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Chaos Expansions : A Review[1]

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Abstract: The aim of this paper is to describe recent results on the Wiener-Itô decomposition. We focus on a survey of new applications of chaos expansions to functionals of Wiener and Gaussian processes arising from different fields of probability and stochastic processes.

Key words: Multiple Wiener Integral, large deviations, self-intersection local time, anticipating stochastic differrential equations, level-crossing counts.

Contents

1 Introduction

Let X be a random variable with normal distribution having zero mean and variance one, and let F be a real valued function in the L^2 space of the Gaussian measure on R, *i.e.* $E\left(F(X)^2\right) < \infty$. It is well known that F admits the L^2−orthogonal expansion

$$F(x) = \sum_{m=0}^{\infty} a_m H_m(x), \tag{1}$$

where H_m, $m \geq 0$, are the orthogonal Hermite polynomials given by

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$$H_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2} \qquad m \geq 1, \quad H_0(x) = 1. \tag{2}$$

Moreover, $E(F(X)) = a_0$ and

$$E\left(F(X)^2\right) = \sum_{m=0}^{\infty} m! a_m^2. \tag{3}$$

The orthogonal expansion (1) has been used in many applications in mathematics and physics. For example, expansions of the type

$$1_{[X>0]} = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{(2m)!}{2^m m!} H_{2m+1}(X) \tag{4}$$

have been very useful in probability theory (see [80]). Similar expressions to (1) and (3) in terms of multivariate Hermite polynomials can be obtained for real valued functions in the L^2 space of the Gaussian measure on R^d. The recent book by Thangavelu [86] is an excellent reference for this subject.

An analogous orthogonal expansion for real valued L^2−functionals of a Wiener process was proved by Itô [31], where the role of the *"infinite dimensional"* Hermite polynomials is played by the so called *multiple Wiener-Itô integrals.* More precisely, for an atomless measure space (T, Γ, ν), let $W(A), A \in \Gamma$, be an orthogonal Gaussian random measure with variance $\nu(A)$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and let $L^2(W) = L^2(\Omega, \mathcal{F}^W, P)$, where $\mathcal{F}^W$ is the σ-field generated by W. Itô [31] proved that any real valued functional $F \in L^2(W)$ admits the L^2−orthogonal expansion

$$F = \sum_{m=0}^{\infty} I_m(f_m), \tag{5}$$

where $I_m(f_m)$ is the m-th multiple Wiener-Itô integral of the *non random kernel* $f_m \in L^2(T^m)$, $m \geq 1$, and $I_0(f_0) = f_0 = E(F)$. This expansion is unique provided the kernels are symmetric. Moreover, resembling (3), it holds that

$$E(F)^2 = \sum_{m=0}^{\infty} m! \left\| \widetilde{f}_m \right\|^2_{L^2(T^m)} < \infty, \tag{6}$$

where $\widetilde{f}_m$ is the symmetrization of f_m.

We shall refer to (5) as the *chaos expansion* of the Wiener functional F. It is also known in the literature as the *Wiener chaos decomposition* or *the chaotic representation.*

Several authors have dealt with chaos expansions in the study of functionals of Wiener and Gaussian processes arising in different fields of probability, stochastic processes, statistics and mathematical physics (see for instance [2], [3], [9], [30],

[40], [44], [45], [47], [55], [57], [60], [65], [79], [80] and [82]). For example, expansions in terms of multiple Wiener-Itô integrals have traditionally been used in the construction of Hida distributions and the White Noise Calculus, as shown in Hida [19] and Hida *et. al* [20]. Other important Wiener distributions, as those in Nualart and Zakai [64], Gorostiza and Nualart [16], Korezlioglu and Üstünel [38], Meyer and Yan [49], Watanabe [91] and references therein, also depend on the Wiener chaos decomposition. Likewise, the recent book by Meyer [51] shows the usefulness of chaos expansions in Quantum Probability, while Hu and Meyer [26] and Johnson and Kallianpur [34] have studied the Feynman integral through chaotic representations.

The purpose of this work is to survey some applications of chaos expansions which have recently been made by several authors. We include the Wiener chaos decomposition of some functionals of Wiener and Gaussian processes appearing in large deviations, stochastic analysis and level crossing counts. The aim is to illustrate how this tool is useful in the study of several problems in probability and stochastic processes. For this review we have drawn freely on the work of several co-authors and colleagues, hoping that this survey will be an incentive for the study of new applications of chaos expansions.

The organization of the paper is as follows. Section 2 presents the basic properties of multiple Wiener-Ito integrals; an anticipating stochastic integral and Sobolev spaces of Wiener functionals in the sense of Watanabe [91]. Section 3 contains results by Pérez-Abreu and Tudor [71] on large deviations for a class of random variables having a special chaos expansion. Section 4 describes the recent work of Nualart and Vives [60], [61], [62] and Imkeller, Pérez-Abreu and Vives [29] on the intersection local time of the Brownian motion and their Wiener chaos decomposition. Section 5 sketches an approach based on chaos expansions for the solution of bilinear stochastic differential equations. We include the case of a random Gaussian drift in the first Wiener chaos, as recently given in León and Pérez-Abreu [42]. Section 6 presents an application due to Üstünel and Zakai [88] on the Wiener chaos expansion for Radon-Nikodym derivatives of some transformations of the Wiener path. Section 7 briefly reviews some of the new work by E. Slud and co-authors ([10], [11], [37], [80], [81]) on the chaos expansion of functionals related to level-crossing counts of a Gaussian process. Finally, Section 8 includes a result of Houdré and Pérez-Abreu [23] on variance inequalities for functionals of the Wiener process.

In this survey we do not include any material on chaos expansions for stochastic processes other than the Wiener and Gaussian. We refer to Ogura [67], Nualart and Vives [59] and Surgailis [83] for the Poisson process; to Itô [33], Segall and Kailath [75] and He and Wang [17] for independent-increment processes; to Azéma and Yor [1] and Emery [12]-[13] for other martingales; to Nualart and Zakai [63] for the multiparameter Wiener process; to Biane [4] for finite Markov chains; to Meyer [50] and Hu and Wang [17] for some discrete time processes; to Kallianpur and Pérez-Abreu [36] for the cylindrical Brownian motion on a Hilbert space, to Hida [18] for the generalized Wiener process, and to Pérez-Abreu [68] for the nuclear space valued Wiener process. Recently, Szulga [84, Sec. 3.3] has offered a general perspective on the study of the infinite chaos order for general processes including stable processes. On the other hand, the book by Kwapień and Woycziński [39] presents an updated account on multiple stochastic integrals and series for L^2 and non-L^2 processes, while Houdré and Pérez-Abreu [24] compile several recent contributions and surveys to the fields of multiple Wiener-Ito integrals, chaos processes and applications in theoretical and applied areas of probability, stochastic processes and statistics.

2 Multiple Wiener-Itô integrals and chaos expansions

In this section we review basic properties and applications of multiple Wiener-Itô integrals. These tools are by now very useful in the study of Wiener functionals and their chaos expansions, as those included in the following sections.

Multiple integrals of Gaussian orthogonal random measures were defined by Itô [31] using the so called special elementary functions in $L^2(T^m)$. These integrals were initially introduced by Wiener [92] and different approaches to their construction and extensions to L^2-processes are presented in Engel [14], Kwapień and Woyczinsky [39], Major [46], Meyer [48], Neveu [54] and Pérez-Abreu [69]. For the standard Wiener process W_t, $t \in T = [0,1]$, the m-th *multiple Wiener-Itô integral* $I_m(f_m)$ of a function $f_m \in L^2(T^m)$ coincides with the iterated Itô stochastic integral, i.e.,

$$I_m(f_m) = m! \int_0^1 \int_0^{t_{m-1}} \cdots \left\{ \int_0^{t_2} \widetilde{f}_m(t_1, ..., t_m) dW_{t_1} \right\} \cdots dW_{t_{m-1}} . dW_{t_m}, \quad (7)$$

where $\widetilde{f}_m$ is the symmetrization of f_m with respect to the m variables $t_1, ..., t_m$. We shall denote by $\widehat{L}^2(T^m)$ the subspace of square integrable symmetric functions of T^m.

