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Bifurcations in Discretized Reaction-Diffusion Equations [1]

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Abstract: It was recently shown by Fiedler and Rocha that the global attractor of the dissipative semiflow generated by a reaction-diffusion equation of the form $u_t = u_{xx} + f(x, u, u_x), 0 < x < 1$, with Neumann boundary conditions is characterized by a permutation defined on the set of its equilibria. This permutation is defined by the braid of the equilibria in the space of (x, u, u_x) and determines the attractor up to connection equivalence. The space discretization of these equations leads to nonlinear ODE systems in $\mathbb{R}^n$, $\dot{u} = f(u)$, which under appropriate conditions are Morse-Smale. We extend to these systems the first steps in the characterization of the global attractors obtained for reaction-diffusion equations.

Key words: Reaction-Diffusion Equations, Attractors, Morse-Smale Systems, Discretization, Transverse Heteroclinic Points.

1. Introduction

By far the best understood infinite dimensional dynamical systems are the ones generated by scalar semilinear parabolic equations $u_t = u_{xx} + f(x, u, u_x)$ defined on an interval with linear separated boundary conditions. When the nonlinearity satisfies adequate growth and dissipative conditions the equation generates a global semigroup with a gradient like structure and a compact attractor in an appropriate Sobolev space (see for example Hale [10]). Moreover, generically the flow has the Morse-Smale property and the attractor decomposes into a set of equilibria and a set of heteroclinic orbits connecting them (Henrÿ [12] and Angenent [1]).

Many authors have worked on the characterization of this attractor, and by now it is almost complete. In particular, for Neumann boundary conditions, Fiedler and Rocha [7] have recently proved that a special algebraic object introduced by Fusco and Rocha [9] with the purpose of characterizing the attractor, completely determines its structure of heteroclinic connections. In fact, this object, a permutation defined on the set of the equilibria, even allows one to attempt the first steps in the classification of attractors for one-dimensional parabolic equations (see Fiedler [6]).

By discretizing in space these scalar parabolic equations we obtain ODE's defining dynamical systems that under certain conditions belong to a class of Morse-Smale systems in $\mathbb{R}^n$. This class was studied by Fusco and Oliva [8] and consists of smooth nonlinear systems $\dot{x} = f(x)$ such that the derivative $f'(x)$ of the vector field has a matrix representation of positive Jacobi type. The main objective of this paper is to present the first steps in the characterization of the

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global attractors for this class of finite dimensional Morse-Smale systems.

The characterization of all heteroclinic connections in this case also involves the introduction of a permutation defined on the set of equilibria for the ODE and is completely analogous to the characterization obtained for the case of a reaction diffusion equation. Here we only consider the characterization of the equilibria and discuss its bifurcation diagrams, leaving the heteroclinic connections to be considered separately.

As in the case of reaction diffusion equations, the characterization of the equilibria is obtained through the use of a shooting method to obtain the stationary solutions satisfying both boundary conditions. However, the equilibria for the discretized problem are obtained as solutions of a set of nonlinear equations instead of a second order differential equation as before. Hence, the shooting process used here involves the use of an iterated map of $\mathbb{R}^2$ that plays the role of a numerical integration algorithm for the differential equation (the same process used by Domokos and Holmes [5] in the context of a discretization of the Euler buckling problem). This algorithm is a symplectic integrator and the map obtained is an area preserving diffeomorphism related to the standard family. Therefore, for a certain range of the parameters some complicated behavior is expected as the number of iterations of this map is increased. In fact, in the last part of this paper we consider odd nonlinearities and show that generically the dynamical system generated by this map has transverse heteroclinic orbits. As a consequence, the bifurcation diagrams for one parameter families has a very complicated behavior with many equilibrium solutions appearing through secondary bifurcations. This solutions are usually interpreted as numerically irrelevant or spurious. However, we show that some of these solutions correspond to stable equilibria of the attractor for the discrete problem. Moreover, by discretizing a reaction-diffusion equation that cannot have stable nonconstant solutions we show that it is possible to obtain an ODE with a large number of sign changing stable equilibria.

2. The continuous problem

In this section we outline some of the earlier results in the characterization of the equilibria for scalar reaction-diffusion equations (see for example [15]) and resume some of the results of Fiedler and Rocha [7] in order to motivate the methods that we use and the results that we expect for the discretized problem. We consider the scalar reaction-diffusion equation

$$(C)\qquad \begin{array}{rcll} u_t &=& u_{xx} + f(x,u,u_x) & , \quad 0 < x < 1, \\ u_x &=& 0 & , \quad x = 0 \text{ or } 1. \end{array}$$

that from here on we refer to as the continuous probem. The nonlinearity $f(x,u,p)$ is assumed twice continuously differentiable, satisfying a dissipative condition of the type

$$f(x,u,0)\cdot u < 0$$

for $|u|$ large enough and a growth condition on p, for example subquadratic, to ensure global existence of solutions of (C). Then, (C) defines a global dissipative semiflow in a Sobolev space X, for example $H^1(0,1)$ (see for instance Henry [11]), with a gradient like structure provided by the existence of a Liapunov functional for each trajectory $u(\cdot,t)$, [17]. By dissipativeness, this flow has a maximal compact invariant set, the global attractor $\mathcal{A}$, consisting of all the globally defined bounded orbits. Since the flow on $\mathcal{A}$ is gradient like with an associated Morse decomposition, we have that $\mathcal{A}$ is composed of the set E of equilibria and the set of heteroclinic orbits connecting them (that we denote $C(v,w)$ for $v,w \in E$)

$$\mathcal{A} = E \cup \bigcup_{v,w \in E} C(v,w) \ .$$

To determine the set E we have to solve

$$\begin{aligned} u_{xx} + f(x,u,u_x) &= 0 \ , \quad 0 < x < 1, \\ u_x &= 0 \ , \quad x = 0 \text{ or } 1. \end{aligned} \tag{2.1}$$

