has produced new insight into the nature of the algebra which, for this class of non-ultralocal models, underlies their Hamiltonian structure in the same way as the classical Yang-Baxter algebra governs the Hamiltonian structure of ultralocal models. As a prerequisite, we found it necessary to compute explicitly the current algebra of classical non-linear sigma models [1] – a calculation which, to our great surprise, has apparently never before been performed in full generality and which can easily be extended to include the energy-momentum tensor [4]. The algebras resulting from these calculations seem to be new and may be of independent interest; it is therefore an instructive exercise to compare them with the analogous algebras in conformal field theories with chiral currents, in particular in the Wess-Zumino-Novikov-Witten models [3, 5]. What remains open at the present time is a solution to the second part of the puzzle, namely an explicit understanding of the way in

[4] A precise definition of this notion will be given below.

which the new algebra emerging from the purely classical analysis carried out in ref. [2] can appear as the classical limit of the quantum Yang-Baxter algebra.

In the following, I want to give a brief account of the results obtained in refs [1, 4] (Sect. 1) and in ref. [2] (Sect. 2)..

1 Extended Current Algebra of Sigma Models

For models of classical field theory which exhibit an invariance under some continuous internal symmetry, the term "current algebra" is generally used to denote the algebra generated, under Poisson brackets, by the components of the Noether current referring to that symmetry. Similarly, I shall employ the term "extended current algebra" as an abbreviation to denote the algebra generated, under Poisson brackets, by the components of that Noether current and the components of the energy-momentum tensor. As an important example where these algebras have a standard structure and can be regarded as well understood, we may mention the class of two-dimensional conformal classical field theories with chiral currents, where the current algebra is a chiral affine Kac-Moody algebra and the extended current algebra its semidirect sum with a chiral Witt algebra (= Virasoro algebra with vanishing central charge). Moreover, in typical examples such as the WZNW (Wess-Zumino-Novikov-Witten) models, the latter can indeed be considered an extension of the former since the light-cone components of the energy-momentum tensor can be written as quadratic forms in the corresponding light-cone components of the Noether current. (The quantum version of this statement is known as the Sugawara construction; see [17] for details.)

The situation for two-dimensional classical non-linear sigma models is different because these models are conformal field theories in which the Noether current fails to be chiral. Correspondingly, the light-cone components of the energy-momentum tensor still generate a chiral Witt algebra, but the current algebra is non-chiral and apparently has nothing to do with affine Kac-Moody algebras.[5] This is a significant result in view of the role of non-linear sigma models as one of the most important and widely studied classes of field theoretical models.

Classically, a non-linear sigma model is simply a field theory of maps between two given pseudo-Riemannian manifolds, defined by an action functional which is purely quadratic in first derivatives and whose stationary points are exactly the pseudo-harmonic maps. (This action functional is, somewhat misleadingly, called energy functional in some of the mathematical literature [18, 19].) The input data to define a specific non-linear sigma model are therefore a pseudo-Riemannian manifold Σ, with metric h, as source space, and a pseudo-Riemannian manifold M, with metric g, as target space. We shall adopt the standard interpretation of the source space as space-time and the target space as an auxiliary space repre-

[5] In particular, the convention occasionally adopted in the literature of using the terms "current algebra" and "affine Kac-Moody algebra" as being synonymous is more than unfortunate and should be avoided.

senting internal degrees of freedom, so h will be a Lorentz metric and g a Riemann metric. (For applications of non-linear sigma models to string theory, however, the interpretation is essentially reversed in that the target space is interpreted as space-time and the source space as an auxiliary space for the parameters of the string world sheet.) The configuration space of the model is then the space of (smooth) maps φ from Σ to M. In the following, we shall employ component notation, that is, we shall use local coordinates x^μ on Σ and u^i on M to express the map φ and its tangent map $T\varphi$ in terms of component fields φ^i and their partial derivatives $\partial_\mu \varphi^i$. Then the action of the model reads

$$S = \tfrac{1}{2} \int d^n x \, \sqrt{|\det(h)|} \, h^{\mu\nu} \, g_{ij}(\varphi) \, \partial_\mu \varphi^i \, \partial_\nu \varphi^j \quad , \tag{1}$$

where $\dim \Sigma = n$, and the Euler-Lagrange equations of motion are

$$h^{\mu\nu} \left(\partial_\mu \partial_\nu \varphi^i + \Gamma^i_{jk}(\varphi) \, \partial_\mu \varphi^j \, \partial_\nu \varphi^k \right) = 0 \ . \tag{2}$$

When dealing with the two-dimensional model, which by definition means that Σ is two-dimensional, we may and shall assume for simplicity that the local coordinates x^μ on Σ are isothermal, so we can write $h_{\mu\nu} = e^{2\sigma} \, \eta_{\mu\nu}$ where the $\eta_{\mu\nu}$ are the coefficients of the standard Minkowski metric and σ is a conformal factor. Then the action simplifies to

$$S = \tfrac{1}{2} \int d^2 x \; \eta^{\mu\nu} \, g_{ij}(\varphi) \, \partial_\mu \varphi^i \, \partial_\nu \varphi^j \quad , \tag{3}$$

and the equations of motion become

$$\eta^{\mu\nu} \left(\partial_\mu \partial_\nu \varphi^i + \Gamma^i_{jk}(\varphi) \, \partial_\mu \varphi^j \, \partial_\nu \varphi^k \right) = 0 \ . \tag{4}$$

The conformal factor σ has dropped out, signalling conformal invariance for $\dim \Sigma = 2$.

The Hamiltonian formulation of the model may be derived from the Lagrangian one by applying standard procedures of classical field theory to pass from the configuration space of the theory to its phase space (which should form a symplectic manifold).[6] Roughly speaking, there are two ways to do this: the phase space can be defined either as the space of all solutions to the equations of motion (covariant approach) or as the space of all initial conditions (non-covariant approach). The first method seems to be relatively recent [20] and is extremely elegant; its application to non-linear sigma models is presently being worked out [21]. The second method is the traditional one and, being easier to handle, the one we shall

[6] The manifolds appearing as configuration spaces or phase spaces in classical field theory are infinite-dimensional and are usually easy to write down as sets, while the definition of their manifold structure – and therefore also the precise definition of their tangent or cotangent bundle – is in general rather involved and quite technical. Fortunately, it is often sufficient to regard the prescription of forming the tangent or cotangent bundle at a purely formal level.

