Resenhas IME-USP 1994, Vol. 1, No. 4, 531 - 547.

Restrictions on the flows of functional differential equations in neighborhoods of singularities [1]

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Abstract: The paper is a report on the work of Faria and Magalhães regarding possible restrictions on the flows defined by scalar retarded Functional Differential Equations (FDEs), locally around certain simple singularities, when compared with the possible flows of Ordinary Differential Equations (ODEs) with the same singularities. For the Hopf and the Bogdanov-Takens singularities there are no restrictions on the local flows defined by scalar FDEs, even when the nonlinearities involve just one delay. On the other hand, for the singularity associated with a zero and a conjugated pair of pure imaginary numbers as simple eigenvalues, there occur restrictions on the flows defined by scalar FDEs with nonlinearities involving just one delay, as well as two delays satisfying a certain resonance condition. These restrictions are of geometric significance, since they amount to the impossibility of observing the homoclinic orbits that occur in arbitrarily small neighborhoods of the singularity for ODEs. Versal unfoldings for the considered singularities by FDEs and the possible restrictions on the associated flows are also studied.

Key words: Functional differential equations, singularities, versal unfoldings, normal forms, locally invariant manifolds.

1. Introduction

When considering the dynamics of a class of infinite dimensional systems, it is natural to enquire about the restrictions on the flows that can be observed on finite dimensional invariant manifolds, relative to flows defined by finite dimensional Ordinary Differential Equations (ODEs). In particular, it is interesting to look for situations where the extension from a finite dimensional phase space to an infinite dimensional one is not associated with an extension of the possible dynamical behavior that can be observed in the considered class of infinite dimensional systems.

We will be concerned with the study of possible restrictions on the local flows of scalar retarded FDEs around equilibrium points, reporting on the work of Faria and Magalhães in [10].

Let us consider $r > 0$ and denote by $C = C([-r, 0]; \mathbb{R})$ the Banach space of the continuous functions from $[-r, 0]$ to $\mathbb{R}$, taken with the uniform norm. We consider scalar retarded Functional Differential Equations (FDEs) in C, of the form

$$\dot{z}(t) = L(z_t) + f(z_t) \quad , \tag{1}$$

[1] Partially supported by FAPESP's Projeto Temático "Transição de Fase Dinâmica e Sistemas Evolutivos" and JNICT-Portugal (STRDA/C/CEN/528/92).

where $z_t \in C$ satisfies $z_t(\theta) = z(t+\theta)$, and L is a bounded linear operator from C to $\mathbb{R}$ and $f \rightarrow \mathbb{R}$ is a C^k function, for an appropriate $k \geq 1$, satisfying $f(0) = 0$ and $Df(0) = 0$. The solutions of the considered FDE define a C^0 semigroup on C which has an equilibrium at the origin, where the linearization of the FDE is

$$\dot{z}(t) = L(z_t) \quad , \tag{2}$$

The center-unstable manifold theorem assures that the local flow defined by the FDE around equilibria is finite dimensional, in the sense that there exists a finite dimensional differential manifold invariant under the flow which attracts exponentially nearby orbits and where the flow can be given by a finite dimensional ODE. It is therefore natural to ask how do local flows of FDEs in finite dimensional manifolds compare with flows of finite dimensional Ordinary Differential Equations (ODEs), and if there are reasonably large classes of FDEs whose flows are restricted when compared to the possible flows of ODEs. Such situations would be particularly interesting, since they would be cases of restricted dynamic behavior, in spite of a drastic enlargement of degrees of freedom, with the phase space passing from finite to infinite dimensional.

The following result of Hale establishes that there occur no restrictions on flows determined by finite jets of vector fields in $\mathbb{R}^m$, provided the linear part does not itself involve restrictions and the nonlinearities are allowed to depend on a sufficiently large number of delayed values of the solutions which, in any case, does not need to be taken larger than m.

Theorem [14,15]: *Any finite jet of vector fields in $\mathbb{R}^m$ at an equilibrium can be realized by a scalar FDE in an appropriate m-dimensional locally invariant manifold if and only if the linear part can be realized.*

Such a realization can be accomplished with nonlinearities in the FDE of the form

$$f(z_t) = F(z(t-r_0), \cdots, z(t-r_{m-q})) \quad , \tag{3}$$

where $q \in \mathbb{N}$ and $0 \leq r_0 < \cdots < r_{m-q} \leq r$ depend only on the linear term in the jet. Adequate values for $r_0, \cdots, r_{m-q}$ can be explicitely computed from the linear term.

This result was extended to systems of FDEs in [9], a paper that also contains the following result regarding the realization of linear ODEs.

Theorem [9]: *A linear ODE $\dot{x} = Bx$ in $\mathbb{R}^m$ is realizable by a scalar FDE if and only if the Jordan canonical forms of B have only one Jordan block for each eigenvalue.*

Remark: Analogous questions for scalar semilinear parabolic PDEs were considered in [11,19,20].