This is done using a shooting method, determining the solutions of the initial value problem

$$\begin{aligned} u_x &= v \\ v_x &= -f(x,u,v) \\ u(0) = u_0 \ &, \ v(0) = 0 \ , \end{aligned} \tag{2.2}$$

that also satisfy the end point boundary condition $v(x) = 0$ at $x = 1$. Here we assume that the solutions of (2.2) exist for $0 \le x \le 1$ (otherwise, since $\mathcal{A}$ is compact, we change f outside some large ball). Let $u = u(x,u_0), v = v(x,u_0)$ denote these solutions parametrized by the initial condition, and define

$$S \stackrel{\text{def}}{=} \{ \ (u(1,u_0), v(1,u_0)) : u_0 \in \mathbb{R} \ \} \subset \mathbb{R}^2 \ .$$

Then, S is a simple curve in the phase plane determined by the solutions of (2.2) at $x = 1$ and parametrized in u_0. We let $p(u_0) = (u(1,u_0), v(1,u_0))$ denote a point of S and let $H = \{(s,0) : s \in \mathbb{R}\}$ denote the axis $\{v = 0\}$.

Proposition 2.1: A solution $u = u(\cdot,u_0)$ of (2.2) is an equilibrium solution of (C) if and only if $p(u_0) \in S \cap H$.

The tangent vector $\vartheta(u_0)$ to S at the point $p(u_0)$ has components $(\overline{u},\overline{v})|_{x=1}$ obtained from the linear variational equation of (2.2) around the solution $u(\cdot,u_0)$

$$\begin{aligned} \overline{u}_x &= \overline{v} \\ \overline{v}_x &= -f_u(x,u,u_x)\overline{u} - f_v(x,u,u_x)\overline{v} \\ \overline{u}(0) = 1 \ &, \ \overline{v}(0) = 0 \ , \end{aligned}$$

and the eigenvalue problem corresponding to the equilibrium $u(\cdot, u_0) \in E$ is given by

$$\begin{array}{ll} w_{xx} + f_v(x, u, u_x)w_x + f_u(x, u, u_x)w = \lambda w \quad , & 0 < x < 1, \\ w_x = 0 \quad , & x = 0\ ,\ 1, \end{array}$$

¿From a comparison between these two problems one obtains a characterization of hyperbolicity for the equilibria.

Proposition 2.2: The equilibrium $u = u(\cdot, u_0) \in E$ is hyperbolic if and only if the corresponding intersection is transverse $S \pitchfork_{p(u_0)} H$.

Let $i(e) =$ dim $W^u(e)$ denote the Morse index of the equilibrium $e \in E$ and let $\theta = \theta(u_0)$ denote the angle formed by $\vartheta(u_0)$ with the axis $\{v = 0\}$ defined continuously for $u_0 \in \mathbb{R}$. Then, from a Sturm oscillation theorem we obtain

Proposition 2.3: The Morse index of the equilibrium $u(\cdot, u_0) \in E$ is given by

$$i(u(\cdot, u_0)) = 1 + [\theta(u_0)/\pi]$$

where $[\cdot]$ denotes the integer part.

Let $z(u(\cdot, t))$ denote the number of zeros of the solution $u = u(\cdot, t)$ for $t > 0$. This zero number is used in the study of the nodal properties of the solutions of (C) and is essential in establishing its dynamic properties providing for a second (discrete) Liapunov function (see for example [2]). Then, the curve S determines the equilibria, their Morse indices and also the intersection number $z(v - w)$ for each pair of equilibria $v, w \in E$. Finally, ordering the set of equilibria first along S and then along H, one defines a permutation π on the set E that determines the heteroclinic connections between equilibria. In fact, the set of equilibria defines a braid in the space (x, u, v) with $0 \leq x \leq 1$ and leads to the permutation π. This permutation corresponds to the ordering of the equilibria by their values at $x = 0$ and at $x = 1$. To compare the flows on the attractors for two different problems one uses the following notion of equivalence.

Definition 2.4: Two attractors $\mathcal{A}_0$ and $\mathcal{A}_1$ are *connection equivalent* if there exists a bijection $\sigma : E_0 \to E_1$ such that:

(i) $i(\sigma(v)) = i(v)$, for each $v \in E_0$;

(ii) $v \searrow w$ if and only if $\sigma(v) \searrow \sigma(w)$, for each pair $v, w \in E_0$;

where $\searrow$ denotes the relation *connects to*.

The permutation π determines all the connections between the equilibria and using the above notion of equivalence one obtains the following characterization of the attractor. Given two problems with permutations π_0 and π_1 we have

Theorem 2.5: ([7], Corollary 6.1)
$\pi_0 = \pi_1 \implies \mathcal{A}_0$ and $\mathcal{A}_1$ are connection equivalent.

3. The discrete problem

In this section we consider the space discretization of the continuous problem, where the sampling is taken at the points $x_j = \frac{j-1/2}{n}$, $j = 1, \ldots, n$. To simplify the notation we consider the case $f = f(x, u)$ and make some remarks about the dependence on u_x where needed.