adopt in the following. Here, the first step is to consider all fields in the theory as being restricted to a fixed Cauchy hypersurface $\mathcal{S}$ in the space-time manifold Σ – a procedure which can be regarded as the passage from the configuration space in the Lagrangian sense to the configuration space in the (non-covariant) Hamiltonian sense: the phase space is then the formal cotangent bundle of the configuration space so defined. For the classical non-linear sigma model on M, with input data as described above, the resulting phase space consists of pairs (φ, π), with φ a (smooth) map from $\mathcal{S}$ to M and π a (smooth) section of the pull-back $\varphi^*(T^*M)$ of the cotangent bundle of M to $\mathcal{S}$ via φ. As before, we can use local coordinates x^μ on Σ and u^i on M to express the map φ and the section π in terms of component fields φ^i and π_i; moreover, we shall arrange the local coordinates x^μ on Σ so that on the hypersurface $\mathcal{S}$, x^0 is a coordinate normal to $\mathcal{S}$ while the remaining coordinates are along $\mathcal{S}$. In the two-dimensional case, we may and shall, as before, assume that the local coordinates x^μ on Σ are isothermal; moreover, as there is in this case just a single coordinate x^1 along $\mathcal{S}$, we shall suppress the index 1. Then using a dot to denote the time derivative (and later on, a prime to denote the spatial derivative), the Legendre transformation derived from the action (3) yields the relation

$$\pi_i \;=\; g_{ij}(\varphi)\,\dot{\varphi}^j \quad , \tag{5}$$

and the canonical Poisson brackets, expressing the fact that the π_i are indeed the canonically conjugate momenta to the φ^i, read

$$\begin{gathered} \{\varphi^i(x)\,,\varphi^j(y)\} = 0 \quad , \quad \{\pi_i(x)\,,\pi_j(y)\} = 0 \quad , \\ \{\varphi^i(x)\,,\pi_j(y)\} \;=\; \delta^i_j\,\delta(x-y) \quad . \end{gathered} \tag{6}$$

The energy-momentum tensor $\theta_{\mu\nu}$ of the theory is most conveniently derived by variation of the Lagrangian appearing in eq. (1) with respect to the metric h on Σ. (For details, see e.g. [22, p. 64 ff] or [23, p. 504 f].) It reads

$$\theta_{\mu\nu} \;=\; g_{ij}(\varphi)\,\partial_\mu\varphi^i\,\partial_\nu\varphi^j \;-\; \tfrac{1}{2}\,\eta_{\mu\nu}\,\eta^{\kappa\lambda}\,g_{ij}(\varphi)\,\partial_\kappa\varphi^i\,\partial_\lambda\varphi^j \quad . \tag{7}$$

In the two-dimensional case, it is obviously traceless,

$$\eta^{\mu\nu}\,\theta_{\mu\nu} \;=\; 0 \quad , \tag{8}$$

signalling, once again, that the theory is conformally invariant for $\dim\Sigma = 2$. Moreover, we shall assume that the theory exhibits a global invariance under some internal symmetry group, represented by a (connected) Lie group G acting on M by isometries; this action will be written as a map

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ (g,m) & \longmapsto & g \cdot m \end{array} \quad . \tag{9}$$

Then every generator X in the Lie algebra $\mathfrak{g}$ of G is represented by a fundamental vector field X_M on M, given by

$$X_M(m) \;=\; \frac{d}{dt}\,(\exp(tX)\cdot m)\Big|_{t=0} \quad , \tag{10}$$

and we have the representation condition

$$[X,Y]_M = -[X_M, Y_M] \; ; \tag{11}$$

moreover, all these vector fields are Killing vector fields. As usual, G-invariance of the action (1) or (3) leads to a conserved Noether current j_μ with values in $\mathfrak{g}^*$ (the dual of $\mathfrak{g}$). Explicitly, for $X \in \mathfrak{g}$,

$$\langle j_\mu, X\rangle = -g_{ij}(\varphi)\,\partial_\mu\varphi^i\, X^j_M(\varphi) \; . \tag{12}$$

Finally, we shall find it convenient to define a scalar field j with values in the second symmetric tensor power of $\mathfrak{g}^*$ by setting, for $X, Y \in \mathfrak{g}$,

$$\langle j, X\otimes Y\rangle = g_{ij}(\varphi)\, X^i_M(\varphi)\, Y^j_M(\varphi) \; . \tag{13}$$

($\langle .\,,.\rangle$ denotes the natural pairing between a vector space and its dual.)

The energy-momentum tensor $\theta_{\mu\nu}$, the Noether current j_μ and the auxiliary field j are composite fields and, when restricted to the hypersurface $\mathcal{S}$ in Σ, can be considered as defining specific function(al)s on the phase space of the theory with well-defined Poisson brackets which can be computed explicitly from the canonical Poisson brackets. These calculations have been carried out in refs [1] and [4], with the following results: in two dimensions ($\dim\Sigma = 2$), the energy-momentum tensor algebra reads [4]

$$\{\theta_{00}(x)\,,\theta_{00}(y)\} = (\theta_{01}(x)+\theta_{01}(y))\,\delta'(x-y) \; , \tag{14}$$

$$\{\theta_{00}(x)\,,\theta_{01}(y)\} = (\theta_{00}(x)+\theta_{00}(y))\,\delta'(x-y) \; , \tag{15}$$

$$\{\theta_{01}(x)\,,\theta_{01}(y)\} = (\theta_{01}(x)+\theta_{01}(y))\,\delta'(x-y) \; , \tag{16}$$

(note $\theta_{11} = \theta_{00}$), while the current algebra is [1]

$$\{\langle j_0(x)\,,X\rangle\,,\langle j_0(y)\,,Y\rangle\} = -\langle j_0(x)\,,[X,Y]\rangle\,\delta(x-y) \; , \tag{17}$$

$$\begin{aligned}\{\langle j_0(x)\,,X\rangle\,,\langle j_1(y)\,,Y\rangle\} = & -\langle j_1(x)\,,[X,Y]\rangle\,\delta(x-y) \\ & + \langle j(y)\,,X\otimes Y\rangle\,\delta'(x-y) \; ,\end{aligned} \tag{18}$$

$$\begin{aligned}\{\langle j_0(x)\,,X\rangle\,,\langle j(y)\,,Y\otimes Z\rangle\} = & -\langle j(x)\;,\,[X,Y]\otimes Z + Y\otimes[X,Z]\rangle \\ & \delta(x-y) \; ,\end{aligned} \tag{19}$$

(the remaining Poisson brackets vanish), and the mixed Poisson brackets are [4]

$$\{\theta_{00}(x)\,,\langle j_0(y)\,,X\rangle\} = \langle j_1(x)\,,X\rangle\,\delta'(x-y) \; , \tag{20}$$

$$\begin{aligned}\{\theta_{00}(x)\,,\langle j_1(y)\,,X\rangle\} = & \langle j_0(x)\,,X\rangle\,\delta'(x-y) \\ & -\langle(\partial_0 j_1 - \partial_1 j_0)(x)\,,X\rangle\,\delta(x-y) \; ,\end{aligned} \tag{21}$$

$$\{\theta_{01}(x)\,,\langle j_0(y)\,,X\rangle\} = \langle j_0(x)\,,X\rangle\,\delta'(x-y) \; , \tag{22}$$

$$\{\theta_{01}(x)\,,\langle j_1(y)\,,X\rangle\} = \langle j_1(x)\,,X\rangle\,\delta'(x-y) \; , \tag{23}$$

$$\{\theta_{00}(x)\,,\langle j(y)\,,X\otimes Y\rangle\} = -\langle\partial_0 j(x)\,,X\otimes Y\rangle\,\delta(x-y) \; , \tag{24}$$

$$\{\theta_{01}(x)\,,\langle j(y)\,,X\otimes Y\rangle\} = -\langle\partial_1 j(x)\,,X\otimes Y\rangle\,\delta(x-y) \; . \tag{25}$$

Moreover, it is also shown in refs [1] and [4] that in higher dimensions ($\dim\Sigma > 2$), the current algebra (eqns (17-19)) and the mixed brackets (eqns (20-25)) remain essentially unchanged, while the energy-momentum tensor algebra (eqns (14-16)) is substantially modified and in fact no longer closes.