It is easily seen that multiple Wiener-Itô integrals have the following properties. Let $f_m \in L^2(T^m)$ and $g_n \in L^2(T^n)$, then $I_m(f_m) = I_m(\widetilde{f}_m)$,

$$E(I_m(f_m)) = 0, \quad (8)$$

$$E(I_m(f_m) I_n(g_n)) = \delta_{mn} m! < \widetilde{f}_m, \widetilde{g}_n >_{L^2(T^m)}, \quad (9)$$

and in particular

$$E(I_m(f_m)^2) = m! \left\| \widetilde{f}_m \right\|^2_{L^2(T^m)}. \quad (10)$$

From (9) we observe that multiple Wiener-Itô integrals of different order are orthogonal.

An important available tool is the so called *product formula for multiple integrals*, first proved by Ito [31] and later generalized by Shigekawa [77]. Namely, for $f_m \in \widehat{L}^2(T^m)$ and $g_n \in \widehat{L}^2(T^n)$ it holds that

$$I_m(f_m)I_n(g_n) = \sum_{r=0}^{\min(m,n)} r! \binom{m}{r} \binom{n}{r} I_{m+n-2r}(f_m \otimes_r g_n), \tag{11}$$

where

$$f_m \otimes_r g_n(t_1, ...t_{m-r}, s_1, ...s_{n-r}) =$$

$$\int_{T^r} \cdots \int f_m(t_1, ..., t_{m-r}, u_1, ...u_r) g_n(s_1, ..., s_{n-r}, u_1, ...u_r) du_1 ... du_r. \tag{12}$$

In particular (see [31] or [39, Th. 10.3.1]):

$$I_m(f_m)I_1(g_1) = I_{m+1}(f_m \otimes g_1) + I_{m-1}(f_m \otimes_1 g_1). \tag{13}$$

A valuable relation between Hermite polynomials and multiple Wiener-Ito integrals should be noted. For orthonormal functions $\phi_1, ..., \phi_n$ in $L^2(T)$ it holds that (see[31]):

$$I_{p_1+\cdots+p_n}(\phi_1^{\otimes p_1} \otimes \cdots \otimes \phi_n^{\otimes p_n}) = \prod_{i=1}^{n} \sqrt{p_i!} H_{p_i}(I_1(\phi_i)).$$

Specially, for a function ϕ with $||\phi||_{L^2(T)} = 1$

$$I_p(\phi^{\otimes p}) = \sqrt{p!} H_p(I_1(\phi)). \tag{14}$$

The chaos expansion (5) was initially proved by K. Ito [31] using the orthogonal expansion of Cameron and Martin [8]. Alternative proofs of this result can be found, for example, in Kallianpur [35], Kwapień and Woyczinski [39] and Neveu [54]. An alternative representation for $F \in L^2(W)$ is given by (see [5] or [66])

$$F = E(F) + \int_0^1 \psi_s^1 dW_s. \tag{15}$$

where $\psi^1 \in L^2([0, 1] \times \Omega)$ is adapted and the stochastic integral is in the sense of Ito.

An important application of chaos expansions is to the construction of a stochastic integral for an anticipating stochastic process $u \in L^2(T \times \Omega)$. The original idea seems to be originated in Hitsuda [21], and was later pursued by Skorohod [79], Berger and Mizel [3] and Nualart and Pardoux [57] among others. The basic essence is the following: $u \in L^2(T \times \Omega)$, it holds from (5) that for each $t \in T$

$$u_t = \sum_{m=0}^{\infty} I_m(u_m^t),$$

where for $m \geq 1$, $u_m \in L^2(T^{m+1})$. Then, it seems natural to define the integral of u (sometimes called the *Skorohod integral)* as

$$\int_0^1 u_s \delta W_s = \sum_{m=0}^{\infty} I_{m+1}(\widetilde{u}_m^{\cdot}), \tag{16}$$

whenever

$$\sum_{m=0}^{\infty} (m+1)! \, \|\widetilde{u}_m^{\cdot}\|^2_{L^2(T^{m+1})} < \infty, \tag{17}$$

where $\widetilde{u}_m^{\cdot}$ denotes the symmetrization of $u_m^{\cdot}$ with respect to their $m+1$ variables, *i.e.*

$$\widetilde{u}_m^t(t_1, ...t_m) = \frac{1}{m+1}\left\{ u_m^t(t_1, ...t_m) + \sum_{i=1}^{m} u_m^{t_i}(t_1, ..., t_{i-1}, t, t_{i+1}, ...t_m) \right\}.$$

If the process u_s is non-anticipating, the stochastic integral (16) coincides with the Itô stochastic integral ([57]).

Nualart and Zakai [63], following the above ideas, present generalizations to multiple stochastic integrals with random integrands. That is, for $k \geq 1$, let $u \in L^2(T^k \times \Omega)$ be such that for $t_1, ..., t_k \in T$

$$u_{t_1 \cdots t_k} = \sum_{m=0}^{\infty} I_m(u_m^{t_1 \cdots t_k}),$$

with

$$\sum_{m=0}^{\infty} (m+k)! \, \|\widetilde{u}_m^{\cdot}\|^2_{L^2(T^{m+k})} < \infty.$$

This leads them to define a multiple stochastic integral of the random process u as

$$\int_{T^k} \cdots \int u_{s_1 \cdots s_k} \delta W_{s_1} \cdots \delta W_{s_k} = \sum_{m=0}^{\infty} I_{m+k}(\widetilde{u}_m^{\cdot}). \tag{18}$$

Other useful application of chaos expansions in stochastic analysis is to the construction of Wiener distributions is the sense of Watanabe [91]. For $\alpha \in R$, the *Sobolev space of order* α of Wiener functionals $D^{2,\alpha}$ is defined (see [5] or [91]) by introducing the norm

$$\|F\|_{2,\alpha}^2 = \sum_{m=0}^{\infty} (m+1)^\alpha m! \left\| \tilde{f}_m \right\|_{L^2(T^m)}^2 \tag{19}$$

on the space of all Wiener functionals having finite chaos expansions (which is dense in $L^2(W)$) and completing with respect to $\|\cdot\|_{2,\alpha}$. The case $\alpha = 1$ corresponds to the domain of the *gradient or Malliavin derivative* D defined for a functional F in its domain (having the chaos expansion (5)) by

$$D_t F = \sum_{m=1}^{\infty} m I_{m-1}(\tilde{f}_m(t,\cdot)). \tag{20}$$

In particular, for a multiple Wiener-Itô integral

$$D_t I_m(f_m) = m I_{m-1}(\tilde{f}_m(t,\cdot)). \tag{21}$$

More generally, for $F \in D^{2,k}$, some integer $k \geq 1$, the k-derivative of F is given by

$$D^k_{t_1 \ldots t_k} F = \sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1) I_{m-k}(\tilde{f}_m(t_1, \ldots t_k, \cdot)). \tag{22}$$

Using (8)-(10), from (22) we obtain that for an $F \in D^{2,k}$ having chaos expansion (5),

$$f_k(t_1, \ldots, t_k) = \frac{1}{k!} E(D^k_{t_1 \ldots t_k} F) \tag{23}$$

and

$$E(D^k_{t_1 \ldots t_k} F)^2 = \sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1) m! \left\| \tilde{f}_m(t_1, \ldots t_k, \cdot) \right\|_{L^2(T^{m-k})}^2 . \tag{24}$$

The case $\alpha < 0$ corresponds to a space of distributions of Wiener functionals. Similar Sobolev spaces are defined for the d-dimensional Wiener processes (see [91]).