Let $u_j(t) = u(x_j, t)$, and define $f_j(\cdot) = f(x_j, \cdot)$. Applying to (C) a semi-implicit Euler discretization and rescaling time by a factor $1/n^2$ we obtain the set of ODE's:

$$u_j' = u_{j-1} - 2u_j + u_{j+1} + \frac{1}{n^2} f_j(u_j) \ , \quad j = 1, \ldots, n$$

where the Neumann boundary conditions are replaced by

$$u_1 = u_0 \quad , \quad u_n = u_{n+1} \ .$$

Defining $U = (u_1, \ldots, u_n)$ we write this as

$$(D) \qquad U' = J_n U + \frac{1}{n^2} F(U)$$

where

$$J_n = \begin{bmatrix} -1 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -1 \end{bmatrix} \quad \text{and} \quad F(U) = (f_1(u_1), \ldots, f_n(u_n)) \ .$$

¿From here on we refer to this ODE in $\mathbb{R}^n$ as the discrete system. The derivative of the vector field at any point $U \in \mathbb{R}^n$ has a tridiagonal matrix representation with positive subdiagonal elements (i.e. a positive Jacobi matrix). Therefore, (D) belongs to the class of systems studied in [8] for which the stable and unstable manifolds of hyperbolic equilibria always intersect transversally. Furthermore, (D) is a gradient flow (see [8] for an explicit Liapunov function) with a compact attractor

$$\mathcal{A} = E \cup \bigcup_{v,w \in E} C(v, w)$$

that, as before, has a Morse decomposition. The main objective of this section is to characterize the set E of equilibria of (D).

Remark 3.1: To include the u_x dependence on f preserving the Morse-Smale property of the flow (D) we assume an upper bound on $\frac{\partial f}{\partial v}$, say $f_v(x, u, v) \leq M$

for all $(x, u, v) \in \mathbb{R}^3$. Then it is sufficient to consider step sizes satisfying the condition $n > M$. This ensures that the derivative of the vector field is a positive Jacobi matrix.

The equilibria $U \in E$ of (D) are the vector solutions of the set of nonlinear equations

$$J_n U + \frac{1}{n^2} F(U) = 0 \; . \tag{3.1}$$

Introducing the auxiliary variables $v_j = u_j - u_{j-1}$, (3.1) can be written as

$$(\Phi_{n,f}) \qquad \begin{cases} u_{j+1} &= u_j + v_{j+1} \\ v_{j+1} &= v_j - \frac{1}{n^2} f_j(u_j) \end{cases} , \quad j = 1, \ldots, n$$

and the boundary conditions become

$$v_1 = v_{n+1} = 0 \; .$$

To obtain the solutions of (3.1) we again use a shooting method. We let $v_1 = 0$, $u_1 = u_0$ with $u_0 \in \mathbb{R}$ taken as a parameter and let

$$(u_{j+1}(u_0), v_{j+1}(u_0)) = \Phi^j_{n,f}(u_0, 0) \; , \quad j = 1, \ldots, n \; , \tag{3.2}$$

where $\Phi^j_{n,f} = \overbrace{\Phi_{n,f} \circ \cdots \circ \Phi_{n,f}}^{j \text{ times}}$. Then, after n iterations of the mapping $\Phi_{n,f}$ we define

$$S \stackrel{\text{def}}{=} \{ \, (u_{n+1}(u_0), v_{n+1}(u_0)) : u_0 \in \mathbb{R} \, \} \subset \mathbb{R}^2 \; .$$

One easily verifies that $\Phi_{n,f}$ is a one to one area preserving diffeomorphism. Hence, S is a smooth simple curve in the plane and, as in the continuous case, provides all the necessary information on the attractor. If we let $p(u_0) = (u_{n+1}(u_0), v_{n+1}(u_0))$ denote a point of S and let H denote the axis $\{v = 0\}$, one easily establishes the following correspondence between equilibria of (D) and intersection points of S and H:

Proposition 3.2: A solution $U(u_0) = (u_1, \ldots, u_n)$ obtained from (3.2) is an equilibrium solution of (D) if and only if $p(u_0) \in S \cap H$.

Remark 3.3: If we consider again the u_x dependence on f, the nonlinear term in $\Phi_{n,f}$ becomes $\frac{1}{n^2} f_j(u_j, n v_j)$. Then, $\Phi_{n,f}$ is no longer area preserving but the condition $n > M$ still ensures that it is one to one and S is a simple curve.

To consider the question of hyperbolicity of equilibria we take again the tangent vector $\vartheta(u_0)$ to S at the point $p(u_0)$ with components (p_{n+1}, q_{n+1}). These are obtained from the derivatives $p_j = \frac{\partial u_j}{\partial u_0}$, $q_j = \frac{\partial v_j}{\partial u_0}$ that satisfy

$$\begin{cases} p_{j+1} = p_j + q_{j+1} \\ q_{j+1} = q_j - \frac{1}{n^2} f_j'(u_j) p_j \end{cases}, \quad j = 1, \ldots, n$$

with initial conditions $p_1 = 1$, $q_1 = 0$. On the other hand, the eigenvalue problem corresponding to the equilibrium $U(u_0) = (u_1, \ldots, u_n) \in E$ is given by

$$\begin{aligned} &w_{j-1} - 2w_j + w_{j+1} + \tfrac{1}{n^2} f_j'(u_j) w_j = \lambda w_j \quad , \quad j = 1, \ldots, n \\ &w_1 = w_0 \quad , \quad w_n = w_{n+1} \,. \end{aligned} \tag{3.3}$$

A comparison between these two problems when $\lambda = 0$ is an eigenvalue of the linearization around $U(u_0) \in E$ establishes the following

Proposition 3.4: The equilibrium $U(u_0) = (u_1, \ldots, u_n) \in E$ of (D) is hyperbolic if and only if the corresponding intersection is transverse $S \pitchfork_{p(u_0)} H$.

Finally, we also obtain an expression for the Morse indices of the equilibria of (D). Let again $\theta = \theta(u_0)$ denote the angle formed by $\vartheta(u_0)$ with the axis $\{v = 0\}$ defined continuously for $u_0 \in \mathbb{R}$.

Proposition 3.5: The Morse index of the equilibrium $U(u_0) \in E$ is given by

$$i(U(u_0)) = 1 + [\theta(u_0)/\pi]$$

where again $[\cdot]$ denotes the integer part.