Returning to the two-dimensional case, it remains to be shown that eqns (14-16) give rise to a chiral Witt algebra. To do so, we first switch to light-cone components,

$$\theta_{++} = \tfrac{1}{2}(\theta_{00} + \theta_{01}) \ , \tag{26}$$

$$\theta_{--} = \tfrac{1}{2}(\theta_{00} - \theta_{01}) \ , \tag{27}$$

(note that $\theta_{+-} = 0$), and then convert the equal-time Poisson bracket relations (14-16) to commutation relations valid on the light cone: this requires invoking the equations of motion for the energy momentum tensor. These are nothing but the conservation law

$$\partial_-\theta_{++} = 0 \quad , \quad \partial_+\theta_{--} = 0 \quad , \tag{28}$$

so θ_{++} depends only on x^+ and θ_{--} depends only on x^-; then eqns (14-16) become

$$\{\theta_{++}(x^+)\,,\theta_{++}(y^+)\} = +\,(\theta_{++}(x^+) + \theta_{++}(y^+))\,\delta'(x^+ - y^+) \ , \tag{29}$$

$$\{\theta_{--}(x^-)\,,\theta_{--}(y^-)\} = -\,(\theta_{--}(x^-) + \theta_{--}(y^-))\,\delta'(x^- - y^-) \ , \tag{30}$$

$$\{\theta_{++}(x^+)\,,\theta_{--}(y^-)\} = 0 \ . \tag{31}$$

When expressed in Fourier components this becomes the well-known Witt algebra for θ_{++} and for θ_{--}.

For the currents, on the other hand, this procedure cannot be carried out in the same way, because the conservation law for the currents is, in light-cone components,

$$\partial_+ j_- + \partial_- j_+ = 0 \ ,$$

which by itself is not sufficient to convert equal-time Poisson brackets to light-cone Poisson brackets.

The net result is that the total algebra is a semidirect product of a chiral Witt algebra with a non-chiral (non Kac-Moody) current algebra.

One very important question to be answered next is how the algebraic structure derived above changes when passing from the classical theory to the quantum theory. Some changes are to be expected due to the phenomenon of dynamical mass generation, which will give the energy-momentum tensor a non-vanishing trace and destroy the chiral nature of the energy-momentum tensor algebra. Our hope is that the non-chiral current algebra should give a clue as to what should be this non-chiral energy-momentum tensor algebra, and ideally that there should exist some non-chiral analogue of the Sugawara construction – which, as is well known, e.g., for the WZNW (Wess-Zumino-Novikov-Witten) models or for certain massless fermionic theories [17], actually derives the chiral energy-momentum tensor algebra of conformal field theory from the corresponding chiral current algebra (two commuting copies of the relevant (untwisted) affine Kac-Moody algebra).

2 Canonical Structure of Integrable Sigma Models

2.1 Integrable Systems in Classical Field Theory

The definition of the notion of an integrable system in classical field theory is, in contrast to that in classical mechanics, not quite unambiguous. In fact, various definitions are available which, as can be shown on explicit examples, are not equivalent. However, there is a condition which, in two space-time dimensions, seems to be a sort of greatest common denominator for all of them, namely the existence of a Lax representation, or zero curvature representation, for the field equations. By this is meant the existence of a composite vector field $\mathcal{A}_\mu$, given as an explicit local functional of the original fields in the theory[7] and taking values in the Lie algebra $\mathfrak{g}$ of an appropriate Lie group G, such that the Euler-Lagrange equations of motion of the theory become equivalent to the statement that the *Lax connection* $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ has zero curvature:

Zero Curvature Representation:

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu , \mathcal{A}_\nu] = 0 . \tag{32}$$

This flat connection can then (at least locally) be gauged away by means of a scalar field U, taking values in G: its definition in terms of the vector field $\mathcal{A}_\mu$ is a system of first-order linear partial differential equations, usually referred to simply as the

Linear System:

$$\partial_\mu U = U \mathcal{A}_\mu . \tag{33}$$

In the Hamiltonian approach, the fields $\mathcal{A}_\mu$ and U are – like all other fields in the theory – considered as being restricted to a fixed Cauchy hypersurface $\mathcal{S}$ in the two-dimensional space-time manifold Σ, and the isothermal local coordinates x^μ on Σ are always chosen such that the spatial coordinate x is the local coordinate along $\mathcal{S}$. The objects of central importance for the analysis of an integrable model as an (infinite-dimensional) Hamiltonian system are then the Lax matrix, the Lax operator, the transition matrix and the monodromy matrix:

[7] In this context, locality of a composite field ϕ simply means that its functional dependence on the original fields ϕ_i in the theory is such that, for any domain Ω in space-time, ϕ will vanish on Ω as soon as all the ϕ_i vanish on Ω. This is the case if (and only if) ϕ is a linear combination of products of the ϕ_i and their partial derivatives.

Lax Matrix: $L \equiv \mathcal{A}_1$,

Lax operator:

$$D(x) \;=\; \frac{\partial}{\partial x} + L(x) \quad , \tag{34}$$

Transition Matrix:

$$T(x,y) \;=\; U(x)^{-1}\,U(y) \;=\; P\exp\int_y^x dz\; L(z) \quad , \tag{35}$$

Monodromy Matrix:

$$T \;=\; \lim_{\substack{x\to+\infty \\ y\to-\infty}} T(x,y) \quad . \tag{36}$$

Apart from their possible implicit dependence on space-time coordinates or spatial coordinates, induced by their functional dependence on the original fields in the theory, these objects usually exhibit an explicit dependence on an additional parameter λ, the so-called *spectral parameter*. In standard cases, λ is simply a complex number, but should better be regarded as a local coordinate on the Riemann sphere since the functions of λ that typically arise in this context are meromorphic.[8] In order not to burden the notation, however, we shall in the following suppress the dependence on the spectral parameter whenever there is no danger of confusion.