There is an important connection (*the duality relation*) between the stochastic integral (16) and the derivative (20). Namely, whenever u satisfies (17), the integral (16) is the unique element in $L^2(W)$ such that

$$E\left(F \int_0^1 u_t \delta W_t\right) = E\left(\int_0^1 D_t F u_t dt\right) \quad \forall\ F \in D^{2,1}. \tag{25}$$

In fact, it can be shown that the integral operator (16) from $L^2(T \times \Omega)$ to $L^2(W)$ is the adjoint of the derivative operator D from $L^2(W)$ to $L^2(T \times \Omega)$. Moreover, under appropriate conditions, the following integration by parts formula holds

$$D_t\left(\int_0^1 u_s \delta W_s\right) = u_t + \int_0^1 D_t u_s \delta W_s. \tag{26}$$

As an application of this duality relation, from (15) it is possible to obtain the so called *Clark-Ocone* formula (see [5] or [66]): For $F \in D^{2,1}$

$$F = E(F) + \int_0^1 E(D_t F \mid \mathcal{F}_t^W) dW_s,$$

where $\{\mathcal{F}_t^W; 0 \le t \le 1\}$ is the filtration of the Wiener process.

The following duality relation between (18) and (22) is sometimes useful:

$$E\left(F \int_{T^k} u_{t_1\cdots t_k} \delta W_{t_1} \cdots \delta W_{t_k}\right) = E \int_{T^k} D^k_{t_1\ldots t_k} F u_{t_1\ldots t_k} dt_1 \ldots dt_k \;\; \forall \;\; F \in D^{2,k}. \tag{27}$$

3 Large deviations for some Wiener functionals

The probability distribution of a chaos random variable $I_m(f_m), f_m \in \widehat{L}^2(T^m)$, is known only in the cases $m = 1$ and 2. It is quite well known that the probability distribution of the Wiener integral $I_1(f_1)$ is Gaussian with zero mean and variance $\|f_1\|^2_{L^2(T)}$. On the other hand, as shown for example in Imkeller [28] or Varberg [90], $I_2(f_2)$ has the distribution of the random variable $V = \sum_{i=1}^{\infty} \lambda_i(\chi_i^2 - 1)$,where $\chi_1^2, \chi_2^2, \ldots$, are independent chi-square random variables with one degree of freedom and $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the integral operator K_2 in $L^2(T)$ defined by f_2, *i.e.*,

$$(K_2 g)(t) = \int_T f_2(s,t) g(s) ds, \quad g \in L^2(T).$$

For a general $m \ge 3$, all the moments of the random variable $I_m(f_m)$ are finite, but Nualart, Üstünel and Zakai [58] have shown that the characteristic function of $I_m(f_m)$ is not analytic. Shigekawa [77] has proved that every multiple Wiener-Itô integral has a density, but no explicit expression for it still remains an open problem to find an expression for it. However, exponential tail estimates for the distribution of $I_m(f_m)$ have been studied by Borell [6], McKean [47] and Plikusas [72] amongst others. Recently, Pérez-Abreu and Tudor [71] have shown that for each $m \ge 1$, $f_m \in \widehat{L}^2(T^m)$, $x > 0$ and $\alpha > 2$, it holds that

$$P(|I_m(f_m)| > x) \le K_\alpha \exp\left\{-\frac{1}{\alpha}\left(\frac{x^2}{m!\,\|f_m\|^2_{L^2(T^m)}}\right)^{1/m}\right\}, \tag{28}$$

where the constant

$$K_\alpha = \sum_{p=0}^{\infty} \frac{(2p)!}{p!(2\alpha)^p} < \infty \tag{29}$$

is independent of m.

By using exponential tail estimates of the type (28), it is possible to prove a large deviations principle for the random variables $\{\epsilon^{m/2} I(f_m); \epsilon > 0\}$. Namely, (see for example [56]) for any $\epsilon > 0$ there exists an x_0 such that for each $x > x_0$

$$\exp(-(\Lambda_+(f_m) + \epsilon)x^{2/m}) \leq P(I(f_m) > x) \leq \exp(-(\Lambda_+(f_m) - \epsilon)x^{2/m}),$$

with a similar estimate holding for the negative tails replacing $\Lambda_+(f_m)$ by $\Lambda_-(f_m)$, where

$$\Lambda_+(f_m) = \frac{1}{2}\left[\sup\left\{\int \cdots \int_{[0,1]^m} f_m(\underline{t})\phi^{\otimes m}(\underline{t})d\underline{t}; \quad \|\phi\|^2_{L^2(T)} = 1\right\}\right]^{-2/m}$$

and

$$\Lambda_-(f_m) = \frac{1}{2}\left[-\inf\left\{\int \cdots \int_{[0,1]^m} f_m(\underline{t})\phi^{\otimes m}(\underline{t})d\underline{t}; \quad \|\phi\|^2_{L^2(T)} = 1\right\}\right]^{-2/m}.$$

On the other hand, the fact that the constant (29) does not depend on m, allows the large deviation principle for the random variables $\{F^\epsilon = \sum_{m=0}^{\infty} \epsilon^{m/2} I_m(f_m) ; \epsilon > 0\}$, where the kernels satisfy that there exists a constant C such that for each $m \geq 1$

$$m! \, \|f_m\|^2 \leq C^m / m!. \tag{30}$$

More precisely, it holds (see [71]) that for every Borel set E in R

$$-\inf_{x \in E^\circ} \Lambda(x) \leq \liminf_{\epsilon \to 0} \epsilon \log P(F^\epsilon \in E) \leq \limsup_{\epsilon \to 0} \epsilon \log P(F^\epsilon \in E) \leq -\inf_{x \in \overline{E}} \Lambda(x),$$

where

$$\Lambda(x) = \frac{1}{2} \inf\left\{\left\|\dot{\theta}\right\|^2_{L^2(T)} ; G(\theta) = x\right\}, \tag{31}$$

$$G(\theta) = \sum_{m=0}^{\infty} \int_{[0,1]^m} \cdots \int f_m(\underline{t}_m)\dot{\theta}^{\otimes m}(\underline{t}_m)d\underline{t}_m < \infty, \tag{32}$$

E° and $\overline{E}$ are the interior and closure of E, and $\dot{\theta}$ denotes the derivative of θ. Chaos random variables for which the kernels satisfy (30) are called *chaos expansions of* *exponential type*. Examples of such functionals are provided by the applications of Section 5.

Ledoux [41] and Nualart *et. al* [56] have proved the large deviations principle for the processes $\{\epsilon^{m/2} I_m(f_m^t), t \in [0,1]; \epsilon > 0\}$ in the space $C([0,1])$. Namely, if the *chaos process* $\{I_m(f_m^t); t \in [0,1]\}$ has a continuous version, then for any Borel set E in $C([0,1])$ and $\epsilon > 0$ small enough, it holds that

$$P(\epsilon^{m/2} I_m(f_m^0) \in E) \approx \exp(-\frac{1}{\epsilon}(\Lambda_m(E))), \tag{33}$$

where $\Lambda_m(E) = \sup_{g\in E} \Lambda_m(g)$ with

$$\Lambda_m(g) = \frac{1}{2}\inf\left\{ \left\|\dot{\theta}\right\|^2_{L^2(T)}; \int \cdots \int_{[0,1]^m} f_m^t(\underline{t}_m)\dot{\theta}^{\otimes m}(\underline{t}_m)d\underline{t}_m = g(t)\right\}.$$

These type of results are useful to prove laws of iterated logarithm for chaos processes, as it has been shown by Mori and Oodaira [52], [53] (see also [15]).

Conditions for path continuity of the chaos process $X_t = I_m(f_m^t)$ are provided in Marcus [43], Mori and Oodaira [52] and Nualart *et. al* [56]. General conditions for the path continuity of a chaos expansion process $X_t = \sum_{m=0}^{\infty} I_m(f_m^t)$ remains an open problem, as well as their corresponding large deviations principles in the space $C([0,1])$.