Proof: The eigenvalue problem (3.3) is written in matrix form as

$$\left[J_n + \frac{1}{n^2} F'(U) - \lambda I\right] W = 0$$

where $W = (w_1, \ldots, w_n)$ and $F'(U) = \operatorname{diag}(f_1'(u_1), \ldots, f_n'(u_n))$. Then, the matrix $\left[J_n + \frac{1}{n^2} F'(U)\right]$ is positive Jacobi and has a real simple spectrum given by $\{\nu_1 > \cdots > \nu_n\}$. Introducing the variables $\overline{p}_j = w_j$ and $\overline{q}_j = w_j - w_{j-1}$ we write the initial value problem corresponding to (3.3) as the iterative linear map

$$\begin{cases} \overline{p}_{j+1} = \overline{p}_j + \overline{q}_{j+1} \\ \overline{q}_{j+1} = \overline{q}_j - \frac{1}{n^2} f_j'(u_j) \overline{p}_j + \lambda \overline{p}_j \end{cases}, \quad j = 1, \ldots, n$$

with initial conditions $\overline{p}_1 = 1$, $\overline{q}_1 = 0$. Using λ as a parameter in addition to u_0, we obtain a solution of (3.3) whenever the condition $\overline{q}_{n+1}(u_0, \lambda) = 0$ is satisfied. Introducing polar coordinates, we let $\overline{p}_j = r_j \cos\theta_j$, $\overline{q}_j = -r_j \sin\theta_j$ where $\theta_j = \theta_j(u_0, \lambda)$ and $r_j = r_j(u_0, \lambda)$ are defined continuously in $\mathbb{R}^2$. Then, the condition $\overline{q}_{n+1}(u_0, \lambda) = 0$ becomes $\theta_{n+1}(u_0, \lambda) = 0 \bmod \pi$. We have that $\theta_1(u_0, \lambda) = 0$ for all $(u_0, \lambda) \in \mathbb{R}^2$ and by a straightforward computation we obtain that θ_j satisfy the following recursive relations

$$\cot\theta_{j+1} = -1 + \left[\frac{1}{n^2} f_j'(u_j) - \lambda + \tan\theta_j\right]^{-1} \quad , \quad j = 1, \dots, n\,.$$

One easily verifies that for u_0 fixed and λ sufficiently large $\theta_{n+1}(u_0, \lambda)$ becomes negative, and that $\theta_{n+1}(u_0, \lambda) \to -\frac{\pi}{4}$ as $\lambda \to +\infty$. Moreover, for $\lambda = 0$ we obtain $\theta_{n+1}(u_0, 0) = \theta(u_0)$ and adapting Proposition 3.4 we have that $\theta_{n+1}(u_0, \lambda) = 0 \bmod \pi$ if and only if $\lambda = \nu_k$ for some $1 \le k \le n$.

Finally, differentiating θ_j with respect to λ we obtain

$$\cos^2\theta_j \, \frac{\partial\theta_{j+1}}{\partial\lambda} = 2\cos^2(\theta_{j+1} - \pi/4)\left[\frac{\partial\theta_j}{\partial\lambda} - \cos^2\theta_j\right] \;, \quad \text{if} \quad \cos\theta_j \neq 0$$

$$\frac{\partial\theta_{j+1}}{\partial\lambda} = \frac{1}{2}\frac{\partial\theta_j}{\partial\lambda} \;, \quad \text{if} \quad \cos\theta_j = 0$$

and we conclude that $\frac{\partial\theta_{j+1}}{\partial\lambda}(u_0, \lambda) \le 0$, for $j = 1, \dots, n$. Hence, as λ decreases from $+\infty$ across the values ν_k, the function θ_{n+1} increases from $-\frac{\pi}{4}$ across the values $(k-1)\pi$ and a simple counting procedure completes the proof.

Remark 3.6: The above characterization of the equilibria of (D) also holds when f depends on u_x under the condition $n > M$.

4. The bifurcation problem

In this section we consider bifurcation diagrams for families of problems of the form (D) and use the characterization obtained for the equilibria to discuss some features that are specific of these discrete problems.

¿From the convergence of the Euler discretization method applied to the initial value problem (2.2) on the fixed interval $x \in [0, 1]$ we conclude that the S curve for the discrete problem is C^1-close to the S curve of the continuous problem for n sufficiently large. Therefore, we expect the corresponding attractors to be connection equivalent when the step size is adequate. However, this convergence cannot be achieved uniformly for the families of problems we want to consider, and since the curve S is obtained by iterating an area preserving diffeomorphism, we expect to obtain complicated bifurcation diagrams very different from the ones obtained for continuous problems.

The simplest continuous bifurcation problem is the Chafee-Infante problem [3]

$$(4.1) \qquad \begin{array}{rcll} u_t & = & u_{xx} + \mu f(u) & , \quad 0 < x < 1, \\ u_x & = & 0 & , \quad x = 0 \text{ or } 1 \end{array}$$

where $f(u) = u - u^3$ and $\mu > 0$. Its bifurcation diagram in terms of the parameter μ is well known. All the equilibria are hyperbolic if $\mu \neq k^2\pi^2$ with $k \in \mathbb{N}$. Moreover, it contains exactly three constant equilibrium solutions, $u^0 \equiv 0$ and $u^\pm \equiv \pm 1$. The equilibria $u^\pm$ are stable for all $\mu > 0$ (these are the only stable equilibria). The equilibrium u^0 is unstable and, if $(k-1)^2\pi^2 < \mu < k^2\pi^2$ is satisfied, its Morse index is given by $i(u^0) = k$. Finally, at the values $\mu = k^2\pi^2$ the trivial equilibrium u^0 undergoes supercritical pitchfork bifurcations, with two nonconstant equilibria $u^{k,\pm}$ arising from u^0 and becoming hyperbolic for all $\mu > k^2\pi^2$ with Morse indices given by $i(u^{k,\pm}) = k$. This completes the description of the bifurcation diagram of (4.1) in what concerns the set of equilibria.