The passage from given initial data on the Cauchy hypersurface $\mathcal{S}$ to the monodromy matrix T is usually referred to as the *direct scattering transformation*. Moreover, there are cases, such as the famous Korteweg – de Vries, non-linear Schrödinger or sine-Gordon equations, where the initial data can in fact be completely recovered from the monodromy matrix by means of the so-called *inverse scattering transformation* and where, in addition, the entries of the monodromy matrix satisfy canonical commutation relations with respect to Poisson brackets; this means that the direct scattering transformation is in fact the analog of Liouville's canonical transformation to action-angle variables in a field-theoretical context: we shall call such systems *completely integrable*. But even for integrable systems which are not completely integrable, the monodromy matrix still plays a central role as the generating functional for higher conserved charges: these arise from expanding its logarithm in power series of the spectral parameter, around some suitably chosen normalization point. The structure of the hidden symmetry group generated by these charges can be found by determining their Lie algebra, or in other words, their commutation relations: this requires, once again, computing the Poisson brackets between the entries of the monodromy matrix.

The strategy for calculating the Poisson brackets between the entries of the monodromy matrix is simple: one first computes the Poisson brackets between the entries of the Lax matrix and, from these, derives the Poisson brackets between

[8] For certain types of integrable systems, the spectral parameter becomes a local coordinate on a more general Riemann surface; we shall not encounter such situations here.

the entries of the transition matrix by using the differential equations with the initial condition that define $T(x, y)$ from $L(z)$; they read

$$\frac{\partial}{\partial x} T(x, y, \lambda) \;=\; -\,L(x, \lambda)\, T(x, y, \lambda) \tag{37}$$

and

$$\frac{\partial}{\partial y} T(x, y, \lambda) \;=\; +\,T(x, y, \lambda)\, L(y, \lambda) \tag{38}$$

with

$$T(x, y, \lambda)\Big|_{x=y} \;=\; 1 \;. \tag{39}$$

For later use, we also note the composition rule

$$T(x, y, \lambda)\, T(y, z, \lambda) \;=\; T(x, z, \lambda) \;, \tag{40}$$

which, together with (39), leads to the inversion formula

$$T(x, y, \lambda)^{-1} \;=\; T(y, x, \lambda) \;. \tag{41}$$

The resulting Poisson bracket relations will in the following simply be called the *fundamental Poisson brackets* for the Lax matrix or Lax operator resp. for the transition matrix. Finally, in the latter one must simply take the limit $x \to +\infty$, $y \to -\infty$ to derive the *fundamental Poisson brackets* for the monodromy matrix.

In order to compute the Poisson brackets between the entries of the Lax matrix, we have to know their explicit form as function(al)s on the phase space of the theory, that is, their functional dependence on the original fields, restricted from Σ to $\mathcal{S}$, and on their canonically conjugate momenta. This dependence is, of course, model dependent. One statement that can however be made in general is that the Poisson brackets between matrix elements of $L(x)$ and $L(y)$, as distributions in the spatial variables x and y (viewed as a pair), must be concentrated at $x = y$ and are therefore finite linear combinations of $\delta(x - y)$ and its derivatives, with coefficients that can be written as distributions in x or in y alone: this is simply a consequence of the axiom of locality and motivates the following (tentative) definition: an integrable system is called *ultralocal* if the Poisson brackets between matrix elements of $L(x)$ and $L(y)$ are products of distributions in x or in y with $\delta(x - y)$, i.e., if no derivatives of $\delta(x - y)$ appear; all other integrable systems are called *non-ultralocal*. More generally, what suggests itself is to classify these commutation relations according to the order of their (strongest) singularity; thus an integrable system is *non-ultralocal of first order* if in the Poisson brackets between matrix elements of $L(x)$ and $L(y)$, no derivatives of $\delta(x - y)$ except the first one appear.

In order to be able to formulate more specific hypotheses about the structure of the fundamental Poisson brackets, we shall find it useful to first of all introduce some notation which will free us from the use of an explicit matrix representation of G, thus showing that the structures to be described below are in fact completely independent of the choice of representation. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and, for any integer $n \geq 1$, let $U(\mathfrak{g})^{\otimes n}$ be its n^{th} tensor power: these are associative algebras with unit. (Explicitly, the universal enveloping algebra of $\mathfrak{g}$ can be obtained from the full tensor algebra over $\mathfrak{g}$ by factoring out the two-sided ideal generated by tensors of the form

$$X \otimes Y \ - \ Y \otimes X \ - \ [X,Y] \quad \text{with} \quad X, Y \in \mathfrak{g} \ . \tag{42}$$

The reader who is unfamiliar with the concept of the universal enveloping algebra of a Lie algebra is invited to continue thinking in terms of an arbitrary matrix representation.) Now for any two integers $m < n$, we have an embedding of $U(\mathfrak{g})^{\otimes m}$ into $U(\mathfrak{g})^{\otimes n}$ given by "putting factors of 1 into the remaining variables". For example, $U(\mathfrak{g})$ and $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ will be embedded into $U(\mathfrak{g})^{\otimes n}$ according to

$$\begin{array}{ccc} U(\mathfrak{g}) & \longrightarrow & U(\mathfrak{g})^{\otimes n} \\ u & \longmapsto & u_k \ = \ 1 \otimes \ldots \otimes u \otimes \ldots \otimes 1 \end{array} \tag{43}$$

(with u appearing in the k^{th} place, $1 \leq k \leq n$) and to

$$\begin{array}{ccc} U(\mathfrak{g}) \otimes U(\mathfrak{g}) & \longrightarrow & U(\mathfrak{g})^{\otimes n} \\ u \otimes v & \longmapsto & (u \otimes v)_{kl} \ = \ 1 \otimes \ldots \otimes u \otimes \ldots \otimes v \otimes \ldots \otimes 1 \end{array} \tag{44}$$

(with u appearing in the k^{th} place and v in the l^{th} place, $1 \leq k, l \leq n$), respectively. Moreover, on $U(\mathfrak{g})^{\otimes n}$, we have an action of the permutation group in n elements which, in the simplest case $n = 2$, reduces to the transposition

$$\begin{array}{ccc} U(\mathfrak{g}) \otimes U(\mathfrak{g}) & \longrightarrow & U(\mathfrak{g}) \otimes U(\mathfrak{g}) \\ u \otimes v & \longmapsto & (u \otimes v)^T \ = \ v \otimes u \end{array} \quad . \tag{45}$$

These operations will in the sequel be applied not only to elements of $U(\mathfrak{g})$ or $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ but also to functions or distributions taking their values in $U(\mathfrak{g})$ or $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. In this case the numerical indices k and l, the operation of transposition etc. will refer not only to the position inside the tensor product but also to the independent variable(s) involved. Thus if u is a function of x with values in $U(\mathfrak{g})$, then u_k will denote the function of $(x_1, \ldots, x_n)$ with values in $U(\mathfrak{g})^{\otimes n}$ given by

$$u_k(x_1, \ldots, x_n) \ = \ (u(x_k))_k \quad , \tag{46}$$

or more explicitly,

$$u_k(x_1, \ldots, x_n) \ = \ 1 \otimes \ldots \otimes u(x_k) \otimes \ldots \otimes 1 \tag{47}$$