4 Intersection local time of a d-dimensional Brownian motion

The study of chaos expansions for the local time of multiparameter and d-dimensional Brownian motions has seen considerable interest in recent years, as shown by the works of Nualart and Vives [60], [61], [62], Imkeller, Pérez-Abreu and Vives [29], and Shieh [76]. These Wiener chaos decompositions give the existence of such functionals as well as their degree of smoothness in the case when they are distributions in the sense on Watanabe [91].

Let $W_t, t \in T = [0,1]$, be a standard Wiener process and consider the occupation measure of W defined by

$$\mu_W^t(A) = \int_0^t 1_A(W_s)ds, \quad A \in B(R).$$

It is well known (see for example [27]) that this measure has a density L_x^t (called *the local time*), which can be formally defined as

$$L_x^t = \int_0^t \delta_x(W_s)ds, \tag{34}$$

where $\delta_x(W_s)$ denotes the composition of the Dirac delta function δ_x with W_s. For s$\neq 0$ this composition is a Wiener distribution in the sense of Watanabe [91].

The basic idea for studying the intersection local time through chaos expansions is for using the Wiener chaos decomposition of $\delta_x(W_s)$ to compute the integral in (34). In the Wiener process case (see [60] or [61]) this integral provides a smoothing effect and therefore the local time L_x^t is well defined and belongs to a Sobolev space of random variables.

More specifically, let $p_\epsilon(x)$ be the one dimensional Gaussian density, *i.e.*,

$$p_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{x^2}{2\epsilon}\right) \quad -\infty < x < \infty, \quad \epsilon > 0. \tag{35}$$

Using the relation (14) between multiple Wiener-Itô integrals and Hermite polynomials, it holds (see [29] or [60]) that for any $0 \neq h \in L^2(T)$, $x \in R$, and $\epsilon > 0$, the chaos expansion of $p_\epsilon(I_1(h) - x)$ is given by

$$p_\epsilon(I_1(h) - x) =$$

$$\sum_{m=0}^{\infty} I_m \left[\frac{1}{\sqrt{m!}} \left(\|h\|^2_{L^2(T)} + \epsilon \right)^{-\frac{m}{2}} H_m \left(\frac{x}{\sqrt{\|h\|^2_{L^2(T)} + \epsilon}} \right) p_{\|h\|^2_{L^2(T)}+\epsilon} h^{\otimes m}) \right].$$

In particular, for $x = 0$ we have

$$p_\epsilon(I_1(h)) = \sum_{m=0}^{\infty} I_{2m} \left[\frac{(-1)^m}{\sqrt{2\pi} m! 2^m} \left(\|h\|^2_{L^2(T)} + \epsilon \right)^{-(m+\frac{1}{2})} h^{\otimes 2m}) \right]. \tag{36}$$

Then (see [60]) it holds that

$$\delta_x(I_1(h)) = \lim_{\epsilon \downarrow 0} p_\epsilon(I_1(h) - x) \ \ in \ D^{2,\alpha} \text{ for any } \alpha < -1/2 \tag{37}$$

and

$$L_x^t = \lim_{\epsilon \downarrow 0} \int_0^t p_\epsilon(W_s) ds \quad in \quad D^{2,\alpha} \text{ for } \alpha < 1/2. \tag{38}$$

Similar results can be obtained for the local time of a multiparameter Wiener process $W_t, t \in [0,1]^k, k \geq 1$ ([60]). For this case, the local time L_x^t belongs to the space $D^{2,\alpha}$ for $\alpha < k - \frac{1}{2}$.

We now describe analogous ideas for the study of the double intersection local time of a d-dimensional Wiener process $W_t = (W_t^1, \ldots W_t^d)$, $0 \leq t \leq 1$, $d \geq 2$, where the single Brownian motions $W_t^1, \ldots W_t^d$ are independent. Using completely different techniques, the double intersection local time of the planar ($d = 2$) and the 3−dimensional Brownian motions is studied in Rosen [73], [74] and Yor [93], among others.

For $x = (x_1, ..., x_d) \in R^d$ consider the *double intersection local time*, formally defined by

$$\eta_x = \iint_{,0\leq s\leq t\leq 1} \prod_{i=1}^{d} \delta_0(W_t^i - W_s^i - x_i)dsdt, \tag{39}$$

and let

$$\eta_x(\epsilon) = \iint_{,0\leq s\leq t\leq 1} \prod_{i=1}^{d} p_\epsilon(W_t^i - W_s^i - x_i)dsdt, \quad \epsilon > 0. \tag{40}$$

Their corresponding renormalizations are:

$$\eta_x^* = \iint_{,0\leq s\leq t\leq 1} \left\{ \prod_{i=1}^{d} \delta_0(W_t^i - W_s^i - x_i) - E\left(\prod_{i=1}^{d} \delta_0(W_t^i - W_s^i - x_i) \right) \right\} dsdt, \tag{41}$$

and

$$\eta_x^*(\epsilon) = \iint_{,0\leq s\leq t\leq 1} \left\{ \prod_{i=1}^{d} p_\epsilon(W_t^i - W_s^i - x_i) - E\left(\prod_{i=1}^{d} p_\epsilon(W_t^i - W_s^i - x_i) \right) \right\} dsdt. \tag{42}$$

By using (37), Imkeller *et. al* [29] have shown that for $x \neq 0$

$$\eta_x = \lim_{\epsilon\downarrow 0} \eta_x(\epsilon) \quad \text{in } D^{\alpha,2} \text{ for } \alpha < \frac{4-d}{2}.$$

Thus, the double intersection local time η_x is a well defined random variable for $d = 2, 3$ and a distribution valued series for d$\geq$ 4. Furthermore

$$\eta_x = \sum_{n_i} \iint_{0\leq s\leq t\leq 1} \prod_{i=1}^{d} \left[\frac{1}{\sqrt{n_i!}} I_n^i \left(\frac{1_{[s,t]}^{\otimes n_i}}{\sqrt{t-s}} \right) H\left(\frac{x_i}{\sqrt{t-s}} \right) p_{t-s}(x_i)dsdt \right], \tag{43}$$

where $I_n^i(\cdot)$ denotes the n-th multiple Wiener-Ito integral with respect to the i-th Brownian motion W^i. Although the existence of η_x for d=2 and 3 is well known ([73], [74], [93]), the chaos expansion gives the expression (43) for η_x as well as its existence for general $d \geq 4$.

For $d = 2$, it was recognized by Varadhan [89] (see also [73]) that $\eta_x \to \infty$ as $|x| \to 0$ and therefore a renormalization is necessary. Using chaos expansions, it was proved in [29] that for $x \neq 0$

$$\eta_x^* = \lim_{\epsilon\downarrow 0} \eta_x^*(\epsilon) \text{ in } D^{2,\alpha} \text{ for } \alpha < 1$$

and moreover, also in $D^{2,\alpha}$ for $\alpha < 1$,

$$\eta_0^* = \lim_{|x|\to 0} \eta_x^*.$$

Similar expansions to (43) are given for η_x^* in [29].

For the case $d = 3$, the Wiener chaos decomposition technique gives (see [29]) that

$$\eta_0^* = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\log\frac{1}{\epsilon}}}\eta_0^*(\epsilon)$$

weakly in $D^{2,\alpha}$ for $\alpha < 1/2$, but not strongly. The extra renormalization factor $1/\sqrt{\log(1/\epsilon)}$ was suggested by Yor [93], who proved convergence in law of $1/\sqrt{\log(1/\epsilon)}\eta_0^*(\epsilon)$.

Finally, for $d \geq 4$ it has been proved in [29] that the net

$$\left\{\epsilon^{(d-3/2)}\eta_0^*(\epsilon); 0 < \epsilon \leq 1\right\}$$

is bounded in $D^{2,\alpha}$ for $\alpha < (4-d)/2$ and therefore weakly sequentially compact. Moreover, the $D^{2,\alpha}$-norms converge to a non-zero quantity as ϵ goes to zero, but eventually existing weak limits of this net have not been identified.