Having on mind the information provided by the two preceding theorems, when looking for restrictions in the flows generated by FDEs it is natural to consider the following program:

1. consider simple nondegenerate linear singularities with one Jordan block for each eigenvalue and finitely determined flows, *i.e.*, flows determined by finite jets,

2. restrict the number of delays in the FDE to be less than the number obtained in Hale's result for realizability cited above,

3. compute Poincaré-Birkhoff normal forms up to the desired order with the coefficients explicitely given in terms of the considered FDE.

In order to accomplish this program we need a theory extending to FDEs the Poincaré-Birkhoff normal forms available for finite dimensional ODEs. Such a normal form theory for FDEs was developed in [7,8].

Let us now be explicit about what we understand by simple nondegenerate singularities. By order of increasing complexity these, the associated eigenvalues and Jordan canonical forms are:

1. Hopf singularity

   eigenvalues: $(i\omega, -i\omega)$, with $\omega \neq 0$

   Jordan form: $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$.

2. Bogdanov-Takens singularity

   eigenvalues: $(0, 0)$

   Jordan form: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

3. singularity with eigenvalues: $(i\omega, -i\omega, 0)$, with $\omega \neq 0$

   Jordan form: $\begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4. singularity with eigenvalues: $(i\omega_1, -i\omega_1, i\omega_2, -i\omega_2,)$, with $\omega_1, \omega_2 \neq 0$

   Jordan form: $\begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix}$.

2. The Hopf singularity

The Hopf singularity is characterized by the infinitesimal generator A_0 for the semigroup defined by the linear equation $\dot{z}(t) = L(z_t)$ having as simple eigenvalues $\pm i\omega$, with $\omega \neq 0$, and no other eigenvalues in the imaginary axis. We denote by P the invariant space for A_0 associated with the set of eigenvalues $\Lambda = \{i\omega, -i\omega\}$. In this case, the function f defining the nonlinearities in the FDE is required to be of class C^3.

The result of Hale recalled in the preceding section assures the realizability of any finite jet of vector fields in $\mathbb{R}^2$ by FDEs with nonlinearities depending on two delays, provided the delays are appropriately chosen in $[0, r]$. Since we are looking for situations with restrictions on the flows observed on a center manifold at zero, we consider FDEs with just one delay in the nonlinearities, as follows.

$$\dot{z}(t) = L(z_t) + F(z(t - r_0)) \quad , \tag{4}$$

with

$$F(u) = \frac{A_2}{2}u^2 + \frac{A_3}{3}u^3 + O(|u|^4) \quad . \tag{5}$$

As the singularity involves a pair of conjugate pure imaginary eigenvalues, to compute a normal form relative to P for this FDE with the help of the theory developed in [7] it is convenient to consider the FDE in $C([-r, 0]; \mathbb{C})$, still denoted here by C, extending L and F to complex functions in the natural way. We then obtain on center manifolds at the origin, in polar coordinates, the normal form

$$\begin{aligned} \dot{\rho} &= K\rho^3 + O(\rho^4) \\ \dot{\xi} &= -\omega + O(\rho) \end{aligned} \quad , \tag{6}$$

where

$$K = \frac{A_3}{2} Re(e^{-i\omega r_0}\psi_1(0)) + A_2^2 \left\{ -\frac{Re(e^{-i\omega r_0}\psi_1(0))}{L(1)} + \frac{1}{2} Re\left[\frac{e^{-3i\omega r_0}\psi_1(0)}{2i\omega - L(e^{2i\omega\theta})}\right]\right\} \quad , \tag{7}$$

with $\psi_1(0) = [1 - L(\theta e^{i\omega\theta})]^{-1}$.

It is known (*cf.* [1,5,16,17]) that generically the flows for ODEs with this singularity are determined at third order, with a normal form given in polar coordinates (ρ, ξ) as in (6).

Changing A_3 generates arbitrary changes in K as given in (7). Consequently, there are no restrictions on the possible phase portraits on center manifolds at the origin, in comparison to what can be generically observed for ODEs with this singularity, even if the nonlinearities of the FDE involve just one delay.

We conclude that there are no restrictions imposed on the possible phase portraits on center manifolds at the origin that result from considering FDEs (4) with nonlinearities involving just one delay.

It is of interest to consider also versal unfoldings of the singularity by families of FDEs. For this, we study a linear perturbation of (4) in the form

$$\dot{z}(t) = \nu z(t - r_0) + L(z_t) + F(z(t - r_0)) \quad ,$$

with $\nu \in \mathbb{R}$. Since we are interested in discussing the Hopf bifurcation, it is necessary to impose the Hopf condition, namely that the infinitesimal generator of $\dot{z}(t) = \nu z(t - r_0) + L(z_t)$ has a pair of conjugate simple eigenvalues, $\gamma(\nu) \pm i\sigma(\nu)$, crossing transversally the imaginary axis at $\nu = 0$, with $\gamma(0) = 0$, $\sigma(0) = \omega$, $\gamma'(0) \neq 0$. In this case, the Hopf condition amounts to $Re\ \psi_1(0)e^{-i\omega r_0} \neq 0$. Under this condition, we obtain from the results established in [8] that the flows on center manifolds at $z = 0$, $\nu = 0$ are given in polar coordinates (ρ, ξ) by