(with u appearing in the k^{th} place, $1 \leq k \leq n$). Similarly, if t is a function of (x, y) with values in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, then t_{kl} will denote the function of $(x_1, \ldots, x_n)$ with values in $U(\mathfrak{g})^{\otimes n}$ given by

$$t_{kl}(x_1, \ldots, x_n) \ = \ (t(x_k, x_l))_{kl} \quad , \tag{48}$$

or more explicitly, when t can be factorized in the form $t(x,y) \;=\; u(x) \otimes v(y)$ where u and v are functions of x and y, respectively, with values in $U(\mathfrak{g})$,

$$t_{kl}(x_1, \ldots, x_n) \;=\; 1 \otimes \ldots \otimes u(x_k) \otimes \ldots \otimes v(x_l) \otimes \ldots \otimes 1 \tag{49}$$

(with u appearing in the k^{th} place and v in the l^{th} place, $1 \leq k, l \leq n$). Finally, if t is a function of (x,y) with values in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, then t^T will denote the function of (x,y) with values in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ given by

$$t^T(x,y) \;=\; (t(y,x))^T \quad , \tag{50}$$

or more explicitly, when t can be factorized in the form $t(x,y) \;=\; u(x) \otimes v(y)$ where u and v are functions of x and y, respectively, with values in $U(\mathfrak{g})$,

$$t^T(x,y) \;=\; v(x) \otimes u(y) \quad . \tag{51}$$

These generalized conventions can in fact be reduced to the previous ones by substituting for $U(\mathfrak{g})$ its tensor product with some appropriate space of functions or distributions, but as there is little to be gained by such a formalization, this will not be attempted here.

What will however be important for the applications that will be encountered below is that $U(\mathfrak{g})$ should contain not only the Lie algebra $\mathfrak{g}$ itself as a subspace but also the Lie group G – or at least the quotient Lie group G/Z, where Z is an appropriately chosen discrete subgroup of the center of G (this is, in general, the best that can be achieved) – as a subgroup of the group of its invertible elements. This will be the case provided we understand $U(\mathfrak{g})$ to be the completion of the purely algebraic universal enveloping algebra $U_{\text{alg}}(\mathfrak{g})$ in some locally convex topology for which a) multiplication is continuous and b) the exponential series is absolutely convergent. One explicit possibility to introduce such a topological version of the universal enveloping algebra of $\mathfrak{g}$, as a Banach algebra $U(\mathfrak{g})$ which is the norm completion of the purely algebraic universal enveloping algebra $U_{\text{alg}}(\mathfrak{g})$, is to choose any inner product on $\mathfrak{g}$, extend it to the tensor powers of $\mathfrak{g}$ in the standard way and then form the orthogonal direct sum of these tensor powers (in the Hilbert space sense). This "Fock space construction" yields a topological version of the tensor algebra over $\mathfrak{g}$, as a Banach algebra $T(\mathfrak{g})$ which is the norm completion of the purely algebraic tensor algebra $T_{\text{alg}}(\mathfrak{g})$, and just as $U_{\text{alg}}(\mathfrak{g})$ is the quotient of $T_{\text{alg}}(\mathfrak{g})$ by the two-sided ideal generated by elements of the form (42), $U(\mathfrak{g})$ will be the quotient of $T(\mathfrak{g})$ by the closed two-sided ideal generated by elements of the form (42). A similar construction, with a slightly modified definition of the ideal to be factored out, can be given to define a topological version of the n^{th} tensor power of the universal enveloping algebra of $\mathfrak{g}$, as a Banach algebra $U(\mathfrak{g})^{\otimes n}$ which is the norm completion of the purely algebraic n^{th} tensor power $U_{\text{alg}}(\mathfrak{g})^{\otimes n}$, such that the embeddings (43) and (44), as well as the transposition (45), and, more generally, the embedding of $U(\mathfrak{g})^{\otimes m}$ into $U(\mathfrak{g})^{\otimes n}$ $(m<n)$ given by "putting factors of 1 into the remaining variables", as well as the action of the permutation group in n elements on $U(\mathfrak{g})^{\otimes n}$, all become isometric.

With these rather technical remarks out of the way, we can proceed to discuss specific forms of the fundamental Poisson brackets, first for ultralocal models and then for non-ultralocal models of first order.

2.1.1 Ultralocal Models

There exists a distinguished class of ultralocal models which has played (and continues to play) a central role in the evolution of the theory of integrable systems, namely those in which the fundamental Poisson brackets can be expressed in terms of what has come to be known as a "classical r-matrix". The standard examples of completely integrable systems such as the Korteweg – de Vries equation, the non-linear Schrödinger equation, the continuous Heisenberg magnet or the sine-Gordon equation, are of this type. On the other hand, the solution of such models is profoundly related to group theory – more precisely, the theory of Lie groups and Lie algebras, applied to the description of symmetries in dynamical systems. (For a discussion of this aspect, the reader is referred to [14, Part 2, Chapter 4].) The starting point for the analysis of this class of ultralocal models are the following

Fundamental Poisson Brackets for the Lax Matrix:

$$\{L(x) \overset{\otimes}{,} L(y)\} \;=\; [\,r\,,\, L(x) \otimes 1 + 1 \otimes L(y)\,]\; \delta(x-y) \quad . \tag{52}$$

Here r is an element of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ [9] which is allowed to be a meromorphic function of the (two) spectral parameters involved (if any are present) but is supposed to be field independent (i.e., to be constant as a function(al) on the phase space of the theory); moreover, r is required to be antisymmetric ($r^T = -r$) in order for eqn (52) to be consistent with the antisymmetry of the Poisson bracket.

An equivalent form of the above fundamental Poisson brackets appears when we consider instead of the Lax matrix L the Lax operator D, namely the

Fundamental Poisson Brackets for the Lax Operator:

$$\{D(x) \overset{\otimes}{,} D(y)\} \;=\; [\,r\,,\, D(x) \otimes 1 \,+\, 1 \otimes D(y)\,] \quad . \tag{53}$$

In the simplified notation introduced through eqns (43)-(49) above, these fundamental Poisson brackets take the form

$$\{D_1\,, D_2\} \;=\; [\,r_{12}\,, D_1 + D_2\,] \quad . \tag{54}$$

On the other hand, it can be shown by an explicit calculation (see, e.g., [14, pp. 189-194]) that eqn (52) implies, without any further assumptions, the following

[9] The commutator on the rhs of eqn (52) is the one induced by the associative product in $U(\mathfrak{g})^{\otimes 2}$.