5 Bilinear stochastic differential equations

The use of chaos expansions together with the anticipating integral (16) and the product formula for multiple Wiener-Ito integrals (11), provide an easy approach to the study of anticipative bilinear stochastic differential equations driven by a Wiener process. Although the idea of the method is somehow elementary, it raises interesting infinite systems of deterministic integral equations which solutions in many cases are still unknown.

As an illustration, let $W_t, t \in T = [0,1]$, be a standard Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. Consider the *anticipative bilinear stochastic differential equation*

$$dX_t = A(t)X_t + B(t)X_t dW_t\ ,\ 0 < t \leq 1,\ X_0 = \xi, \tag{44}$$

where $B(t), t \in [0,1]$, is a suitable family of deterministic functions and the *random drift* $A(t)$ lives in a finite chaos, *i.e.*, for an $n \geq 1$, there exist kernels $a \in L^2(T^{n+1})$, such that $A(t) = I_n(a^t), t \in T$.

It is said that the equation (44) has a strong solution $X = (X_t, t \in T)$, if the stochastic process X. is in $L^2(T\times\Omega)$ and satisfies that for each $t \in T$

$$X_t = X_0 + \int_0^t A(s)X_s ds + \int_0^1 1_{[0,t]}(s)B(s)X_s\delta W_s\ , \tag{45}$$

where the last integral is in the sense of (16).

Assume that the initial condition has the Wiener chaos decomposition

$$X_0 = \sum_{m=0}^{\infty} I_m(g_m), \quad g_m \in \widehat{L}^2(T^m), \quad m \geq 1, \quad g_0 = E(X_0). \tag{46}$$

It is important to observe that we do not assume that X_0 is $\mathcal{F}_0^W$−measurable nor that $A(s)$ is not anticipating.

Let the potential solution X. have the following chaos expansion for each $t \in T$

$$X_t = \sum_{m=0}^{\infty} I_m(f_m^t), \tag{47}$$

where $f_m^t \in \widehat{L}^2(T^m)$, $m \geq 1$, $f_0^t = E(X_t)$. Then, writing (46) and (47) in (45) we obtain

$$\sum_{m=0}^{\infty} I_m(f_m^t) = \sum_{m=0}^{\infty} I_m(g_m) + \int_0^t I_n(a^s) \sum_{m=0}^{\infty} I_m(f_m^s) ds$$
$$+ \int_0^1 1_{[0,t]}(s) B(s) \sum_{m=0}^{\infty} I_m(f_m^s) \delta W_s. \tag{48}$$

We first apply the product formula (13) in the second and third integral of the right hand side of (48). Then we use the stochastic integral (16) in the last integral of this equation. Finally, by the uniqueness of chaos expansions up to symmetric kernels, from both sides of (48) it is possible to identify kernels of the same order.

For example, in the case $n = 1$, *i.e.*, $A(t) = I_1(a^t)$, we obtain the following infinite system of deterministic integral equations (see [42]):

$$f_0^t = g_0 + \int_0^t \int_0^1 a^r(s) f_1^r(s) ds dr,$$

$$f_m^t(t_1, ..., t_m) = g_m(t_1, ..., t_m) + \frac{1}{m} \sum_{i=1}^{m} \int_0^t a^r(t_i) f_{m-1}^r(t_1, ..., t_{i-1}, t_{i+1}, ..., t_m) dr$$
$$+ (m+1) \int_0^t \int_0^1 a^r(s) f_{m+1}^r(s, t_1, ..., t_m) ds dr$$
$$+ \frac{1}{m} \sum_{i=1}^{m} B(t_i) 1_{\{t_i < t\}} f_{m-1}^{t_i}((t_1, ..., t_{i-1}, t_{i+1}, ..., t_m)), \quad m \geq 1. \tag{49}$$

Under the assumption that the initial condition X_0 has a finite chaos expansion, León and Pérez-Abreu [42] have shown that the above system (49) has a solution. Moreover, there is an explicit expression for the kernels of the unique solution X of (45), and for each $t \in T$ these kernels are of exponential type (30) and they satisfy

$$\sum_{m=0}^{\infty}(m+1)!\,\|f_m\|^2_{L^2(T^{m+1})} < \infty.$$

In a personal communication, C. Tudor has observed that the above results still hold if the initial condition X_0 has an infinite chaos expansion of exponential type (30).

For the case when the drift $A(t)$ is not random, *i.e.*, $n = 0$, the Wiener-Itô decomposition approach gives in a straightforward manner the kernels for the chaos expansion of the solution of (44), as Shiota [78], Pérez-Abreu [70] and Tang [85] show.

For drifts leaving in higher finite chaos, *i.e.*, $n \geq 2$, the question of existence and uniqueness of solutions to the corresponding infinite systems of deterministic integral equations is an open problem. On the other hand, although the above approach is simple, it does not lead to a general theory of anticipative stochastic differential equations. The latter could be achieved with the help of an anticipative Girsanov's transformation as presented by Buckdahn [7] and Üstünel and Zakai [87]. However, the chaos expansion approach could be still useful in the case when $W_t, t \in T$, is a multiparameter Wiener process (see [85]) and a Girsanov's type theorem is not available or when the assumptions of the latter theorem are not satisfied.

6 Wiener chaos expansion for Radon-Nikodym derivatives

In this section we present, in a few words, an application of chaos expansions to the study of Radon-Nikodym derivatives of transformations of the Wiener measure. The result is due to Ustunel and Zakai [88] and we give here the main ideas.

Let $T = [0,1], \Omega = C(T), \mathcal{F} = B(\Omega)$, and P be the Wiener measure in Ω. Let H be the Cameron-Martin space, i.e.,

$$H = \left\{ f \in L(T); \quad f(t) = \int_0^t f'(s)ds \quad and \quad \int_0^{t1} (f'(s))^2 ds < \infty. \right\}$$

Consider the transformation of the Wiener path $\mathbf{T}\omega = \omega + u(\omega)$, $u \in H$, and let $P \circ \mathbf{T}^{-1}$ denote the corresponding induced measure on $\mathcal{F}^W$.

The idea presented in Section 2 for defining the Malliavin derivative and the Watanabe distributions for real valued chaos random variables extends to $H-$valued random elements (indeed for an arbitrary separable Hilbert space). We do not pursue this generality here, but in the remaining of this section we will need of this more general set up (see [5, C. 3]). We shall proceed in a formal way using the fact that expressions (5), (16), (22), (23) and (27) also hold in this case. We denote by $u^{\otimes m}$ the mth fold tensor product of $u \in H$ with itself.

Under suitable assumptions on u and $P \circ \mathbf{T}^{-1}$, Üstünel and Zakai [88] have proved that the Radon-Nikodym derivative $\frac{dP\circ\mathbf{T}^{-1}}{dP}$ has the following chaos expansion:

$$\frac{dP \circ \mathbf{T}^{-1}}{dP} = 1 + \sum_{m=1}^{\infty} I_m(f_m), \tag{50}$$

where

$$f_m(t_1, ..., t_m) = \frac{1}{m!} E\left(\sum_{j=0}^{m} \binom{m}{j} D^{m-j}_{t_1 \dots t_m} u^{\otimes j}\right), \quad m \geq 1. \tag{51}$$

A sketch of the proof of (50) and (51) is the following (see [88] for details as well as for a more general result). Let $g_m(\omega) = \phi(I_1(e_1), ..., I_1(e_m))$ where $\phi(\cdot, ..., \cdot)$ is a polynomial of order m or less and $e_i \in H$. Then it can be shown that

$$g_m(\omega + u(\omega)) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle D^n g_m(\omega), u^{\otimes n}\rangle_{H^{\otimes n}} \quad .$$

