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Dissipative Mechanical Systems

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Abstract: The dissipative mechanical systems are second order vector fields on the tangent bundle of the configuration space, a compact Riemannian manifold ; they are obtained by the addition of a dissipative field of forces to a conservative one. The main results deal with generic properties and structural stability of these mechanical systems.

Key words: Strongly dissipative forces, Newton's law, transversality, generic properties, structural stability.

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Logical scheme in Section 3:

$$\left.\begin{array}{r} \left.\begin{array}{r}\text{Lemma 3.6}\\ \text{Lemma 3.7}\\ \text{Lemma 3.8}\end{array}\right\} \Rightarrow \text{Prop. 3.5} \Rightarrow \begin{array}{c}\text{Prop. 3.3}\\ \text{Prop. 3.4}\end{array} \\ \text{Lemma 3.11} \Rightarrow \text{Lemma 3.10} \Rightarrow \text{Lemma 3.9} \Rightarrow \text{Theor. 1.5} \end{array}\right\} \Rightarrow \text{Theor. 1.6}$$

0. Introduction.

The dissipative mechanical systems are second order vector fields on the tangent bundle TM of a given compact Riemannian manifold M (see [1], p.19) and are obtained by the addition of a dissipative field of forces to a conservative one. The dissipative forces are velocity dependent and slow down the system in such a way that the mechanical energy decreases along the non trivial integral curves, making the non-wandering set a collection of singular points. Shashahani in 1972 started a geometric study of the dissipative mechanical systems [13]; later on, in [3], 1986, dissipative systems with constraints were considered. The dissipative mechanical systems are parametrized by a pair (V, D) where V, the potential of the conservative forces, is a smooth real function defined on M and D is the dissipative force. Among the dissipative mechanical systems there are the strongly dissipative ones for which V is a Morse function and D is a strongly dissipative force i.e. satisfies a strongly dissipative condition (see Def. 1.3); they have very simple properties that we will describe below.

There are two well known results in the geometric theory of dynamical systems (see [9], [14]); the so called theorem of Kupka and Smale ([7], [11], [14]) and the theorems of Palis and Smale ([8], [10]) on the structural stability of the Morse–Smale systems (including gradient systems). We cannot apply directly the theorems of Kupka and Smale presented in [7], [11] and also the results in [12] for dissipative mechanical systems; the local perturbation arguments used to prove these theorems are not valid since the class of dissipative systems is too small. On the other hand, in spite of the fact that TM is not compact, we will see, in the last section, that many of the arguments used in [8] can be adapted to prove the structural stability of a certain class of complete strongly dissipative mechanical systems (see Theorem 1.7).

Later, Takens ([15], 1983) obtained other generic results on gradient systems with a fixed Riemannian metric and on mechanical (conservative) systems in the special case of zero curvature metric.

In many physical applications the ambient space where the evolution takes place and the geometry of the system cannot be changed. Hence it is meaningful to analyse properties of dissipative systems (V, D) where the friction forces D, corresponding to the action of the ambient space and the Riemannian structure, representing the geometry and distribution of masses, are fixed. One can also act on the system with small controlling forces or have situations with variable conservative forces; hence the potential V can be changed.

In the present paper the main results deal with generic properties and structural stability of dissipative mechanical systems. Theorem 1.4 proves that in the case of strongly dissipative mechanical systems the non-wandering set consists of hyperbolic singular points only and determines the structure of the invariant manifolds.

The C^r–Whitney topology is introduced in the set of all strongly dissipative mechanical systems with a fixed strongly dissipative force D (resp. with a fixed potential V); the Theorems 1.5 and 1.6 state that the collection of the strongly dissipative ones such that the invariant manifolds are in general position is an open dense subset. The Theorem 1.7 proves that the complete systems belonging to these open dense sets are structurally stable.

In proving transversality, it is easy to put the invariant manifolds in general position perturbing D and leaving fixed the potential V; as a matter of fact, this follows from arguments used in the Kupka-Smale result for first order systems (see [7], [9], [11]) together with the same result for general second order vector fields (see [12], p.267). On the other hand, it is much harder to prove the generic transversality of stable and unstable manifolds of the dissipative systems (V, D) if we keep D fixed and allow only V to vary. This is due to the fact that no perturbation of V is local on the tangent structure TM of M since if we change V in some arbitrarily small open set w of M, it will still affect the evolution of the system on the whole tangent space Tw of w. For more details see the proof of Proposition 3.5.