$$\begin{aligned} \dot{\rho} &= \nu Re\left(\psi_1(0)e^{-i\omega r_0}\right)\rho + K\,\rho^3 + O(\nu^2\rho + |(\rho,\nu)|^4) \\ \dot{\xi} &= -\omega + O(|(\rho,\nu)|) \quad , \end{aligned}$$

where K is given by (7). Therefore, under the generic conditions on the linear operator L

$$Re\left(\frac{e^{-i\omega r_0}}{1 - L(\theta e^{i\omega\theta})}\right) \neq 0 \quad ,$$

the one-parameter family of FDEs

$$\dot{z}(t) = \nu z(t - r_0) + L(z_t) + (z(t - r_0))^3$$

is a versal unfolding of the Hopf singularity for the flows on center manifolds at the origin. Again, there are no restrictions imposed on the possible phase portraits on center manifolds at the origin that result from considering scalar FDEs that provide a versal unfolding of the Hopf singularity with nonlinearities depending on just one delayed value of the solutions.

3. The Bogdanov-Takens singularity

The Bogdanov-Takens singularity is characterized by the infinitesimal generator A_0 having 0 as a double eigenvalue and no other eigenvalues in the imaginary axis, but such that the restriction of A_0 to its invariant space P associated with the set of eigenvalues $\Lambda = \{0\}$ is not zero. In this case, the function f defining the nonlinearities in the FDE is required to be of class C^2.

As before for the Hopf singularity, the result of Hale recalled in the introduction assures the realizability of any finite jet of vector fields in $\mathbb{R}^2$ by FDEs with nonlinearities depending on two delays, provided the delays are appropriately chosen in $[0, r]$. So, we consider FDEs with just one delay in the nonlinearities, as follows.

(8) $$\dot{z}(t) = L(z_t) + F(z(t - r_0)) \quad ,$$

where

$$F(u) = \frac{A_2}{2} u^2 + O(|u|^3) \quad , \tag{9}$$

with $A_2 \in \mathbb{R}$.

From the results on normal forms for FDEs established in [7], we know that a normal form relative to P is of the form

$$\begin{aligned} \dot{x}_1 &= x_2 & &+ O(|x|^3) \\ \dot{x}_2 &= a\, x_1^2 + b\, x_1 x_2 & &+ O(|x|^3) \quad , \end{aligned} \tag{10}$$

where

$$\begin{aligned} a &= -A_2\; L(\theta^2)^{-1} \\ b &= \frac{2}{3}\; A_2\; L(\theta^2)^{-2}\; L(\theta^3) + 2r_0 A_2 L(\theta^2)^{-1} \quad . \end{aligned} \tag{11}$$

It is known [2,3,21] that, provided $ab \neq 0$, the flow of (10) in a neighborhood of the origin is completely determined by the terms up to second order. From (11) it is clear that changing A_2 and r_0 allows arbitrary changes in a, b. Consequently, there are no restrictions imposed on the possible phase portraits on center manifolds at the origin that result from considering FDEs of this type.

In [7], versal unfoldings for this singularity by families of FDEs were obtained by two-parameter families of the form

$$\dot{z}(t) = \nu_1 z(t) + \nu_2 z(t - r_0) + L(z_t) + F(z(t - r_0)) \quad . \tag{12}$$

A corresponding normal form relative to P is

$$\begin{aligned} \dot{x}_1 &= x_2 & &+ O((|\mu| + |x|)|x|^2) \\ \dot{x}_2 &= \mu_1 x_1 + \mu_2 x_2 + a x_1^2 + b x_1 x_2 & &+ O((|\mu| + |x|)|x|^2) \quad , \end{aligned} \tag{13}$$

with a, b given as in (11) and $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ satisfying

$$\begin{aligned} \mu_1 &= -2(\nu_1 + \nu_2)\; L(\theta^2)^{-1} \\ \mu_2 &= \frac{2}{3}(\nu_1 + \nu_2)\; L(\theta^2)^{-2}\; L(\theta^3) + 2\nu_2 r_0\; L(\theta^2)^{-1} \quad . \end{aligned} \tag{14}$$

It is known [2,3,21] that the terms up to second order in (13) define a versal unfolding for the Bogdanov-Takens singularity in $\mathbb{R}^2$, under the generic condition $ab \neq 0$, so that (12) is a versal unfolding under the generic condition $abr_0 \neq 0$, where a, b are given by (11). In particular a generic versal unfolding is

$$\dot{z}(t) = \nu_1 z(t) + \nu_2 z(t - r_0) + L(z_t) + \frac{A_2}{2}(z(t - r_0))^2 \quad , \tag{15}$$

provided $r_0, A_2 \neq 0$. Again, in the considered conditions, we observe that there are no restrictions imposed on the possible phase portraits on center manifolds at the origin for a versal unfolding of the Bogdanov-Takens singularity that result from scalar FDEs with nonlinearities involving just one delay.

4. The singularity with a pure imaginary pair and a zero as simple eigenvalues

We consider now the case where A_0 has 0 and $\pm i\omega$, with $\omega \neq 0$, as simple eigenvalues and no other eigenvalues in the imaginary axis, and $f : C \to \mathbb{R}$ is a C^2 function. We denote by P the invariant space for A_0 associated with the set of eigenvalues $\Lambda = \{i\omega, -i\omega, 0\}$.