Fundamental Poisson Brackets for the Transition Matrix: Given $x > y$ and $u > v$, put $a = \min\{x, u\}$ and $b = \max\{y, v\}$. Then

$$\begin{aligned} &\{T(x,y) \overset{\otimes}{,} T(u,v)\} \\ &= \; + (T(x,a) \otimes T(u,a)) \; r \; (T(a,y) \otimes T(a,v)) \\ &\quad\; - (T(x,b) \otimes T(u,b)) \; r \; (T(b,y) \otimes T(b,v)) \end{aligned} \tag{55}$$

if $a \geq b$, i.e., if the intervals $[y, x]$ and $[v, u]$ have a non-trivial intersection (namely the interval $[b, a]$), while

$$\{T(x,y) \overset{\otimes}{,} T(u,v)\} \; = \; 0 \tag{56}$$

if $a < b$, i.e., if the intervals $[y, x]$ and $[v, u]$ are disjoint. Essentially the same formulae continue to hold even when the constraints $x > y$ and $u > v$ are dropped, with the following modifications: the definition of a and b must be modified to read

$$a = \min\{\max\{x,y\}, \max\{u,v\}\} \quad \text{and} \quad b = \max\{\min\{x,y\}, \min\{u,v\}\} \quad ,$$

respectively, and the entire rhs of eqn (55) must be multiplied by a factor $\epsilon(x-y)$ $\epsilon(u-v)$, where ϵ denotes the usual sign function:

$$\epsilon(z) \;=\; \begin{cases} +1 & \text{for} \quad z > 0 \\ \;\;0 & \text{for} \quad z = 0 \\ -1 & \text{for} \quad z < 0 \end{cases} \tag{57}$$

Indeed, it is easily verified that with these modifications, the rhs of eqn (55) vanishes for $x = y$ or $u = v$, so as to be consistent with the normalization condition (39), and has the correct behavior under the exchange of x with y and/or of u with v, as dictated by the inversion formula (41) and the composition rule (40):

$$\begin{aligned} \{T(x,y) \overset{\otimes}{,} T(u,v)\} &= -(T(x,y) \otimes 1) \{T(y,x) \overset{\otimes}{,} T(u,v)\} (T(x,y) \otimes 1) \quad , \\ \{T(x,y) \overset{\otimes}{,} T(u,v)\} &= -(1 \otimes T(u,v)) \{T(x,y) \overset{\otimes}{,} T(v,u)\} (1 \otimes T(u,v)) \quad . \end{aligned}$$

In eqn (56), one can safely put $x = u$ and $y = v$ and take the limit $x \to +\infty$, $y \to -\infty$ to obtain the following

Fundamental Poisson Brackets for the Monodromy Matrix:

$$\{T \overset{\otimes}{,} T\} \; = \; [\, r \, , T \otimes T \,] \quad . \tag{58}$$

In the simplified notation introduced through eqns (43-49) above, these fundamental Poisson brackets take the form

$$\{T_1 \, , T_2\} \; = \; [\, r_{12} \, , T_1 T_2 \,] \quad . \tag{59}$$

Finally, consistency of eqns (54) and (59) with the Jacobi identity for Poisson brackets is guaranteed by the following structure equation for the r-matrix:

Classical Yang-Baxter Equation:

$$[\, r_{12} \, , r_{13} \,] + [\, r_{12} \, , r_{23} \,] + [\, r_{13} \, , r_{23} \,] \; = \; 0 \quad . \tag{60}$$

2.1.2 Non-Ultralocal Models of First Order

The theory of non-ultralocal models, even those of first order, is considerably less well understood than that of ultralocal models. We shall describe here an approach first proposed by Maillet [25] and further developed by the authors of ref. [2], which starts out from the following

Fundamental Poisson Brackets for the Lax Matrix:

$$\begin{aligned}\{L(x) \overset{\otimes}{,} L(y)\} \;&=\; -[\,r(x)\,,\, L(x)\otimes 1 + 1\otimes L(y)\,]\;\delta(x-y) \\ &\quad+[\,s(x)\,,\, L(x)\otimes 1 - 1\otimes L(y)\,]\;\delta(x-y) \\ &\quad-(r(x)+s(x)-r(y)+s(y))\;\delta'(x-y)\;\;. \end{aligned} \tag{61}$$

Here r and s are elements of $U(\mathfrak{g})\otimes U(\mathfrak{g})$ [10] which are allowed to be meromorphic functions of the (two) spectral parameters involved (if any are present) and, in addition, will in general be field dependent (i.e., will be non-trivial function(al)s on the phase space of the theory) and hence, implicitly, will be functions on $\mathcal{S}$; moreover, r is supposed to be antisymmetric ($r^T = -r$) while s is supposed to be symmetric ($s^T = s$) in order for eqn (61) to be consistent with the antisymmetry of the Poisson bracket.

An equivalent but significantly more concise form of the above fundamental Poisson brackets appears when we consider instead of the Lax matrix L the Lax operator D and combine the (antisymmetric) r-matrix and the (symmetric) s-matrix into an (asymmetric) d-matrix according to

$$d(x) \;=\; (s-r)(x) \quad,\quad d(x,y) \;=\; d(x)\,\delta(x-y)\;\;, \tag{62}$$

with

$$d(x)^T \;=\; (s+r)(x) \quad,\quad d^T(x,y) \;=\; d(x)^T\,\delta(x-y)\;\;. \tag{63}$$

Then eqn (61) becomes equivalent to the

Fundamental Poisson Brackets for the Lax Operator:

$$\{D(x) \overset{\otimes}{,} D(y)\} \;=\; [\,d(x,y)\,,\, D(x)\otimes 1\,] \;-\; [\,d^T(x,y)\,,\, 1\otimes D(y)\,]\;\;. \tag{64}$$

In the simplified notation introduced through eqns (43)-(49) above, these fundamental Poisson brackets take the form

$$\{D_1\,,D_2\} \;=\; [\,d_{12}\,,D_1\,] \;-\; [\,d_{21}\,,D_2\,]\;\;. \tag{65}$$

On the other hand, it can be shown by an explicit calculation (see, e.g., [2, Sect. 3]) that eqn (61) implies, without any further assumptions, the following

[10] The commutator on the rhs of eqn (61) is the one induced by the associative product in $U(\mathfrak{g})^{\otimes 2}$.