The corresponding discrete Chafee-Infante problem in $\mathbb{R}^n$ is given by

$$U' = J_n U + \frac{\mu}{n^2} F(U) \tag{4.2}$$

with $F(U) = (f(u_1), \ldots, f(u_n))$. One easily shows that $U^0 = (0, \ldots, 0)$ and $U^\pm = (\pm 1, \ldots, \pm 1)$ belong to the set E of equilibria for this problem. Moreover, denoting by $\{\nu_1^n > \nu_2^n > \cdots > \nu_n^n\}$ the spectrum of J_n we have that

$$\nu_j^n = -4 \sin^2 \frac{(j-1)\pi}{2n} \ , \quad j = 1, \ldots, n \ .$$

Then, since $\nu_1^n = 0$, we conclude that $U^\pm$ are stable and hyperbolic for all $\mu > 0$ and that U^0 is unstable. Furthermore, U^0 is hyperbolic if $\mu \neq -n^2\nu_j^n$ with Morse index $i(U^0) = j$ if $-n^2\nu_j^n < \mu < -n^2\nu_{j+1}^n$, and $i(U^0) = n$ if $\mu > -n^2\nu_n^n$. Also, it can be shown that U^0 undergoes supercritical pitchfork bifurcations at $\mu = -n^2\nu_j^n$, $j = 2, \ldots, n$.

This is in complete agreement with what happens in the continuous problem up to the limitation in the dimension of the phase space $\mathbb{R}^n$. However, the similarities stop here and the bifurcation diagrams for both problems are very different. In particular, the bifurcation diagram for the discrete problem shows an incredibly large number of secondary bifurcations as μ increases. In Figures 1-5 we illustrate the appearance of these secondary bifurcations. The figures were obtained using Mathematica and represent the curve S for the discrete Chafee-Infante problem with $n = 4$ and different values of μ.

The secondary bifurcations are related to the properties of the area preserving map $\Phi_{n,\mu f}$ that determines the curve S:

$$(\Phi_{n,\mu f}) \qquad \begin{cases} u_{j+1} &= u_j + v_{j+1} \\ v_{j+1} &= v_j - \frac{\mu}{n^2} f(u_j) \ . \end{cases}$$

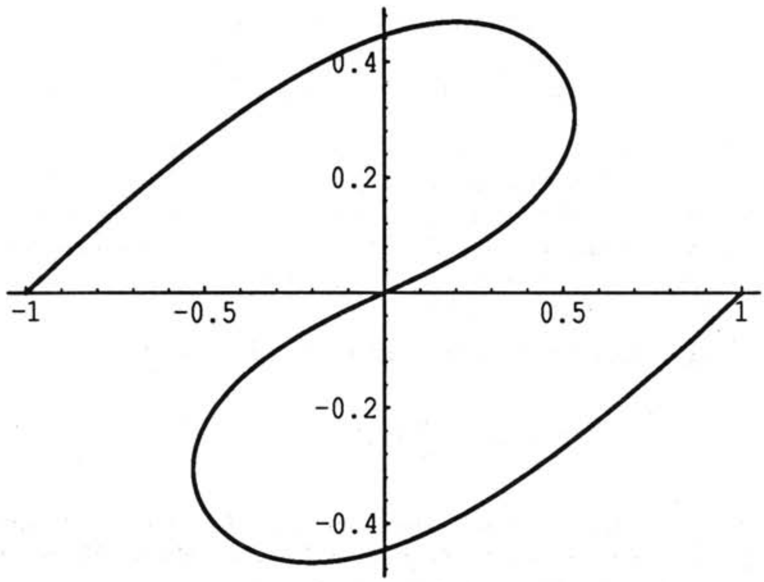

Figure 1: S curve for the discrete problem with $n = 4$ and $\mu = 7$.

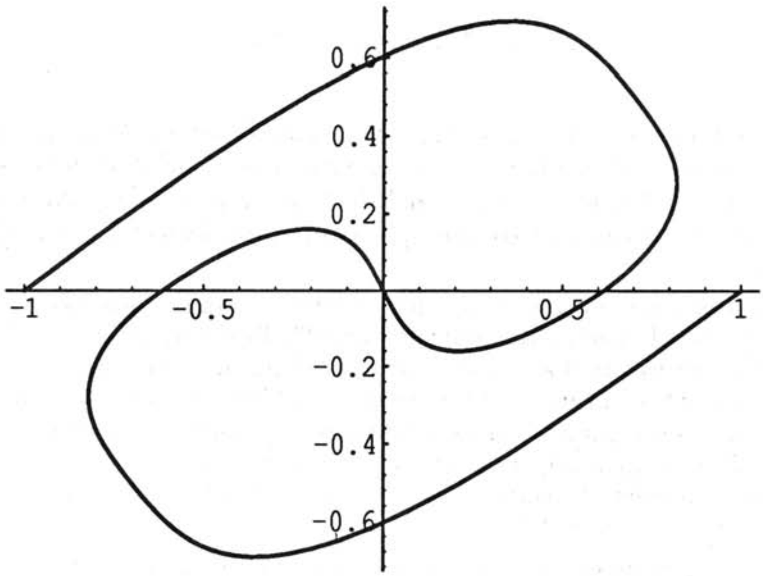

Figure 2: S curve for the discrete problem with $n = 4$ and $\mu = 14$.