1 - Statements of the Results

Throughout the paper $(M, <,>)$ will be a C^∞ compact connected Riemannian manifold, without boundary. We call M the **configuration space**. The C^∞ metric $<,>$ defines the **kinetic energy** $K : TM \to I\!R$ by $K(v_p) = \frac{1}{2} < v_p, v_p >$, $v_p \in T_pM$. The associated Levi-Civita[E0 covariant derivative will be denoted by ∇. The motivation to introduce the Levi-Civita connection is to enable us to express conveniently the Newton's law which governs the evolution of our systems. A **potential** V is a C^{r+1} function, $r \geq 1$, $V : M \to I\!R$ and the **mechanical energy** is $E_V : TM \to I\!R$ defined by $E_V(v) = K(v) + V(\pi_M(v))$, (TM, π_M, M) being the tangent bundle of M. Let O_M denote the zero section, that is, the set of all zero vectors of this vector bundle and $TM \backslash O_M = (TM)_o$ be the set of all non zero vectors. A C^r second order vector field on TM is a vector field X on TM such that $(d\pi_M) \circ X$ is the identity mapping of TM where $d\pi_M : TTM \to TM$ is the tangent mapping of π_M.

Definition 1.1. *A dissipative force* D *is a* C^r *map* $D : TM \to TM$ *which* *preserves each fiber and such that* $< D(v), v > < 0$ *for all* $v \in (TM)_o$.

We easily see that if D is a dissipative force, then for all $0 \in O_M$ one has $D(0) = 0$.

Definition 1.2. *A dissipative mechanical system on the configuration space* M *is a pair* (V, D) *of a* C^{r+1} *potential* V *and a* C^r *dissipative force* D, $r \geq 1$. *The pair* (V, D) *defines a second order* C^r *vector field on* TM *(sometimes also denoted by* (V, D)*). If* z *is a trajectory of* (V, D) *and* q *its projection on* M, *then* $z = \frac{dq}{dt} = \dot{q}$ *and* q *satisfies the equation*

$$\nabla_{\dot{q}}\dot{q} = -(\text{grad } V)(q) + D(\dot{q}).$$

The curve $t \mapsto q(t) \in M$ *verifying that law is called a motion and -grad* V *is called the conservative field of forces.*

The equation above is just the statement of the Newton's law on the manifold M. Recall that grad V is the vector field on M characterized by:

$$dV(v) =< (\text{grad } V)(p), v > \quad \text{for all } p \in M \text{ and all } v \in T_pM.$$

Let us denote by DMS the set of all vector fields $X \in C^r(TM, TTM)$ such that X is defined by a dissipative mechanical system (V, D) as in Definition 1.2.

It is useful to remark that the mechanical energy decreases along non trivial integral curves of any mechanical system (V, D). In fact, we have:

$$\frac{d}{dt}E_V(\dot{q}(t)) = \frac{d}{dt}[\frac{1}{2} < \dot{q}, \dot{q} > + V(q(t))] =< D(\dot{q}), \dot{q} >$$

which shows that E_V decreases on all integral curves not reduced to a singular point. Note also that the integral curves of the system are the derivatives of the motions of the system and its singular points lie on the zero section O_M. Moreover $O_p \in (T_pM) \cap O_M$ is a singular point if, and only if, $(\text{grad } V)(p) = 0$, that is, $p \in M$ is a critical point of V.

We recall that a function $V \in C^{r+1}(M, \mathbb{R})$ is said to be a **Morse function** if the Hessian of V at each critical point is a non-degenerate quadratic form. It is well known that the set of all Morse functions is an open dense subset of $C^{r+1}(M, \mathbb{R})$ with the standard C^{r+1} topology.

Definition 1.3. *A dissipative mechanical system* (V, D) *is said to be strongly dissipative if* V *is a Morse function and* D *is a strongly dissipative force i.e. satisfies the following additional condition: for all* $p \in M$ *and all* $w \in$

$(TM)_o \cap T_pM$ one has $< d_vD(O_p)w, w > < 0$ where d_vD denotes the vertical differential of D.

Note that we assume V to be a Morse function for technical reasons only. From now on we denote by $SDMS$ the set of all $X \in DMS$ such that $X = (V, D)$ is strongly dissipative and by $\mathcal{D}$ the set of all strongly dissipative forces.

Theorem 1.4. *Let (V, D) be a strongly dissipative mechanical system. Then the following properties hold:*

(i) The singular points of (V, D) are hyperbolic.