This singularity, referred to below as the singularity $(\pm i\omega, 0)$, is studied along the lines followed above for the Hopf and the Bogdanov-Takens singularities.

From the results of Hale recalled in the Introduction, we know that there are no restrictions on the possible finite jets that can be observed in the ODEs associated with the flows on center manifolds that result from considering FDEs whose nonlinearities involve three appropriately chosen delays. So, we consider FDEs with nonlinearities involving just two or one delays.

Let us first consider FDEs in C of the form

$$\dot{z}(t) = L(z_t) + F(z(t-r_0), z(t-r_1)) \quad , \tag{16}$$

where

$$F(v_1, v_2) = \frac{1}{2}(A_{20}v_1^2 + A_{11}v_1v_2 + A_{02}v_2^2) + O(|(v_1, v_2)|^3) \quad . \tag{17}$$

with $A_{20}, A_{11}, A_{02} \in \mathbb{R}$. As for the case of Hopf bifurcation above, due to the presence of the pair of conjugated eigenvalues $\pm i\omega$, it is convenient to complexify the equation before computing a normal form. A normal form relative to P can be obtained by applying the theory in [7], leading to

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x, 0) + O(|x|^3) \quad ,$$

where

$$B = \mathrm{diag}\,(i\omega, -i\omega, 0)$$

and

$$g_2^1(x,0) =$$

$$= \begin{bmatrix} \mathcal{S}\left(\psi_1(0)[2A_{20}e^{-i\omega r_0} + A_{11}(e^{-i\omega r_0} + e^{-i\omega r_1}) + 2A_{02}e^{-i\omega r_1}]x_1x_3\right) \\ \psi_3(0)\left\{2[A_{20} + A_{11}Re(e^{-i\omega(r_0-r_1)}) + A_{02}]x_1x_2 + [A_{20} + A_{11} + A_{02}]x_3^2\right\} \end{bmatrix},$$

with $\mathcal{S}$ being an operator acting on the space of second order homogeneous polynomials in $\mathbb{R}^2$ such that $\mathcal{S}(h+k) = \mathcal{S}(h) + \mathcal{S}(k)$ and

$$\mathcal{S}(c\, x_1^{q_1}\, x_2^{q_2}\, x_3^{q_3}) = \begin{bmatrix} c\, x_1^{q_1}\, x_2^{q_2}\, x_3^{q_3} \\ \\ \bar{c}\, x_1^{q_2}\, x_2^{q_1}\, x_3^{q_3} \end{bmatrix} , \quad c \in \mathbb{C}\,,\ (q_1, q_2, q_3) \in \mathbb{N}_0^3,\ \ |(q_1, q_2, q_3)| = j\,.$$

This normal form can now be written in real coordinates w, through the change of variables $x_1 = w_1 - iw_2$, $x_2 = w_1 + iw_2$, $x_3 = w_3$, and changing to cylindrical coordinates according to $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, $w_3 = \zeta$, we obtain

$$\begin{aligned} \dot{\rho} &= a_1\rho\zeta + O(\rho|(\rho,\zeta)|^2) \\ \dot{\zeta} &= b_1\rho^2 + b_2\zeta^2 + O(|(\rho,\zeta)|^3) \\ \dot{\xi} &= -\omega + O(|(\rho,\zeta)|) \quad , \end{aligned} \tag{18}$$

with

$$\begin{aligned} a_1 &= Re\left\{\psi_1(0)[A_{20}e^{-i\omega r_0} + \frac{A_{11}}{2}\,(e^{-i\omega r_0} + e^{-i\omega r_1}) + A_{02}e^{-i\omega r_1}]\right\} \\ b_1 &= \psi_3(0)[A_{20} + A_{11}\ Re(e^{i\omega(r_1 - r_0)}) + A_{02}] \\ b_2 &= \frac{\psi_3(0)}{2}[A_{20} + A_{11} + A_{02}] \quad , \end{aligned} \tag{19}$$

where $\psi_1(0) = (1 - L(\theta\ e^{i\omega\theta}))^{-1}$ and $\psi_3(0) = (1 - L(\theta))^{-1}$.

Writing equation (18) up to second order terms and eliminating the equation in ξ, since the right hand side of (18) is independent of this variable, we get the equation in the plane (ρ,ζ)

$$\begin{aligned} \dot{\rho} &= a_1\ \rho\ \zeta \\ \dot{\zeta} &= b_1\ \rho^2 + b_2\ \zeta^2 \quad . \end{aligned} \tag{20}$$

As it is shown in [21], the singularity considered is determined to second order, provided $a_1, b_1, b_2 \neq 0$ and $a_1 \neq b_2$.