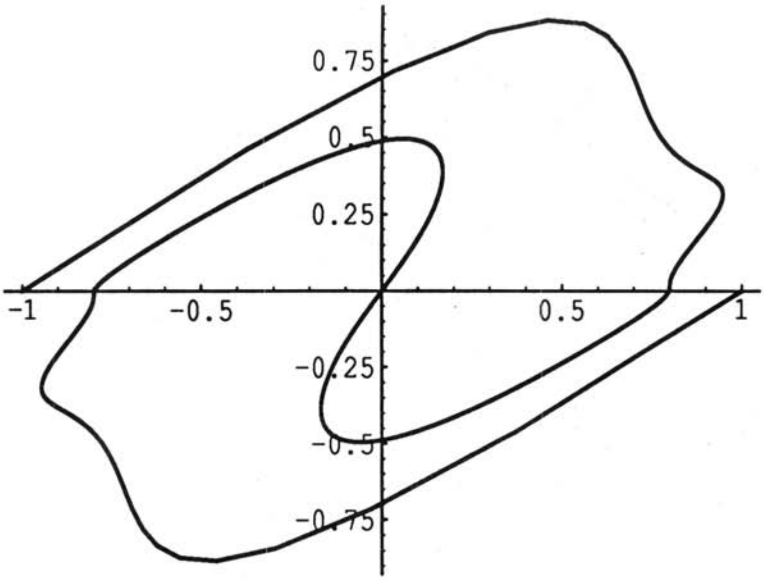

Figure 3: S curve for the discrete problem with $n = 4$ and $\mu = 21$.

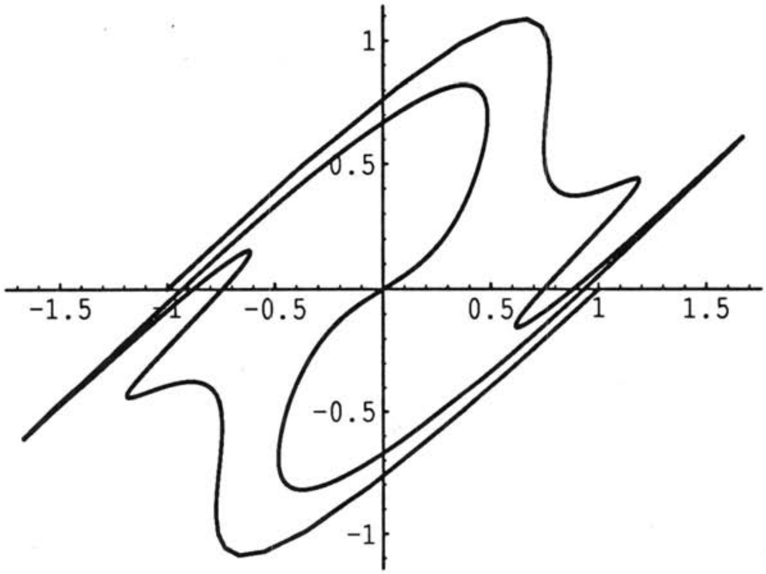

Figure 4: S curve for the discrete problem with $n = 4$ and $\mu = 29$.

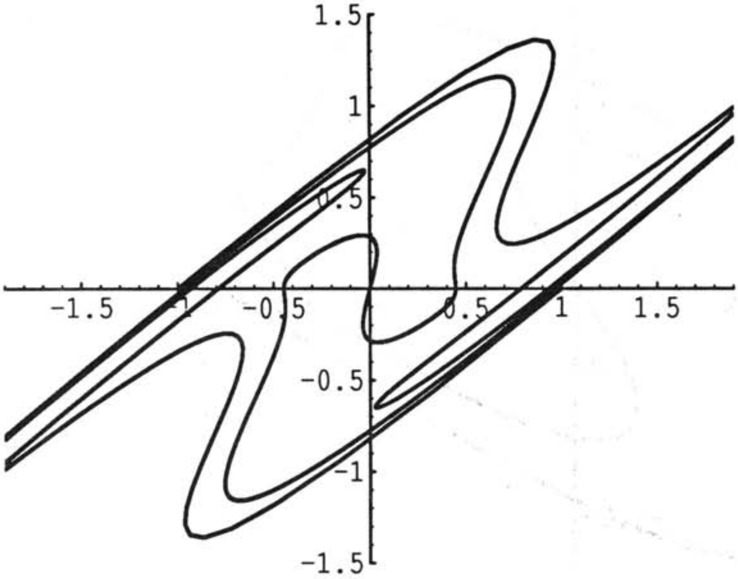

Figure 5: S curve for the discrete problem with $n = 4$ and $\mu = 40$.

We remark that for the nonlinearity f given by $f(u) = \sin u$ if $|u| \leq \pi$, this map is known in the literature as the *standard map* [4]. The bifurcation diagram for this map is studied in [5] where it appears in the context of a discrete model for the Euler buckling rod. In this reference the complexity of the bifurcation diagram for the standard map is very well illustrated. Moreover, some particular results on the secondary bifurcations are also presented.

The standard map and the mapping $\Phi_{n,\mu f}$ for the cubic nonlinearity are qualitatively similar, sharing some important properties. For example, both maps exhibit a pair of hyperbolic saddles with heteroclinic points. As it is well known, the behavior of a discrete dynamical system generated by a diffeomorphism is very complicated in the presence of transversal heteroclinic points. However, there are very few results in the literature on the existence of transversal heteroclinic points for specific maps ([13] contains a partially analytical proof for the standard map). Therefore, we are going to look for generic results in the class of planar diffeomorphisms of the form $\Phi_{n,\mu f}$ with cubic like f's.

Let $\mathcal{F}$ denote the space of C^2 odd functions with the strong C^2-topology. Let $f \in \mathcal{F}$ have exactly three simple zeros and satisfy $f'(0) > 0$, and let $\Phi_f \in \mathrm{Diff}^2(\mathbb{R}^2)$ denote the following associated C^2 area preserving diffeomorphism:

$$\Phi_f(u, v) = (u + v - f(u), v - f(u)) \ .$$

Let also $\mathcal{N}_f \subset \mathcal{F}$ denote a neighborhood of f containing only functions with the same qualitative behavior, that is, having exactly three simple zeros and positive derivative at the origin. Then, we have

Theorem 4.1: The subset $\mathcal{H} \subset \mathcal{F}$ of functions g such that Φ_g has transversal heteroclinic points is residual in $\mathcal{N}_f$.