(ii) The stable and unstable manifolds $W^s(O)$ and $W^u(O)$ of a singular point O are properly embedded.

(iii) $\dim W^u(O)$ *is the Morse index of V at $\pi_M(O)$.*

(iv) $\dim W^u(O) \leq \dim M \leq \dim W^s(O)$.

Two submanifolds S_1 and S_2 of a manifold $\mathcal{F}$ are said to be in **general position** or **transversal** if either $S_1 \cap S_2$ is empty or at each point $x \in S_1 \cap S_2$ the tangent spaces T_xS_1 and T_xS_2 span the tangent space $T_x\mathcal{F}$.

Let us denote by $SDMS(D)$ the set of all C^r strongly dissipative mechanical systems $X = (V, D)$ with a fixed D. Analogously we introduce the set $SDMS(V)$.

All the subsets of DMS are endowed with the topology induced by the C^r–**Whitney topology** of $C^r(TM, TTM)$. This topology possesses the Baire property (see [11], p.224, for a definition of the Whitney topology and the proof of this fact).

Theorem 1.5. *The set of all systems X in $SDMS$ such that their stable and unstable manifolds are pairwise transversal is open in $SDMS$.*

Theorem 1.6. *Assume* $\dim M > 1$ *and* $r > 3(1 + \dim M)$ *and let $\mathcal{G}$ be the subset of $SDMS(D)$ (resp. $SDMS(V)$) of all systems X such that their invariant manifolds are pairwise transversal. Then $\mathcal{G}$ is open dense in $SDMS(D)$ (resp. $SDMS(V)$).*

As usual we say that $X \in SDMS$ is **structurally stable** if there exists a neighbourhood W of X (in the Whitney C^r–topology) and a continuous map h from W into the set of all homeomorphisms of TM (with the compact open topology), such that:

1) $h(X)$ is the identity map;

2) $h(Y)$ takes orbits of X into orbits of Y, for all $Y \in W$, that is, $h(Y)$ is a **topological equivalence** between X and Y.

If the topological equivalence $h(Y)$ preserves the time, that is, if X_t (resp. Y_t) is the flow map of X (resp. Y) and $h(Y) \circ X_t = Y_t \circ h(Y)$ for all $t \in \mathbb{R}$, then we say that $h(Y)$ is a **conjugacy** between X and Y.

As we will see in Proposition 4.3 the subset of all **complete** C^r vector fields of a manifold $\mathcal{F}$ is open in the set of all C^r vector fields with the Whitney C^r-topology.

Theorem 1.7. *Any complete strongly dissipative mechanical system such that all the stable and unstable manifolds of singular points are in general position is structurally stable and the topological equivalence is a conjugacy.*

The Theorems 1.6 and 1.7 have also the flavour of an interesting theorem proved by D. Henry ([5]) for a dynamical system in infinite dimensions. On the Sobolev space $H_0^1 = H_0^1((0, \pi), \mathbb{R})$ he considered the following parabolic PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda f(u)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f(0) = 0$, $f'(0) = 1$, $sf''(s) < 0$ if $s \neq 0$, and λ is a real positive parameter.

Theorem (D.Henry). *If $\sqrt{\lambda}$ is not a positive integer, then all stable and unstable manifolds of the flow defined on H_0^1 by the PDE above are in general position.*

If in Theorem 1.7 we do not assume the mechanical system to be complete, the same arguments used in the proof also show that the corresponding time one map is a Morse–Smale map in the sense of [4], then stable with respect to the attractor $\mathcal{A}(V, D)$, which in this case is the union of the unstable manifolds of all singular points of (V, D).

Let us consider an example of a strongly dissipative mechanical system which does not satisfy the conclusions of Theorem 1.6 in the sense that it does not belong to $\mathcal{G}$; it is the system which describes the motions of a particle (unit mass) constrained on the surface M of a symmetric vertical solid torus of $\mathbb{R}^3$ obtained by the rotation, around the x-axis, of a circle defined by the equations $y = 0$ and $x^2 + (z-3)^2 = 1$. The potential V is proportional to the height function of M and the dissipative force D is given by $D(v) = -cv$, $c > 0$, for all $v \in TM$. These data define a strongly dissipative mechanical system with M as the configuration space. The metric of M is the one induced by the usual inner product of $\mathbb{R}^3$ and the potential is a well known Morse function with four critical points. The symmetry of the problem shows that the unstable manifold of dimension one of a saddle is contained in the stable manifold of dimension 3 of the other saddle and hence they are not in general position since dim $TM = 4$.