Using the analogous duality relation of (27) we obtain

$$E(g_m(\omega + u(\omega)) = E\left(\sum_{n=0}^{\infty} \frac{1}{n!} g_m(\omega) \int \dots \int_{T^k} u^{\otimes n}(s_1, ..., s_n) \delta W_{s_1} \cdots \delta W_{s_n}\right).$$

Since

$$E(g_m(\omega + u)) = E\left(g_m \frac{dP \circ \mathbf{T}^{-1}}{dP}\right),$$

we have that

$$E\left(g_m \frac{dP \circ \mathbf{T}^{-1}}{dP}\right) =$$

$$E\left(g_m(\omega) \sum_{n=0}^{m} \frac{1}{n!} \int \dots \int_{T^k} u^{\otimes n}(s_1, ..., s_n) \delta W_{s_1} \cdots \delta W_{s_n}\right),$$

which means that the projection of $dP \circ \mathbf{T}^{-1}/dP$ on the m-th Wiener chaos is the same as that of

$$F_m = \sum_{n=0}^{m} \frac{1}{n!} \int_{T^k} \dots \int u^{\otimes n}(s_1, ..., s_n) \delta W_{s_1} \cdots \delta W_{s_n}.$$

Then, from the analogous of (22) we have that

$$f_m(t_1,\ldots,t_m)=\frac{1}{m!}E(D^m_{t_1\cdots t_m}F_m).$$

Additional computations yield (51).

7 Chaos expansions for functionals of level-crossing counts

We now give a brief account of the work by E. Slud ([80],[81]) on chaos expansions for the number of level-crossing counts of a Gaussian process and related functionals.

Let $X_t, t\geq 0$, be a stationary Gaussian process with zero mean, variance one, and correlation function

$$r(t)=E(X_tX_0)=\int_0^\infty \exp(ixt)\sigma(dx), \tag{52}$$

with σ a nonatomic probability measure on **R**. It is assumed that $r(t)$ is twice differentiable and we shall denote $\rho^2=-r''(0)$. Let $\psi : R\to R$ be a continuously differentiable function (which in many applications is the zero function) and let $N_\psi(T)$ be the continuous time number of crossings of ψ by the Gaussian process $X_t, 0\leq t\leq T$.

Slud [80], [81] proved that $N_\psi(T)$ is a well defined random variable having the chaos expansion

$$N_\psi(T)=E\left(N_\psi(T)\right)+\sum_{m=1}^\infty I_m(f_m), \tag{53}$$

where for $m\geq 1$

$$\begin{aligned} f_m(t_1,\ldots,t_m) &= \int_0^T \exp(is(t_1+\ldots+t_m))[\frac{\rho}{\pi}H_m(u)\exp(-u^2/2) \\ &- \frac{\exp(-u^2/2)}{\pi}\sum_{j=1}^m H_{m-j}(u)\frac{i^j}{j!}\sum_{1\leq l_1<\cdots<l_{j_m}} t_{l_1}\ldots t_{l_m} \\ &\left\{\int_0^{1/\rho}\exp(-z^2y^2/2)H_j(-zy)y^{j-2}dy]_{u=\psi(s),z=\psi'(s)}ds\right\}. \end{aligned} \tag{54}$$

The main idea of Slud [80], [81] for the proof of (53) and (54), is to exhibit the crossings-indicator $1_{[(X_t-\psi(t))(X_{t+h}-\psi(t+h)<0]}$ first using the Hermite polynomials expansion (similar to (4)) of the indicator $1_{[X_0\geq c]}$ for an arbitrary level c, and then applying the product formula (11).

Strictly speaking, all the results in this section have to consider the complex domain. The underlying idea is to use the spectral representation of X_t as the Fourier transform of a complex Gaussian measure with both real and imaginary parts being independent-increments real Gaussian processes. In this case there is an analogous theory available for complex multiple Wiener-Itô integrals, as presented, for example, in Itô [32] and Major [46].

Chambers and Slud [10], [11] and Slud [80], [81] have used chaos expansions similar to (53) and (54) to prove central and noncentral limit theorems for functionals related to level crossings counts of X. Similar kind of limit theorems for nonlinear functionals of a Gaussian process have been a customary application of the Wiener chaos decomposition. For expository works to this particular subject, the reader is referred to the excellent book by Major [46], the recent above papers by D. Chambers and E. Slud, and references therein.

Recently, Kedem and Slud [37] (see also [81]) have used the above expansions in the study of some statistical problems of Gaussian processes. We hope that more statistical applications will appear in the near future.

8 Variance inequalities for Wiener functionals

We finally present some inequalities, due to Houdré and Pérez-Abreu [23], for the variance of functionals of the Wiener process $W_t, t \in T = [0,1]$. Although (6) provides an identity for the variance of the functional (5), these inequalities give useful approximations to the variance.

Let $F \in D^{2,k}, k = 1, \ldots, 2n-1$ (respectively $k = 1, \ldots, 2n$) for some $n \geq 1$. Then, the following right (respectively left) inequality holds:

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} E \left\| D^k F \right\|^2_{L^2(T^k)} \leq Var(F) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E \left\| D^k F \right\|^2_{L^2(T^k)}, \quad (55)$$

where

$$E \left\| D^k F \right\|^2_{L^2(T^k)} = E \int_{T^k} \cdots \int \left(D^k_{t_1 \ldots t_k} F \right)^2 dt_1 \ldots dt_k.$$

Houdré and Pérez-Abreu [23] have obtained (55) as particular cases of a general covariance identity for functionals of the Wiener process. Hu [25] has recently generalized these inequalities to diffusion processes. We here present a simple proof of (55) due to C. Houdré (see also [22]) using chaos expansions.

Let F have the Wiener chaos decomposition (5). Using (21), (22) and (24) on the right hand side of (55) we have

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E \left\| D^k F \right\|^2_{L^2(T^k)} = \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1) j! \left\| \widetilde{f}_j \right\|^2_{L^2(T^j)}$$

$$= \sum_{j=1}^{2n-1} j! \left\| \widetilde{f}_j \right\|^2_{L^2(T^j)} \sum_{k=1}^{j} \frac{(-1)^{k+1}}{k!} j(j-1)\cdots(j-k+1)$$

$$+ \sum_{j=2n}^{\infty} j! \left\| \widetilde{f}_j \right\|^2_{L^2(T^j)} \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} j(j-1)\cdots(j-k+1).$$

Now, from the facts that $\sum_{k=1}^{j} \frac{(-1)^{k+1}}{k!} j(j-1)\cdots(j-k+1) = 1$ and

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} j(j-1)\cdots(j-k+1) - 1 \geq 0,$$

by using (6) we obtain

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E \left\| D^k F \right\|^2_{L^2(T^k)} - Var(F) \geq 0.$$

This proves the right hand side inequality in (55) and a similar argument shows the lower inequality.

References

[1] Azéma, J. and M. Yor (1989). *Étude d'une martingale remarquable.* In Sem. Probab. XXIII, Lecture Notes in Mathematics 1372, pp 89-129, Springer-Verlag, New York.

[2] Berger, M. A. (1988). *A Malliavin-type anticipative stochastic calculus.* Ann. Probab. **16**, 231-245.

[3] Berger, M. A. and V. J. Mizel (1982). *An extension of the stochastic integral.* Ann. Probab. **10**, 435-450.

[4] Biane, P. (1990). *Chaotic representation for finite Markov chains.* Stochastics and Stochastics Reports **30**, 61-68.

[5] Bouleau, N. and F. Hirsch (1991). *Dirichlet Forms and Analysis on Wiener Space.* W. de Gruyter, Berlin.

[6] Borell, C. (1986). *Tail probabilities in Gauss space.* In: Vector Space Measures and Applications, Dublin 1977. Lecture Notes in Mathematics **644**, pp 73-82, Springer-Verlag, New York.

[7] Buckdahn, R. (1991). *Linear stochastic differential equations.* Probab. Th. Rel. Fields **90**, 223-240.