Proof: Let 0 and $\pm r$ denote the three simple zeros of f. Then, $O = (0,0)$ and $R^{\pm} = (\pm r, 0)$ are the only fixed points of the diffeomorphism Φ_f, and the condition $f'(\pm r) < 0$ implies that $R^{\pm}$ are hyperbolic saddles. In order to study their stable and unstable manifolds we explore the symmetries of the system. Let $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ denote the transformation $\rho(u,v) = (-u,-v)$. Since f is odd, we have that $W^u_{\Phi_f}(R^+) = \rho W^u_{\Phi_f}(R^-)$ and $W^s_{\Phi_f}(R^+) = \rho W^s_{\Phi_f}(R^-)$. Hence, if P is a heteroclinic point of Φ_f satisfying $P \in W^u_{\Phi_f}(R^+) \cap W^s_{\Phi_f}(R^-)$, then ρP also is and satisfies $\rho P \in W^u_{\Phi_f}(R^-) \cap W^s_{\Phi_f}(R^+)$. We conclude that heteroclinic points arise in pairs. To determine a convenient heteroclinic point we look for a relation between the stable and unstable manifolds of the fixed points $R^{\pm}$. Here it is more convenient to introduce a change of coordinates and study the diffeomorphism Φ_f in a different form. Under the transformation $\Sigma(u,v) = (u+v, 2v)$ the diffeomorphism Φ_f is conjugated to $\Psi_f = \Sigma^{-1} \circ \Phi_f \circ \Sigma$ given by

$$\Psi_f(u,v) = \left(u + 2v - \frac{1}{2}f(u+v), v - \frac{1}{2}f(u+v)\right)$$

We remark that $\Sigma(R^{\pm}) = R^{\pm}$ and we let $W^s_{\Psi_f}(R^{\pm}) = \Sigma^{-1}(W^s_{\Phi_f}(R^{\pm}))$, $W^u_{\Psi_f}(R^{\pm}) = \Sigma^{-1}(W^u_{\Phi_f}(R^{\pm}))$ denote the stable and unstable manifolds of the fixed points $R^{\pm}$ for the diffeomorphism Ψ_f. Due to the oddness of f, it turns out that the reflection $\tau(u,v) = (-u,v)$ conjugates Ψ_f to its inverse

$$\Psi_f^{-1}(u,v) = \left(u - 2v - \frac{1}{2}f(u-v), v + \frac{1}{2}f(u-v)\right) = \tau \circ \Psi_f \circ \tau$$

This provides the required relation between the stable and unstable manifolds of Ψ_f. In fact, we conclude that

$$W^u_{\Psi_f}(R^+) = \tau W^s_{\Psi_f}(R^-)\ ,\ W^u_{\Psi_f}(R^-) = \tau W^s_{\Psi_f}(R^+)\ . \tag{4.3}$$

Therefore, a point of intersection of $W^u_{\Psi_f}(R^{\pm})$ with the axis $\{u = 0\}$ belongs to $W^u_{\Psi_f}(R^{\pm}) \cap W^s_{\Psi_f}(R^{\mp})$ and corresponds to a point of intersection of $W^u_{\Phi_f}(R^{\pm})$ with $W^s_{\Phi_f}(R^{\mp})$. From the linearization of Ψ_f around R^- we have that in a small neighborhood of R^- there is a point $Q \in W^u_{\Psi_f}(R^-)$ with coordinates $Q = (\alpha, \beta)$ such that $r^- < \alpha < 0$ and $\beta > 0$. Then, one easily verifies that $\Psi_f^k(Q)$ enters the quadrant $\{u > 0, v > 0\}$ for some finite $k \in \mathbb{N}$. We conclude that, in fact, there is a point of intersection $B \in W^u_{\Psi_f}(R^-) \cap W^s_{\Psi_f}(R^+)$ on the axis $\{u = 0\}$.

In the following we prove that $\mathcal{H}$ is dense in $\mathcal{N}_f$. Assume that the heteroclinic point $B \in W^u_{\Psi_f}(R^-) \cap W^s_{\Psi_f}(R^+)$ is not transverse. Then, B has coordinates $B = (0,b)$ with $b > 0$ and the tangent space $T_B W^u_{\Psi_f}(R^-)$ at B is spanned by a vector $(\overline{b}, 0)$. Let $A = \Psi_f^{-1}(B)$ denote the inverse image of B, with coordinates $A = (a_1, a_2)$, and let $(\overline{a}_1, \overline{a}_2)$ be a vector spanning $T_A W^u_{\Psi_f}(R^-)$. Then,

$$D_A \Psi_f(\overline{a}_1, \overline{a}_2) = (\overline{b}, 0)$$

where

$$D_A \Psi_f = \begin{bmatrix} 1 - \frac{1}{2} f'(a_1 + a_2) & 2 - \frac{1}{2} f'(a_1 + a_2) \\ -\frac{1}{2} f'(a_1 + a_2) & 1 - \frac{1}{2} f'(a_1 + a_2) \end{bmatrix}$$