A dissipative force D is said to be **complete** if, for any Morse function V, the vector field associated to (V, D) is complete, that is, all of its integral curves are defined for all time.

Let us consider a linear dissipative field of forces, that is, a function D defined by

$$D(v) = -c(\pi_M(v))v, \quad \text{for all} \quad v \in TM,$$

where $c : M \to \mathbb{R}$ is a strictly positive C^r function. It is a simple matter to show that D is a strongly dissipative force. We will show that D is complete. If this were not the case, there would exist a smooth function $V : M \to \mathbb{R}$ and a motion $t \to q(t)$ of (V, D) whose maximal interval of existence is $]\alpha, +\infty[$ with $-\infty < \alpha < 0$. We know that $\frac{d}{dt}E_V(\dot{q}(t)) = < D(\dot{q}), \dot{q} >$ is negative and also that

$$0 < | < D(\dot{q}), \dot{q} > | \leq \mu|\dot{q}|^2 \leq 2\mu(E_V(\dot{q}) + k)$$

where $\mu > 0$ is the maximum of the function c on M and $k = |\nu|$, ν being the minimum of V on M (recall that M is compact). For all t, $\alpha < t < 0$, we may write

$$-2\mu(E_V(\dot{q}) + k) \leq \frac{d}{dt}E_V(\dot{q}) = \frac{d}{dt}(E_V(\dot{q}) + k) < 0$$

or

$$\frac{d(E_V(\dot{q}) + k)}{E_V(\dot{q}) + k} \geq -2\mu dt \quad \text{which implies} \quad E_V(\dot{q}) + k \leq (E_V(\dot{q}(0)) + k)e^{-2\mu t}$$

and then $E_V(\dot{q}(t))$ is bounded and strictly decreasing, so that there exists $\lim_{t\to\alpha_-} E_V(\dot{q}(t)) = L < +\infty$.

This shows that $|\dot{q}|^2 = 2(E_V(\dot{q}) - V(q(t)))$ is also bounded because V is bounded; now it is immediate that we have a contradiction.

2 - Proof of Theorem 1.4.

Let p be a point of M and U an open neighbourhood of p in M such that there exists a trivialization of TM over U, i.e., $\phi : \pi_M^{-1}(U) \to U \times \mathbb{R}^m$, m being the dimension of M. Let x and v be the projections onto U and $\mathbb{R}^m$. The vector field associated to (V, D) has the following expression on $U \times \mathbb{R}^m$:

$$\begin{cases} \dfrac{dx}{dt} = v \\[2mm] \dfrac{dv}{dt} = -(\text{grad } V)(x) + D(x, v) - \Gamma(x, v)v \end{cases}$$

where $\Gamma : U \times I\!R^m \to \mathrm{End}(I\!R^m)$ is the difference between the Levi-Civita connection and the trivial connection defined by ϕ. Then, it is clear that the singular points of (V, D) are the vectors $O_p \in O_M \cap T_pM$ such that $(\mathrm{grad}\, V)(p) = 0$. In such a singular point, the linear part of the system is $L : T_pM \times I\!R^m \to T_pM \times I\!R^m$ given by

$$L = \begin{bmatrix} 0 & I \\ -H & \Delta \end{bmatrix}$$

where $I : I\!R^m \to T_pM$ is the canonical isomorphism defined by the trivialization, H is the Hessian of V at p and Δ is the vertical differential $d_vD(O_p)$ of D at O_p. The first statement of Theorem 1.4 follows from the next lemma:

Lemma 2.1. *Let* $\overline{L} : I\!R^m \times I\!R^m \to I\!R^m \times I\!R^m$ *be a linear map given by*

$$\overline{L} = \begin{bmatrix} 0 & Id \\ -\overline{H} & \overline{\Delta} \end{bmatrix}$$

with $\overline{H}$ *symmetric,* $\det \overline{H} \neq 0$, *and* $\overline{\Delta}$ *negative definite:* $(\overline{\Delta}v, v) < 0$ *for all* $v \in I\!R^m$, *((,) is the usual inner product of* $I\!R^m$*). Then the eigenvalues of* $\overline{L}$ *have non zero real parts.*

Proof. If $i\beta \neq 0$ (the case $\beta = 0$ is excluded otherwise $\overline{H}$ would have a zero eigenvalue) is eigenvalue of $\overline{L}$, there exist $u \in \mathbf{C}^m$, $u = y + iw \neq 0$, $y, w \in I\!R^m$, such that $(i\beta)^2u - (i\beta)\overline{\Delta}u + \overline{H}u = 0$, or equivalently

$$\begin{cases} -\beta^2 y + \beta\overline{\Delta}w + \overline{H}y = 0 \\ -\beta^2 w - \beta\overline{\Delta}y + \overline{H}w = 0. \end{cases}$$

The symmetry of $\overline{H}$ implies $\beta[(\overline{\Delta}y, y) + (\overline{\Delta}w, w)] = 0$, which is a contradiction. This proves (i).