Fundamental Poisson Brackets for the Transition Matrix: Given $x > y$ and $u > v$, put $a = \min\{x, u\}$ and $b = \max\{y, v\}$, and let ϵ denote the usual sign function (cf. eqn (57)). Then

$$\begin{aligned} &\{T(x,y) \overset{\otimes}{,} T(u,v)\} \\ &= \ +\left(T(x,a) \otimes T(u,a)\right)\left(r(a) \ + \ \epsilon(x-u)s(a)\right)\left(T(a,y) \otimes T(a,v)\right) \\ &\quad -\left(T(x,b) \otimes T(u,b)\right)\left(r(b) \ - \ \epsilon(y-v)s(b)\right)\left(T(b,y) \otimes T(b,v)\right) \end{aligned} \tag{66}$$

if $a \geq b$, i.e., if the intervals $[y, x]$ and $[v, u]$ have a non-trivial intersection (namely the interval $[b, a]$), while

$$\{T(x,y) \overset{\otimes}{,} T(u,v)\} \ = \ 0 \tag{67}$$

if $a < b$, i.e., if the intervals $[y, x]$ and $[v, u]$ are disjoint. Essentially the same formulae continue to hold even when the constraints $x > y$ and $u > v$ are dropped, with the following modifications: the definition of a and b must be modified to read

$$a = \min\{\max\{x,y\}, \max\{u,v\}\} \quad \text{and} \quad b = \max\{\min\{x,y\}, \min\{u,v\}\} \ ,$$

respectively, the factors $\epsilon(x - u)$ and $\epsilon(u - v)$ appearing on the rhs of eqn (66) must be replaced by $\epsilon(\max\{x, y\}, \max\{u, v\})$ and $\epsilon(\min\{x, y\}, \min\{u, v\})$, respectively, and finally the entire rhs of eqn (66) must be multiplied by a factor $\epsilon(x-y)\,\epsilon(u-v)$. Indeed, it is easily verified that with these modifications, the rhs of eqn (66) vanishes for $x = y$ or $u = v$, so as to be consistent with the normalization condition (39), and has the correct behavior under the exchange of x with y and/or of u with v, as dictated by the inversion formula (41) and the composition rule (40):

$$\begin{aligned} \{T(x,y) \overset{\otimes}{,} T(u,v)\} &= -\left(T(x,y) \otimes 1\right)\{T(y,x) \overset{\otimes}{,} T(u,v)\}\left(T(x,y) \otimes 1\right) \ , \\ \{T(x,y) \overset{\otimes}{,} T(u,v)\} &= -\left(1 \otimes T(u,v)\right)\{T(x,y) \overset{\otimes}{,} T(v,u)\}\left(1 \otimes T(u,v)\right) \ . \end{aligned}$$

In eqn (66), unfortunately, one can no longer simply put $x = u$ and $y = v$ and then take the limit $x \to +\infty$, $y \to -\infty$, because the result depends on whether one takes this limit with $x < u$ or with $x > u$ and with $y < v$ or with $y > v$. In fact, let ϵ_1 and ϵ_2 take the values $+1$ and -1 (independently), then defining

$$\{T \overset{\otimes}{,} T\}_{\epsilon_1,\epsilon_2} \ = \lim_{\substack{x,u \to +\infty \ , \ \epsilon(x-u)=\epsilon_1 \\ y,v \to -\infty \ , \ \epsilon(y-v)=\epsilon_2}} \{T(x,y) \overset{\otimes}{,} T(u,v)\} \ , \tag{68}$$

we obtain

$$\{T \overset{\otimes}{,} T\}_{\epsilon_1,\epsilon_2} \ = \ (r + \epsilon_1 s)(+\infty)\,(T \otimes T) \ - \ (T \otimes T)\,(r - \epsilon_2 s)(-\infty) \ . \tag{69}$$

The definition of Poisson brackets between monodromy matrices, and more generally between transition matrices for coinciding or adjacent intervals, is therefore not completely straightforward (as it is for ultralocal models) but requires some kind of regularization. As a result, there arises the problem of finding a regularization scheme such that the basic algebraic properties of Poisson brackets, viz. the derivation rule and the Jacobi identity, remain valid for their regularized counterparts. A priori, it is not even clear whether such a regularization scheme exists at all, and indeed it has been suggested in the earlier literature that this is not the case [24]. Later, a prescription which amounts to a "total symmetrization over all possible boundary values" was found to meet these requirements [25], but it is a multi-step regularization in the sense that regularization of multiple Poisson brackets with a given number of factors cannot be reduced to repeated regularization of multiple Poisson brackets with a smaller number of factors. For a detailed discussion of this aspect, the reader is referred to ref. [2].

Apart from this regularization problem, which appears as soon as the matrix s does not vanish identically and which seems to indicate that in this case the monodromy matrix, when viewed as a function(al) on the phase space of the theory, is no longer smooth but develops some kind of singularity, additional difficulties arise due to the fact that the matrices r, s and d are field dependent. This implies that they will satisfy Poisson bracket relations of their own, which must be specified in order to close the algebra. Guided by the results obtained in the case of integrable sigma models, the authors of ref. [2] have postulated additional Poisson brackets which, in the simplified notation introduced through eqns (43)-(49) above, take the form

$$\{D_1\,,\,d_{23}\} \;=\; [\,c_{12}+c_{13}\,,\,d_{23}\,] \quad , \tag{70}$$

together with

$$\{d_{12}\,,\,d_{34}\} \;=\; 0 \quad . \tag{71}$$

Here c is an element of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ [11] which is allowed to be a meromorphic function of the (two) spectral parameters involved (if any are present) but is supposed to be field independent (i.e., to be constant as a function(al) on the phase space of the theory); moreover, no symmetry requirement is imposed on c.

Taken together, the fundamental Poisson bracket relations (65), (70) and (71) for non-ultralocal models of first order can be considered as an analogue of the fundamental Poisson bracket relations (54) for ultralocal models. A proper understanding from the point of view of Lie algebra theory, however, is still lacking. It is not even clear what is the underlying abstract algebra and what is its structural equation – in contrast to the ultralocal case, where the underlying abstract algebra is the classical Yang-Baxter algebra defined by the classical r-matrix (whose entries can be regarded as the structure constants of this algebra) and the corresponding structural equation is the classical Yang-Baxter equation (60) (as a quadratic constraint on these structure constants). All that can be said at this stage is that for non-ultralocal models of first order, D and d taken together should

[11] The commutator on the rhs of eqn (70) is the one induced by the associative product in $U(\mathfrak{g})^{\otimes 3}$.

provide a representation of this algebra (by function(al)s on the phase space of the theory) – just as for ultralocal models, D alone provides a representation of the classical Yang-Baxter algebra (by function(al)s on the phase space of the theory). But even the status of eqns (65), (70) and (71) as representation conditions may be called into question because of the fact that D and d are both field dependent, so that eqn (65) yields a "quadratic Lie algebra": the generators of the Lie algebra in question will therefore have to involve products of D's and d's, with at most one D-factor but with an arbitrary number of d-factors. The resulting algebra is sufficiently complicated to justify the suspicion that eqns (65), (70) and (71) may not even be the right starting point for the whole problem! Possible alternatives are presently under investigation, but no definite conclusion has so far been reached.