[8] Cameron, R. H. and W. T. Martin (1947). *The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals.* Ann. Math. **48**, 385-392.

[9] Carmona, R. A. and D. Nualart (1992). *Traces of random variables on Wiener space and the Onsager-Machlup functional.* J. Funct. Anal. **107**, 402-438.

[10] Chambers, D. and E. Slud (1989a). *Central limit theorems for nonlinear functionals of stationary Gaussian processes.* Probab. Th. Rel. Fields. **80**, 323-346.

[11] Chambers, D. and E. Slud (1989b). *Necessary conditions for nonlinear functionals of Gaussian processes to satisfy central limit theorems.* Stochastic Processes Appl. **32**, 93-107.

[12] Emery, M. (1989). *On the Azéma martingales.* In Sem. Probab. XXIII, Lecture Notes in Mathematics **1372**, pp 65-87, Springer-Verlag, New York.

[13] Emery. M. (1991). *Quelques cas de représentation chaotique.* In Sem. Probab. XXV, Lecture Notes in Mathematics **1485**, pp 10-23, Springer-Verlag, New York.

[14] Engel, D. D. (1982). *The Multiple Stochastic Integral.* Mem. Amer. Math. Soc. **38.**

[15] Goodman, V. and J. Kuelbs (1992). *Gaussian chaos and self-similar processes.* Preprint.

[16] Gorostiza, L. and D. Nualart (1993). *Nuclear Gelfand triples on Wiener space and applications to trajectorial fluctuations of particle systems.* Preprint.

[17] He, S. W. and J. G. Wang (1989). *Chaos decomposition and property of predictable representation.* Science in China, Ser. A, **32**, 397-407.

[18] Hida, T. (1978). *Generalized multiple Wiener integrals.* Proc. Japan Acad. **54** 55-58.

[19] Hida, T. (1980). *Brownian Motion.* Springer-Verlag, New York.

[20] Hida, T., H. H. Kuo, J. Potthoff and L. Streit (1993). *White Noise: An Infinite Dimensional Calculus.* Kluwer, Boston.

[21] Hitsuda, M. (1972). *Formula for Brownian partial derivatives.* Second Japan-U.S.S.R. Symposium on Probability Theory, 111-114.

[22] Houdré, C. and A. Kagan (1992). *Variance inequalities for functions of Gaussian variables.* Preprint.

[23] Houdré, C. and V. Pérez-Abreu (1994). *Covariance identities and inequalities for functionals on Wiener and Poisson spaces.* Ann. Probab. To appear.

[24] Houdré, C. and V. Pérez-Abreu (1994). *Chaos Expansions, Multiple Wiener-Itô Integrals and Their Applications.* Series in Probability and Stochastics, CRC Press Inc., Florida. To appear.

[25] Hu, Y. Z. (1993). *Note on covariance inequalities for Gaussian and diffusions.* Preprint.

[26] Hu, Y. Z. and P. Meyer (1988). *Chaos de Wiener et intégrale de Feynman.* In: Sem. Probab. X, Lecture Notes in Mathematics **1321**, pp 51-71. Springer-Verlag, New York.

[27] Ikeda, N. and S. Watanabe (1981). *Stochastic Differential Equations and Diffusion Processes.* North Holland, Amsterdam.

[28] Imkeller P. (1994). *On exact tails for limiting distributions of U-statistics in the second Gaussian chaos.* In: Chaos Expansions, Multiple Wiener-Ito Integrals and Their Applications. C. Houdré and V. Pérez-Abreu (Eds). Series in Probability and Stochastics, CRC Press Inc., Florida. To appear.

[29] Imkeller P., V. Pérez-Abreu and J. Vives (1993). *Chaos expansions of double intersection local time of Brownian motion in R^d and renormalization.* Preprint.

[30] Isobe, E. and S. Sato (1983). *Wiener-Hermite expansion of a process generated by an Itô differential equation.* J. Appl. Probab. **20**, 754-765.

[31] Itô, K.(1951). *Multiple Wiener integral.* J. Math. Soc. Japan **3**, 157-169.

[32] Itô , K.(1952). *Complex multiple Wiener integral.* Japan. J. Math. **22**, 53-86.

[33] Itô , K. (1956). *Spectral type of shift transformation of differential processes with stationary increments.* Trans. Amer. Math. Soc. **81**, 106-134.

[34] Johnson, G. W. and G. Kallianpur (1994). *Homogeneous chaos, p-forms, scaling and the Feynman integrals.* Trans. Amer. Math. Soc. To appear.

[35] Kallianpur, G. (1981). *Stochastic Filtering Theory.* Springer-Verlag, New York.

[36] Kallianpur, G. and V. Pérez-Abreu (1992). *The Skorohod integral and the derivative operator of functionals of a cylindrical Brownian motion.* Appl. Math. Optim. **25**, 11-29.

[37] Kedem, B. and E. Slud (1991). *On autocorrelation estimation in mixed-spectrum Gaussian processes.* Stochastic Processes Appl. To appear.

[38] Korezlioglu, H. and A. S. Üstünel (1990). *Distributions, Girsanov and degree theorems on Wiener space.* In White Noise Analysis. T. Hida, H. H. Kuo, J. Potthoff and L. Streit (Eds). pp 231-245. World Scientific, Singapore.

[39] Kwapień, S. and W. A. Woyczińiski (1992). *Random Series and Stochastic Integrals: Single and Multiple.* Birkauser, Boston.

[40] Léandre, R. and P. A. Meyer (1989). *Sur le developpement d'une diffusion en chaos de Wiener.* In: Sem. Probab. XXIII. Lecture Notes in Mathematics **1372**, pp 161-164, Springer-Verlag, New York.

[41] Ledoux, M. (1990). *A note on large deviations for Wiener chaos.* In: Sem. Probab. XXIV, Lecture Notes in Mathematics **1426**, pp 1-14, Springer-Verlag, New York.

[42] León, J. A. and V. Pérez-Abreu (1993). *Strong solutions of stochastic bilinear equations with anticipating drift in the first Wiener chaos.* In: Stochastic Processes, A Festschrift in Honour of Gopinath Kallianpur. S. Cambanis, J. K. Ghosh, R. L. Karandikar and P. K. Sen (Eds). pp 236-243. Springer-Verlag, New York.

[43] Marcus, M. (1994). *Continuity of some Gaussian chaoses.* In: Chaos Expansions, Multiple Wiener-Itô Integrals and Their Applications. C. Houdré and V. Pérez-Abreu (Eds). Series in Probability and Stochastics, CRC Press Inc., Florida. To appear.

[44] Maruyama, G. (1982). *Applications of the multiplication of the Itô-Wiener expansions to limit theorems.* Proc. Japan Acad. Sci. 58A, 388-390.

[45] Maruyama, G. (1985). *Wiener functionals and probability limit theorems.* Osaka Math. J. **22**, 697-732.

[46] Major, P. (1981). *Multiple Wiener-Itô Integrals.* Lecture Notes in Mathematics **849**, Springer-Verlag, New York.

[47] McKean, H. M. (1974). *Wiener's theory of nonlinear noise.* In: Stochastic Differential Equations, Proc. SIAM-AMS **6**, pp 191-289.

[48] Meyer, P. (1976). *Notions sur les intégrales multiples.* In: Sem. Probab. X, Lecture Notes in Mathematics **511**, pp 321-331. Springer-Verlag, New York.

[49] Meyer, P. (1987). *A propos des distributions sur l'espace de Wiener.* In: Sem. Probab. XXI, Lecture Notes in Mathematics **1247**, pp 8-26. Springer-Verlag, New York.

[50] Meyer, P. (1989). *Un cas de representation chaotique discrete.* In: Sem. Probab. XIII, Lecture Notes in Mathematics **1372**, pp 146. Springer-Verlag, New York.

[51] Meyer, P. (1993). *Quantum Probability for Probabilists.* Lecture Notes in Mathematics **1538**. Springer-Verlag, New York.