The second statement of Theorem 1.4 follows from the fact that the energy E_V decreases strictly along non trivial integral curves (see, for instance, [6] Th. 6.1.10). For the last two statements one considers a path of matrices:

$$\mu \begin{bmatrix} 0 & Id \\ -\overline{H} & -Id \end{bmatrix} + (1-\mu) \begin{bmatrix} 0 & Id \\ -\overline{H} & \overline{\Delta} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & Id \\ -\overline{H} & -\mu I_d + (1-\mu)\overline{\Delta} \end{bmatrix}.$$

Since $-\mu I_d + (1-\mu)\overline{\Delta}$ is negative definite for all μ, $0 \leq \mu \leq 1$, the continuity of the spectrum enables us to consider the case

$$N = \begin{bmatrix} 0 & Id \\ -\overline{H} & -Id \end{bmatrix}.$$

The eigenvalues λ of N are given by

$$\det \begin{bmatrix} -\lambda Id & Id \\ -\overline{H} & -(1+\lambda)Id \end{bmatrix} =$$

$$\det \begin{bmatrix} 0 & Id \\ -\overline{H} - \lambda(1+\lambda)Id & -(1+\lambda)Id \end{bmatrix} = 0$$

or, equivalently, by $\det[-\overline{H} - \mu Id] = 0$ where $\mu = \lambda(1+\lambda)$.

But, in the very beginning, we may assume that the trivialization is chosen in such a way that $-\overline{H}$ is a diagonal matrix. Then, for each positive eigenvalue μ of $-\overline{H}$ (the total number is the Morse index of V) corresponds a positive root of N. Thus (iii) is proved. The proof of (iv) is now evident. ■

3 - Proofs of Theorems 1.5 and 1.6

Although we do not need the next proposition for the proofs of Theorems 1.5 and 1.6 we present it for a sake of completeness.

Proposition 3.1. *$SDMS$ is an open dense subset of DMS.*

Proof. Since the set of Morse functions if open and dense in $C^{r+1}(M, \mathbb{R})$ and

$$< d_v D(0_p)w, w > < 0 \quad \text{on} \quad A = \{w \in TM \mid |w| \leq 1\}$$

is an open condition one sees that the openess of $SDMS$ is trivial. We only have to prove the density. Given any neighbourhood of a vector field of DMS, parametrized by (V, D), we construct a strongly dissipative force $\overline{D}$, which is equal to $D - \delta I$ on the compact set A and equal to D outside of a neighbourhood of A, choosing a C^∞ bump function and a small $\delta > 0$, properly. This and the density of the set of Morse functions give the proof. ■

In the case of a fixed dissipative force we cannot prove the density statement in Proposition 3.4 below for an arbitrary system because perturbing the potential is not a local process on TM. Hence we have to restrict ourselves to systems

for which the projections on M of two distinct trajectories have few intersection points. In fact it would be enough to consider the systems such that the projections of the trajectories have few self intersections. More precisely, let $X = (V, D)$ be an element of $SDMS$. By a trajectory of X we understand a maximal solution. Given two trajectories $y :]a_-, +\infty[\to TM$,
$z :]b_-, +\infty[\to TM$ of X, we denote by $C(y,z)$ the set of all pairs $(t_1, t_2) \in I\!R^2$ such that $a_- < t_1$, $b_- < t_2$, $y(t_1) \neq z(t_2)$ and $\pi_M(y(t_1)) = \pi_M(z(t_2))$. Let $p = \pi_M \circ y$, $q = \pi_M \circ z$. The projection of the set $C(y,z)$, that is the set $\{p(t_1) \mid (t_1,t_2) \in C(y,z)\}$, is the intersection set of the projections p and q of y and z. The next proposition clarifies the structure of $C(y,z)$.