By changing $A_{20}, A_{11}, A_{0,2}$, the values of a_1, b_1, b_2 can be made to assume arbitrary values, provided:

$$\begin{aligned} &i) && \tan\omega(r_0 + r_1) \neq L(\theta\sin\omega\theta)/[1 - L(\theta\cos\omega\theta)] \ , \\ \text{and}\ \ &ii) && \omega(r_0 - r_1) \neq 2k\pi, \ \text{ with } k \in \mathbf{Z} \quad . \end{aligned} \tag{21}$$

So, under these generic conditions and considering situations for which $a_1, b_1, b_2 \neq 0$, we do not observe restrictions on the possible flows of (16), locally on center manifolds at the origin, in comparison to ODEs with the same singularity.

Whenever the second condition in (21) fails, so that $\omega(r_0 - r_1) = 2k\pi, k \in \mathbf{Z}$, the equations (19) lead to the constraints in the coefficients of the normal form

$$\begin{aligned} b_1 &= 2b_2 \\ a_1 &= 2[1 - L(\theta)] Re\left(\frac{e^{-i\omega r_0}}{[1-L(\theta e^{i\omega\theta})]}\right) b_2 \quad . \end{aligned} \tag{22}$$

Before considering the implications of these restrictions for the flows that can be observed, we consider the case of FDEs with just one delay in the nonlinearities, in the form

$$\dot{z}(t) = L(z_t) + F(z(t - r_0)) \quad , \tag{23}$$

where

$$F(u) = \frac{1}{2} A_2 u^2 + O(|u|^3) \quad .$$

Since this is a particular case of the situation considered above for nonlinearities with two delays, we obtain the same normal form, but now the formulas (19) for the coefficients become (it is enough to make in (19) $A_{20} = A_2, A_{11} = A_{02} = 0$)

$$\begin{aligned} a_1 &= A_2 \; Re \; (\psi_1(0) e^{-i\omega r_0}) \\ b_1 &= A_2 \; \psi_3(0) \\ b_2 &= \frac{A_2}{2} \psi_3(0) \quad . \end{aligned} \tag{24}$$

Now we have always the conditions in (22) satisfied, while b_2 can assume arbitrary values by changing A_2. Thus, for nonlinearities involving only one delay we observe always the restrictions in the possible phase portraits around the singularity that occur for nonlinearities depending on two values of the solution in the case of resonant delays satisfying $\omega(r_0 - r_1) = 2k\pi$, with $k \in \mathbf{Z}$.

The restrictions mentioned impose drastic limitations on the flows that can be observed. In fact, the possible topological types of the phase portraits around the origin for equation (20) with $b_1 > 0$ are sketched in [21] as appears in Figure 1, and for $b_1 < 0$ can be obtained by just reversing the arrows. It follows from (22) that the cases corresponding to $b_1 b_2 < 0$ cannot occur for FDEs whose nonlinearities involve just one delay, as well as two delays r_0, r_1 in resonance, *i.e.*, $\omega(r_0 - r_1) = 2k\pi$ with $k \in \mathbf{Z}$. So, of the five phase portraits sketched in Figure 1, only the first two can occur for FDEs whose nonlinearities have just one delay or two delays in resonance. In particular, for ODEs with the singularity considered we can observe homoclinic orbits in arbitrary small neighborhoods of the origin, while homoclinics to the origin in arbitrarily small neighborhoods of this equilibrium are ruled out for FDEs with the preceding properties. In fact, these do not have global orbits in sufficiently small neighborhoods of the origin except the origin itself.

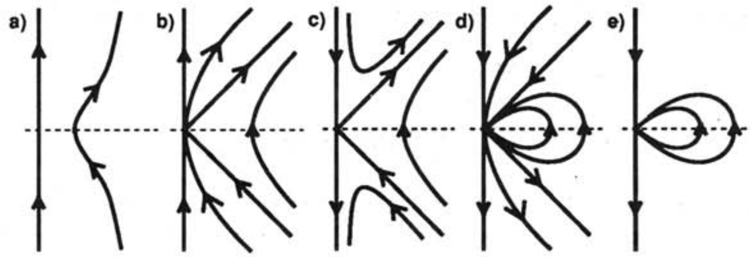

Figure 1: **a)** $b_1b_2 > 0, b_1(b_2 - a_1) > 0$, **b)** $b_1b_2 > 0, b_1(b_2 - a_1) < 0, b_1a_1 > 0$, **c)** $b_1b_2 < 0, b_1(b_2 - a_1) < 0, b_1a_1 > 0$, **d)** $b_1b_2 < 0, b_1(b_2 - a_1) < 0, b_1a_1 < 0$, **e)** $b_1b_2 < 0, b_1(b_2 - a_1) > 0$.

For studying the possibility of obtaining versal unfoldings for the singularity $(\pm i\omega, 0)$ by families of FDEs with nonlinearities involving one delay , we need to study versal unfoldings for equation (20), with $b_1 = 2b_2, b_2 \neq 0, a_1 \neq 0$, by families of ODEs in the plane (ρ, ζ) which are invariant under the transformation $\rho \mapsto -\rho$.