This implies that

$$\overline{a}_2 = \frac{1}{2} f'(a_1 + a_2)(\overline{a}_1 + \overline{a}_2) \tag{4.4}$$

and $\overline{a}_1 + \overline{a}_2 \neq 0$ since otherwise we have $\overline{a}_2 = 0$ from (4.4) and as a consequence $(\overline{a}_1, \overline{a}_2) = (0, 0)$ contrary to the hypothesis that it spans $T_A W^u_{\Psi_f}(R^-)$. Moreover, from the expression of Ψ_f^{-1} we also have that $a_1 + a_2 = -b < 0$. The rest of the argument is similar to an argument in [14], Chapter 4.4. Let $g \in \mathcal{F}$ denote a small C^2 odd perturbation of f such that $g(b) = f(b)$, $g'(b) = f'(b) + \varepsilon$ and $g = f$ outside small neighborhoods of the points $\pm b \in \mathbb{R}$. Then, the diffeomorphism Ψ_g still has $R^\pm$ as hyperbolic fixed points and the unstable manifold of R^- (denoted $W^u_{\Psi_g}(R^-)$) still contains the points A and B. Also, we still have $(\overline{a}_1, \overline{a}_2) \in T_A W^u_{\Psi_g}(R^-)$ and a simple computation yields

$$D_A \Psi_g(\overline{a}_1, \overline{a}_2) = \left(\overline{b} - \varepsilon \frac{\overline{a}_1 + \overline{a}_2}{2}, -\varepsilon \frac{\overline{a}_1 + \overline{a}_2}{2}\right) \in T_B W^u_{\Psi_g}(R^-) \,.$$

This shows that for $\varepsilon \neq 0$ the heteroclinic point B of Ψ_g is transverse. Likewise, $\Sigma(B)$ is a transversal heteroclinic point for Φ_g proving denseness of $\mathcal{H}$ in $\mathcal{N}_f$. Finally, due to the continuity of the maps $g \mapsto W^u_{\Phi_g}(R^\pm)$ we have that $\mathcal{H}$ is open in $\mathcal{N}_f$ completing the proof of this theorem.

Remark 4.2: Using the symmetry in the orbit structure of the diffeomorphism Ψ_g given by the relation (4.3), the perturbation argument used in this Theorem can be extended to make $W^u_{\Phi_g}(R^\pm)$ and $W^s_{\Phi_g}(R^\mp)$ transversal at all points.

A similar problem has been considered by Ushiki [16] for an area preserving diffeomorphism of the plane derived from a numerical integration scheme for the logistic differential equation. In this reference the analyticity of the map is used to establish the existence of chaotic orbits without assuming transversal intersections for invariant curves.

Consider again the ordinary differential equation

$$U' = J_n U + \frac{\mu}{n^2} F(U) \tag{P}$$

with $F(U) = (f(u_1), \ldots, f(u_n))$ and $f \in \mathcal{F}$ a cubic like nonlinearity as above. We are now able to prove that generically in f the bifurcation diagram of this discrete problem contains stable sign changing equilibria. Moreover, the number of these equilibria is very large if the discretization step size $\frac{1}{n}$ is taken sufficiently small.

As before, let $f \in \mathcal{F}$ have exactly three simple zeros and satisfy $f'(0) > 0$. In addition, let Φ_f have the stable and unstable manifolds of its saddle fixed points transversal at every point.

Theorem 4.3: Under the above assumptions for f, given $N \in \mathbb{N}$ there is a positive integer $n = n(N)$ such that the bifurcation diagram of (P) at $\mu = n^2$ contains at least $2N$ stable sign changing equilibria.

Proof: This result follows from an application of the Inclination Lemma, [14]. Let $R^{\pm}$ denote again the saddle fixed points of Φ_f, and let $W^s(R^{\pm})$, $W^u(R^{\pm})$, denote the corresponding stable and unstable manifolds. Then, we take $\mu = n^2$ and remark that $\Phi_{n,\mu f} = \Phi_f$. As seen in the previous section, the characterization of the equilibria of (P) is obtained from the curve S. This curve is given by $S = \Phi_f^n(H)$ where H is the axis $\{v = 0\}$. Moreover, we have that $H \pitchfork W^s(R^{\pm})$ at the points $R^{\pm}$. Hence, by the Inclination Lemma, if $V(R^-)$ denotes a small neighborhood of R^-, the image of $H \cap V(R^-)$ under Φ_f^n becomes C^1-close to $W^u(R^-)$ as $n \to +\infty$. By our assumption the manifolds $W^u(R^-)$ and $W^s(R^+)$ are transversal. If P denotes the point $P = \Sigma(B) \in W^u(R^-) \cap W^s(R^+)$ defined in the proof of the previous Theorem, there is a sequence of points $P_j = \Phi_f^j(P)$ converging to R^+ such that $W^u(R^-) \pitchfork_{P_j} W^s(R^+)$. In a small neighborhood $V(R^+)$ of R^+ we have that the arcs of $W^u(R^-) \cap V(R^+)$ containing the points P_j in $V(R^+)$ are C^1-close to $W^u(R^+)$. This implies the existence of a sequence of points Q_j of transverse intersection of $W^u(R^-)$ with H in $V(R^+)$. Since the arc of S given by $\Phi_f^n(H \cap V(R^-))$ as n increases becomes C^1-close to the manifold $W^u(R^-)$, for n sufficiently large we can find N points in $V(R^+)$ of transverse intersection of S with H. By Proposition 3.2 these points correspond to equilibria of (P), and by Proposition 3.4 they are hyperbolic. Furthermore, we can choose these N points in such way that the corresponding angles θ_k, $k = 1, \ldots, N$, formed by the tangent vectors ϑ_k to S with H are negative. Proposition 3.5 then implies the stability of the corresponding equilibria. Each of these equilibria have negative first component u_1 and positive last component u_n (close to the corresponding zeros $\pm r$ of f), hence they are sign changing equilibria. Finally, the invariance of S under the transformation ρ implies that each of these N equilibria has a corresponding pair with the same properties, concluding this proof.

As a final remark we point out that Figures 4 and 5 illustrate well the appearance of a pair of such stable sign changing equilibria in the case of the cubic nonlinearity.

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