Despite all these drawbacks, I want to conclude this presentation by showing how the integrable sigma models – which after all have served as the basic motivation and guiding line for analyzing the structure of non-ultralocal models of first order – fit into the above framework. It should be clear that proposals for modifications of this framework must accomodate this class of models equally well.

2.2 Integrable Sigma Models

As is well known, the classical non-linear sigma models which have been shown to be integrable are precisely the ones defined on Riemannian symmetric spaces (cf. refs [26-28] and, for a review, ref. [29]). For the derivation of the corresponding Lax representation, it is important to realize that a Riemannian symmetric space M is homogeneous (under its group of isometries $I(M)$ as well as under the connected 1-component $I_0(M)$ of $I(M)$, i.e., these groups both act transitively on M). Thus M can be written as the quotient space of some (connected) Lie group G, with Lie algebra $\mathfrak{g}$, modulo some compact subgroup $H \subset G$, with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, and there exists an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$, with commutation relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \quad , \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \quad , \tag{72}$$

such that $\mathfrak{g}$ is the (vector space) direct sum of $\mathfrak{h}$ and $\mathfrak{m}$:

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \mathfrak{m} \ . \tag{73}$$

The corresponding projections from $\mathfrak{g}$ onto $\mathfrak{h}$ along $\mathfrak{m}$ and from $\mathfrak{g}$ onto $\mathfrak{m}$ along $\mathfrak{h}$ will be denoted by $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{m}}$, respectively. Moreover, it will be required that the $\mathrm{Ad}(H)$-invariant positive definite inner product $(.\,,.)$ on $\mathfrak{m}$, corresponding to the given G-invariant Riemannian metric on M, is induced from an $\mathrm{Ad}(G)$-invariant non-degenerate inner product $(.\,,.)$ on $\mathfrak{g}$, corresponding to a G-biinvariant pseudo-Riemannian metric on G, so that the direct decomposition (73) is orthogonal. (This additional hypothesis, which for Riemannian symmetric spaces of the compact or non-compact type may always be imposed without loss of generality, amounts essentially to requiring that M be naturally reductive; we refer to [29] for a detailed discussion.) Using this scalar product to identify the Lie algebra $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$, we can view the Noether current j_μ defined in eqn (12) as a vector field with values in $\mathfrak{g}$ and the scalar field j defined in eqn (13) as a scalar field with values in the second symmetric tensor power of $\mathfrak{g}$ or, alternatively, in the space $L(\mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}^*$ of linear transformations on $\mathfrak{g}$. Then using that eqn (10) can be rewritten as

$$X_M(gH) \;=\; g\,\pi_{\mathfrak{m}}(\mathrm{Ad}(g)^{-1}X) \quad , \tag{74}$$

and employing the gauge dependent formulation developed in refs [26-29], where the basic field φ with values in M is (at least locally) represented in terms of a field g with values in G, determined modulo a field h with values in H by the condition that $\varphi = gH$, we can express the Noether current j_μ in terms of the covariant derivative $D_\mu g$ of g, defined by

$$D_\mu g \;=\; g\,\pi_{\mathfrak{m}}(g^{-1}\,\partial_\mu g) \quad , \tag{75}$$

according to

$$j_\mu \;=\; -\,D_\mu g\, g^{-1} \quad . \tag{76}$$

As usual, the equations of motion imply (and in the models considered here are in fact equivalent to) current conservation

$$\partial^\mu j_\mu \;=\; 0 \quad . \tag{77}$$

In addition, the commutation relations (72) guarantee that $2j_\mu$ also satisfies the zero curvature condition, i.e.,

$$\partial_\mu j_\nu \;-\; \partial_\nu j_\mu \;+\; 2\,[j_\mu, j_\nu] \;=\; 0 \quad . \tag{78}$$

Together, eqns (77) and (78) are equivalent to the zero curvature condition (32) for the Lax connection $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ defined by[12]

$$\mathcal{A}_\mu \;=\; \left(1 \,-\, \tfrac{1}{2}\left(\lambda + \lambda^{-1}\right)\right) j_\mu \;+\; \tfrac{1}{2}\left(\lambda - \lambda^{-1}\right) \epsilon_{\mu\nu} j^\nu \quad . \tag{79}$$

As far as the dependence of $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ on the spectral parameter λ is concerned, no generally accepted convention for the concrete choice of λ seems to exist so far. After all, one is always free to subject this parameter to a Möbius transformation, so as to shift the position of its poles.[13] The invariant content of eqn (79) is that this Lax connection, as a function of its spectral parameter, is a meromorphic function on the Riemann sphere with two simple poles, and the advantage of the choice made here is that these poles are located at $\lambda = 0$ and at $\lambda = \infty$, which seems to be the most appropriate choice for establishing a connection to loop algebras.

[12] Our convention for the ϵ-tensor is $\epsilon_{01} = -1$.

[13] For example, the spectral parameter λ adopted in eqn (79) is the spectral parameter γ of refs [26-29] and is related to the spectral parameter $\tilde{\lambda}$ – employed, e.g., in ref. [2] – by the Möbius transformation $\lambda = (\tilde{\lambda}-1)/(\tilde{\lambda}+1)$, or equivalently, $\tilde{\lambda} = (1+\lambda)/(1-\lambda)$.

In particular,

$$L(x,\lambda) \;=\; \left(1 - \tfrac{1}{2}\left(\lambda + \lambda^{-1}\right)\right) j_1(x) \;+\; \tfrac{1}{2}\left(\lambda - \lambda^{-1}\right) j_0(x) \;. \tag{80}$$

Then as shown in ref. [2], the fundamental Poisson brackets of the theory can be brought into the form described in the previous subsection (cf. eqns (61) and (64-71)) with

$$r(x,\lambda,\mu) \;=\; \frac{(1-\lambda^2)(1-\mu^2)}{2(\lambda^2-\mu^2)}\, C \;+\; \frac{(1+\lambda\mu)(1-\lambda)(1-\mu)(\lambda-\mu)}{4\lambda\mu(\lambda+\mu)}\, j(x) \tag{81}$$

and

$$s(x,\lambda,\mu) \;=\; -\,\frac{(1-\lambda\mu)(1-\lambda)(1-\mu)}{4\lambda\mu}\, j(x) \tag{82}$$

or equivalently (cf. eqns (62,63)),

$$d(x,\lambda,\mu) \;=\; -\,\frac{(1-\lambda)(1-\mu)}{\lambda+\mu}\left(\frac{(1+\lambda)(1+\mu)}{2(\lambda-\mu)}\, C \;+\; \frac{1-\mu^2}{2\mu}\, j(z)\right) \tag{83}$$

and finally

$$c(\lambda,\mu) \;=\; -\,\frac{1-\lambda^2}{2\lambda}\, C \;. \tag{84}$$

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Michael Forger
Instituto de Matemática e Estatística
Universidade de São Paulo
e-mail: forger@ime.usp.br
Brasil