As in [12,13], we consider unfoldings for equation (20), with $b_1 = 2b_2$, in the form

$$
\begin{aligned}
\dot{\rho} &= \mu_1\rho + a_1\rho\zeta \\
\dot{\zeta} &= \mu_2 + b_2(2\rho^2 + \zeta^2) \quad ,
\end{aligned} \tag{25}
$$

with $\mu_1, \mu_2 \in \mathbb{R}$. In order to analyze equation (25), we consider separately the cases $a_1b_2 > 0$ and $a_1b_2 < 0$. Since changing the sign of b_2 amounts to changing the signs of μ_2, ζ and a_1, we take $b_2 > 0$.

Let us first consider the case $a_1b_2 > 0$. The possible topological types of the phase portraits in a neighborhood of the origin and the corresponding parameter regions are given in [12,13] as sketched in Figure 2. From the Poincaré–Bendixson theory we know that there are no periodic orbits in sufficiently small neighborhoods of the origin which are compatible with the sketched flows. On the other hand, the equilibrium points are all hyperbolic except at the curves separating the four regions indicated. As the picture in Figure 2 is persistent under small perturbations, we conclude that (25) is a versal unfolding for equation (20), with $a_1b_2 > 0$.

For the case $a_1b_2 < 0$, the bifurcation analysis of equilibrium points is similar to that of the case $a_1b_2 > 0$ and appears in [12,13] as well, leading to the diagrams sketched in Figure 3. In contrast with the previous case, at this stage the phase portraits in a neighborhood of the origin cannot be considered complete

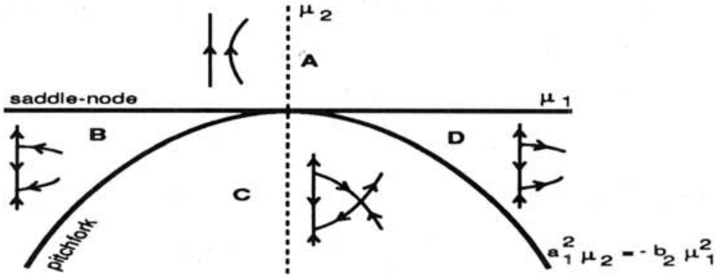

Figure 2: Phase portraits for (25) with $a_1 > 0, b_2 > 0$.

and require further study, due to the following facts:

- along the line $\mu_1 = 0, \mu_2 < 0$ the Hopf condition is satisfied at the equilibrium with $\rho \neq 0$, with a conjugated pair of eigenvalues of the linearization at this equilibrium crossing the imaginary axis from negative to positive real parts as the separating line is crossed from C to D,
- over the line $\mu_1 = 0, \mu_2 < 0$ the equation admits a first integral and the flow for these values of the parameters can be shown to have a heteroclinic loop through the equilibrium points in the ζ–axis encircling the equilibrium point outside this axis, with all the other orbits in the region bounded by the heteroclinic loop being periodic.

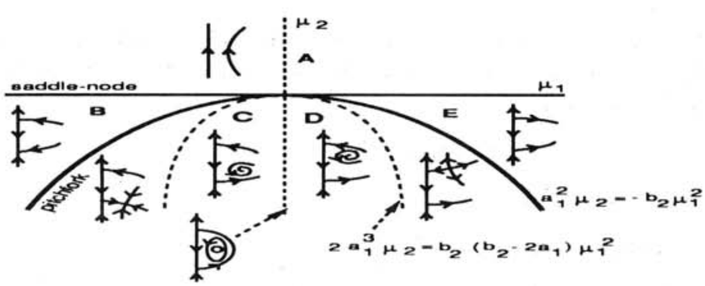

Figure 3: Phase portraits for (25) with $a_1 < 0, b_2 > 0$.

To pursue the analysis in search of a versal unfolding, in the case $a_1 b_2 < 0$, we must add higher order terms to the right hand side of the equation (25). Motivated by the computation of normal forms for the singularity $(\pm i\omega, 0)$ in $\mathbb{R}^3$, we consider, as in [12,13], the cubic equation

$$\begin{aligned} \dot{\rho} &= \mu_1\rho + a_1\rho\zeta + a_2\rho^3 + a_3\rho\zeta^2 \\ \dot{\zeta} &= \mu_2 + b_2(2\rho^2 + \zeta^2) + b_3\rho^2\zeta + b_4\zeta^3 \quad . \end{aligned} \tag{26}$$

For this equation there occurs Hopf bifurcation at a curve

$$\mu_1 = \left[\frac{a_2}{2b_2} - \frac{a_1(2a_2 + b_3)}{4b_2^2}\right]\mu_2 + O(\mu_2^2) \quad ,$$

as $\mu_2 \to 0$. In order to complete the phase portraits in a neighborhood of the origin, we now need to know the number of periodic orbits that may occur an to study the occurrence of heteroclinic loops and their bifurcations.

The equation (26) is considered in [6] where it is shown that when periodic orbits exist in a neighborhood of the origin they are unique, using a technique based on the study of certain Abelian integrals obtained from a first integral associated with the equation after appropriate changes of variables.