[52] Mori, T. and H. Oodaira (1986). *The law of the iterated logarithm for self-similar processes represented by multiple Wiener integrals.* Probab. Th. Rel. Fields **71**, 367-391.

[53] Mori, T. and H. Oodaira (1987). *The functional iterated logarithm law for stochastic processes represented by multiple Wiener integrals.* Probab. Th. Rel. Fields **72**, 299-310.

[54] Neveu, J. (1968). *Processus Aléatories Gaussiens.* Les Presses de l'Université de Montréal, Montréal.

[55] Nualart, D. (1989). *Une remarque sur le développement en chaos d'une diffusion.* In: Sem. Probab. XXIII. Lecture Notes in Mathematics **1372**, pp 165-168. Springer-Verlag, New York.

[56] Nualart, D., E. Mayer-Wolf and V. Pérez-Abreu (1992). *Large deviations for multiple Wiener-Itô integral processes.* In: Sem. Probab. XXVI, Lecture Notes in Mathematics **1526**, pp 11-31. Springer-Verlag, New York.

[57] Nualart, D. and E. Pardoux (1988). *Stochastic calculus with anticipating integrands.* Probab. Th. Rel. Fields. **78**, 535-581.

[58] Nualart, D., A. S. Üstünel and M. Zakai (1988). *On the moments of a multiple Wiener-Itô integral and the space induced by the polynomials of the integral.* Stochastics **25**, 233-340.

[59] Nualart, D. and J. Vives (1990). *Anticipative calculus for the Poisson process based on the Fock space.* In: Sem. Probab. XXIV, Lecture Notes in Mathematics **1426**, pp 154-165. Springer-Verlag, New York.

[60] Nualart, D. and J. Vives (1992a). *Chaos expansions and local times.* Publicacions Matematiques **36**, 827-836.

[61] Nualart, D. and J. Vives (1992b). *Smoothness of Brownian local times and related functionals.* Potential Anal. **1**, 257-263.

[62] Nualart, D. and J. Vives (1994). *Smoothness of local time and related Wiener functionals.* In: Chaos Expansions, Multiple Wiener-Ito Integrals and Their Applications. C. Houdré and V. Pérez-Abreu (Eds). Series in Probability and Stochastics. CRC Press Inc., Florida. To appear.

[63] Nualart, D. and M. Zakai (1988). *Generalized multiple stochastic integrals and the representation of Wiener functionals.* Stochastics **23**, 311-330.

[64] Nualart, D. and M. Zakai (1989). *Generalized Brownian functionals and the solution to a stochastic partial differential equation.* J. Func. Anal. **84**, 279-296.

[65] Ocone, D. (1983). *Multiple integral expansions for nonlinear filtering.* Stochastics **10**, 1-30.

[66] Ocone, D. (1988). *A guide to stochastic calculus of variations.* Lecture Notes in Mathematics **1316**, pp 1-79. Springer-Verlag, New York.

[67] Ogura, H. (1972). *Orthogonal functionals of the Poisson process.* Trans. IEEE Inf. Theory **IT-18**, 473-481.

[68] Pérez-Abreu, V. (1987). *Multiple Wiener integrals and nonlinear functionals of a nuclear space valued Wiener process.* Appl. Math. Optim. **16**, 227-245.

[69] Pérez-Abreu, V. (1991). *On the L^2-theory of product stochastic measures and multiple Wiener-Itô integrals.* Stoch. Anal. Appl. **9**, 53-70.

[70] Pérez-Abreu, V. (1992). *Anticipating solutions of stochastic bilinear equations in Hilbert spaces.* In: Contribuciones en Probabilidad y Estadística Matemática **3**. Eds: L. Gorostiza and J. León. Proceedings IV CLAPEM, pp 297-312. México.

[71] Pérez-Abreu, V. and C. Tudor (1994). *Large deviations for a class of chaos expansions.* J. of Theoretical Probab. To appear.

[72] Plikusas, A. (1981). *Some properties of the multiple Itô integral.* Lithuanian Math. J. **21**, 184-191.

[73] Rosen, J. (1986a). *Tanaka's formula and renormalization for intersections of planar Brownian motion.* Ann. Probab. **14**, 1245-1251.

[74] Rosen, J. (1986b). *A renormalized local time for multiple intersections of planar Brownian motion.* In: Sem. Probab. XX, Lecture Notes in Mathematics **1204**, pp 515-531. Springer-Verlag, New York.

[75] Segall, A. and T. Kailath. (1976). *Orthogonal functionals of independent-increments processes.* Trans. IEEE Inf. Theory **IT-22**, 287-298.

[76] Shieh, N. R. (1991). *White noise analysis and Tanaka formula for intersections of planar Brownian motion.* Nagoya Math. J. **122**, 1-17.

[77] Shigekawa, I. (1980). *Derivatives of Wiener functionals and absolute continuity of induced measures.* J. Math. Kyoto Univ. **20**, 263-289.

[78] Shiota, Y. (1986). *A linear stochastic integral equation containing the extended Itô integral.* Math. Rep. Toyama Univ. **9**, 43-65.

[79] Skorohod, A. V. (1975). *On a generalization of a stochastic integral.* Th. Probab. Appl. **20**, 219-233.

[80] Slud, E. (1991). *Multiple Wiener-Itô integral expansions for level-crossing-count functionals.* Probab. Th. Rel. Fields. **87**, 349-364.

[81] Slud, E. (1994). *MWI Expansions for functionals related to level-crossing counts.* In: Chaos Expansions, Multiple Wiener-Itô Integrals and Their Applications. C. Houdré and V. Pérez-Abreu (Eds). Series in Probability and Stochastics, CRC Press Inc., Florida. To appear.

[82] Stroock, D. (1985). *Homogeneous chaos revisited.* In: Sem. Probab. XXI, Lecture Notes in Mathematics **1247**, pp 1-7. Springer-Verlag, New York.

[83] Surgailis, D. (1984). *On multiple Poisson stochastic integrals and associated Markow semigroups.* Probab. and Math. Statist. **3**, 217-239.

[84] Szulga, J. (1994). *The state of decoupling.* In: Chaos Expansions, Multiple Wiener-Itô Integrals and Their Applications. C. Houdré and V. Pérez-Abreu (Eds). Series in Probability and Stochastics, CRC Press Inc., Florida. To appear.

[85] Tang, L.Q. (1992). *Results in Anticipative Stochastic Calculus.* Ph.D. Thesis, University of Alberta, Canada.

[86] Thangavelu, S. (1993). *Lectures on Hermite and Laguerre Expansions.* Mathematical Notes **42**, Princeton University Press. New Jersey.

[87] Üstünel, A. S. and M. Zakai (1992). *Transformation of Wiener measure under anticipative flows.* Probab. Th. Rel. Fields **93**, 91-136.

[88] Üstünel, A. S. and M. Zakai (1993). *The Wiener chaos expansion of certain Radon-Nikodym derivatives.* Proceedings of the 3rd Silvri Conference. To appear.

[89] Varadhan, S.R.S (1969). *Appendix to Euclidean Quantum Field Theory,* by K. Szymanzik. In Local Quantum Theory. Ed: R. Jost. Academic Press, New York.

[90] Varberg, D. E. (1966). *Convergence of quadratic forms in independent random variables.* Ann. Statist. **37**, 567-576.

[91] Watanabe, S. (1984). *Lectures on Stochastic Differential Equations and Malliavin Calculus.* Springer-Verlag, New York.

[92] Wiener, N. (1938). *The homogeneous chaos.* Amer. J. Math. **60**, 897-936.

[93] Yor, M. (1985). *Renormalisation et convergence en loi pour les temps locaux d'intersection du mouvement Brownien dans* R^3. In: Sem. Probab. XIX, Lecture Notes in Mathematics **1123**, pp. 350-365. Springer-Verlag, New York.

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