Similar methods can be applied to obtain a curve of heteroclinic bifurcation

$$\mu_1 = \frac{1}{\sqrt{2b_2}(2b_2 - 3a_1)}\left[\frac{2(b_2-a_1)^2a_2}{b_2} - \frac{(b_2-a_1)a_1b_3}{b_2} - 2a_1a_3 + \frac{3a_1^2b_4}{b_2}\right]\mu_2 + O(\mu_2^{3/2}).$$

It follows that the possible topological types of the phase portraits in a neighborhood of the origin and the corresponding parameter regions are, in the present case, as sketched in Figure 4. In particular, the line where a heteroclinic loop occurs and the Hopf condition is satisfied for the quadratic normal form is widened to a sector where one periodic orbit occurs for the cubic normal form. Of course, the relative position of the curves of Hopf bifurcation and of heteroclinic loop may need to be interchanged in particular cases, with the stability of the associated periodic orbit and focus changed accordingly.

Now, we can try to obtain versal unfoldings for the considered singularity by families of FDEs. Here, we only refer to the case where the nonlinearity involves just one delay, in the form

$$\dot{z}(t) = L(z_t) + F(z(t - r_0)) \quad . \tag{27}$$

We attempt unfoldings of the form

$$\dot{z}(t) = \nu_1 + \nu_2\, z(t - r_0) + L(z_t) + F(z(t - r_0)) \quad , \tag{28}$$

where $\nu_1, \nu_2 \in \mathbb{R}$. Computing a normal form relative to P in cylindrical coordinates, we get

$$\begin{aligned} \dot{\rho} &= Re(\psi_1(0)e^{-i\omega r_0})\nu_2\rho + a_1\rho\zeta + \ldots \\ \dot{\zeta} &= -\psi_3(0)\nu_1 + \psi_3(0)\nu_2\zeta + b_1\rho^2 + b_2\zeta^2 + \ldots \\ \dot{\xi} &= -\omega + \ldots \quad , \end{aligned} \tag{29}$$

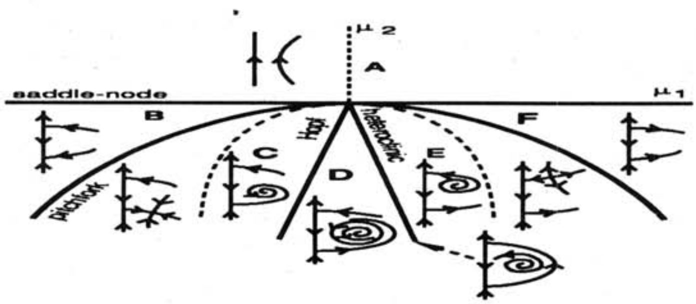

Figure 4: Phase portraits for (26) with $a_1 < 0, b_2 > 0$.

where a_1, b_1, b_2 are as in (24), and the dots stand for higher order terms.

This normal form differs from the second order normal form previously considered because it includes a term in $\nu_2\zeta$ in the second equation. In order to eliminate this term, we consider instead the family of FDEs

$$\dot{z}(t) = \nu_1 + \nu_2[z(t-r_0) - z(t-r_1)] + L(z_t) + F(z(t-r_0)) \quad , \tag{30}$$

with $r_1 \in [0, r] \setminus \{r_0\}$. This involves two delays in the terms with the unfolding parameters (anyway, L must involve more than one delay for $0, \pm i\omega$, with $\omega \neq 0$, to be eigenvalues of A_0).

Computing a normal form relative to P for the last equation in cylindrical coordinates, we get

$$\begin{aligned}
\dot{\rho} &= [1 + \alpha_1\mu_1]\mu_1\rho + [a_1 + \alpha_2\mu_1]\rho\zeta + a_2\rho^3 + a_3\rho\zeta^2 + \dots \\
\dot{\zeta} &= \mu_2 + [b_1 + \beta_1\mu_1]\rho^2 + [b_2 + \beta_2\mu_1]\zeta^2 + b_3\rho^2\zeta + b_4\zeta^3 + \dots \\
\dot{\xi} &= -\omega + \dots \quad ,
\end{aligned} \tag{31}$$

where a_1, b_1, b_2 are given by (24), a_2, a_3, b_3, b_4 can be expressed explicitely in terms of the linear part of the equation and the coefficients in the Taylor expansion for F, $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers that depend on L, F, r_0, r_1, ω and

$$\begin{aligned}
\mu_1 &= Re[\psi_1(0)(e^{-i\omega r_0} - e^{-i\omega r_1})]\nu_2 \\
\mu_2 &= -\psi_3(0)\nu_1 \quad .
\end{aligned}$$

The parameters μ_1, μ_2 can be varied independently by varying ν_1, ν_2, provided the following nonresonance condition between the delays r_0, r_1 and the angle ω is satisfied

$$Re[\psi_1(0)(e^{-i\omega r_0} - e^{-i\omega r_1})] \neq 0 \quad . \tag{32}$$

We conclude that the two-parameter family of FDEs (30) provides indeed an unfolding of the singularity $(\pm i\omega, 0)$ in the class of scalar FDEs with nonlinearities having just one delay which in the $\rho\zeta$-plane is a versal unfolding for the corresponding singularity, as established in the preceding section, provided the nonresonance condition between the delays just mentioned holds. For the case $a_1 b_1 < 0$ this is not a versal unfolding, since the occurrence of periodic orbits and heteroclinic loops in the plane results in the structural instability of the two-parameter family (31) in $\mathbb{R}^3$. However, for the case, $a_1 b_1 > 0$, the considered family does indeed provide a structurally stable two-parameter family of ODEs in $\mathbb{R}^3$, as well as does the family obtained by cutting off the cubic terms in (31). Consequently, the two-parameter family of scalar FDEs (30) provides a versal unfolding for the singularity $(\pm i\omega, 0)$ in the class of scalar FDEs with nonlinearities having just one delay and satisfying $a_1 b_1 > 0$, where a_1 and b_1 are related to each other by the formulas in (22). It can be readily observed that the condition $a_1 b_1 > 0$ only restricts the linear part of the FDEs (27) that can be considered. The fact that a_1, b_1, b_2 in (24) have all the same sign, as happens in this situation, implies severe restrictions on the possible phase portraits in a neighborhood of the origin, in comparison with the situation for ODEs. In particular, the flow of saddle loops sketched in Figure 5 and described in [12,13] does not occur in this case, as well as the more complex behavior associated with the occurrence of infinitely many Smale horseshoes which is described for ODEs in $\mathbb{R}^3$ by Broer and Vegter in [4].

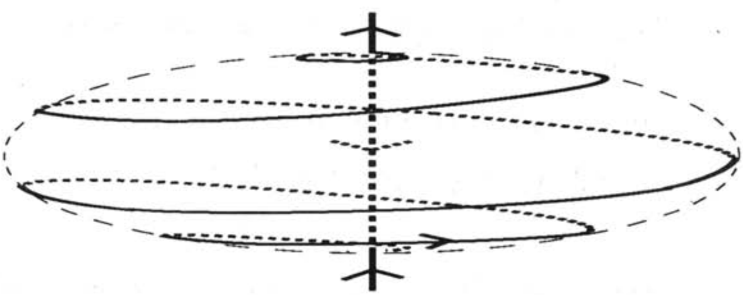

Figure 5: Saddle loops.

As an illustration of the results obtained, we consider an example.

Example:

Since, in the class of scalar linear differential-delay equations $\dot{z}(t) = L(z_t)$, the singularity $(\pm i\omega, 0)$ can only be realized by equations with at least two delays, the simplest situations to be considered are for

$$L(\varphi) = A_0\varphi(-r_0) + A_1\varphi(-r_1) \quad .$$

For $\pm i\omega, 0$, with $\omega \neq 0$, to be characteristic values of the equation $\dot{z}(t) = L(z_t)$ we must have

$$A_1 = -A_0$$

$$A_0 = -\tfrac{\omega}{2\sin\omega r_0}$$

$$\omega(r_0 + r_1) = 2k\pi, k \in \mathbb{N}$$

$$\omega r_0 \neq j\pi, j \in \mathbb{N}_0 \quad .$$

The nonresonance condition is then automatically satisfied since

$$\psi_1(0)(e^{-i\omega r_0} - e^{-i\omega r_1}) = \frac{i\omega}{A_0(1 - L(\theta e^{i\omega\theta}))} \quad ,$$

and, consequently, the nonresonance condition is equivalent to $L(\theta \sin\omega\theta) \neq 0$, which is verified because $L(\theta\sin\omega\theta) = (r_0 + r_1)\sin\omega r_0 \neq 0$. On the other hand, from (22) we have

$$\begin{aligned} \operatorname{sgn} a_1 b_1 &= \operatorname{sgn} Re\left(\frac{[1 - L(\theta)]e^{-i\omega r_0}}{[1 - L(\theta e^{i\omega\theta}]}\right) \\ &= \operatorname{sgn}\ ([\sin 2\omega r_0 - \omega r_0 + (2k\pi - \omega r_0)\cos 2\omega r_0] \\ &\qquad [2\sin\omega r_0 - 2\omega r_0 + 2k\pi]) \quad . \end{aligned}$$

Analyzing the right hand side as a function of ωr_0, it is then easy to conclude that $a_1 b_1$ alternates signs as ωr_0 grows from 0 to $+\infty$, but that it is always positive for $\omega r_0 \in (0, \pi/4]$.

In particular, we conclude that, if the equation $\dot{z}(t) = az(t - r_0) + bz(t - r_1)$, with $a, b \in \mathbb{R}$, $r_0, r_1 > 0$, realizes the singularity $(\pm i\omega, 0)$ and $\omega r_0 \in (0, \pi/4]$, then the two-parameter family of FDEs

$$\dot{z}(t) = \nu_1 + (a + \nu_2)z(t - r_0) + (a - \nu_2)z(t - r_1) + [z(t - r_0)]^2 \quad ,$$

with $\nu_1, \nu_2 \in \mathbb{R}$ close to zero, is a versal unfolding of the singularity $(\pm i\omega, 0)$ in the class of scalar differential–delay equations of the form

$$\dot{z}(t) = C + Az(t - r_0) + Bz(t - r_1) + F(z(t - r_0)) \quad ,$$

where $A, B, C \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ is a C^3 function with $F(0) = F'(0) = 0$, $F''(0) \neq 0$.

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