Definition 3.2. *Let $X \in SDMS$. Then:*

(i) *We say that X has the property GI if, for any two non singular trajectories, $y :]a_-, +\infty[\to M$, $z :]b_-, +\infty[\to M$ of X, the set $C(y,z)$ is discrete in the quadrant $]a_-, +\infty[\times]b_-, +\infty[$ of $I\!R^2$.*

(ii) *We say that X has the property GIW (weak GI) if, at any accumulation point (t_1,t_2) of $C(y,z)$, at least one of the points $y(t_1), z(t_2)$ lies on the zero section O_M of TM.*

Proposition 3.3.

(i) *If the dimension m of M is greater than 2 and $r > 4m+5$, for any strongly dissipative force $D \in C^r(TM, TM)$, there exists a Baire subset $GI(D)$ of $SDMS(D)$ all of whose elements X have the property GI.*

(ii) *If the dimension m of M is greater than 1 and $r > 3m+3$, we have a similar statement replacing GI by GIW and $GI(D)$ by $GIW(D)$.*

Proof. For simplicity we shall assume $r = \infty$ in the proof. But the proof is still valid if we replace everywhere ∞–jet by r–jet and "is flat" by "has zero r–jet".

Let (t_1,t_2) be an accumulation point of $C(y,z)$ in $]a_-, +\infty[\times$
$]b_-, +\infty[$. Then $p(t_1) = q(t_2)$. We have to distinguish several cases. First assume that $y(t_1) \neq z(t_2)$. Then one of the vectors $y(t_1), z(t_2)$ is not zero. Permuting the roles of y and z if necessary, we can assume that $y(t_1) \neq 0$. We claim there exist an open interval δ containing t_2 and a smooth mapping $\sigma : \delta \to I\!R$ such that the ∞–jets of q and $p \circ \sigma$ at t_2 are equal. To see this, choose a coordinate system $x^1, x^2, \ldots, x^m : O \to I\!R$ (m=dim M) in an open neighbourhood O of $p(t_1)$ such that $(x^1 \circ p)(t) = t$ and $x^k \circ p = 0$ if $2 \leq k \leq m$, for all t in an open interval δ_1 containing t_1. There exists a sequence $\{(t_1(n), t_2(n)) \mid n \geq 1\}$ in $C(y,z)$ converging to (t_1,t_2). For all k, $2 \leq k \leq m$, $x^k \circ q(t_2(n)) = x^k \circ p(t_1(n)) = 0$, for all $n \geq 1$. Hence all the functions $x^k \circ q$, $2 \leq k \leq m$, are flat at t_2. σ will denote the restriction of $x^1 \circ q$ to δ_1 and ρ the composition $p \circ \sigma$. Then for any $n \geq 1$, $\sigma(t_2(n)) = x^1 \circ q(t_2(n)) = x^1 \circ p(t_1(n)) = t_1(n)$ and $\rho(t_2(n)) = p(\sigma(t_2(n))) = p(t_1(n)) = q(t_2(n))$. So ρ and q have the same ∞–jet

at t_2.

As one easily sees, $z(t_2) = y(t_1)\dot{\sigma}(t_2), \dot{\sigma}(t_2) = \frac{d\sigma}{dt}(t_2)$. We have already assumed that $\dot{\sigma}(t_2)$ cannot be equal to 1. Now we shall distinguish three cases:

1) $\dfrac{d\sigma}{dt}(t_2) \neq 0$ or -1.

2) $\dfrac{d\sigma}{dt}(t_2) = -1$.

3) $\dfrac{d\sigma}{dt}(t_2) = 0$.

q and p satisfy the following relations:

$$\nabla_{\dot{p}}\dot{p} - D(\dot{p}) + \text{grad } V(p) = 0, \qquad E1$$

$$\nabla_{\dot{q}}\dot{q} - D(\dot{q}) + \text{grad } V(q) = 0. \qquad E2$$

Since ρ and q have the same ∞–jet at t_2, ρ satisfies the relation

$$\nabla_{\dot{\rho}}\dot{\rho} - D(\dot{\rho}) + \text{grad } V(\rho) = \lambda \qquad E3$$

where λ is flat at t_2.

Explicitating $E3$, after setting $\ddot{\sigma} = \frac{d^2\sigma}{dt^2}$, we get:

$$\dot{\sigma}^2(\nabla_{\dot{p}}\dot{p})(\sigma) + \ddot{\sigma}\dot{p}(\sigma) - D(\dot{\sigma}\dot{p}(\sigma)) + \text{grad } V(p(\sigma)) = \lambda. \qquad E4$$

In the first and second cases above, σ is a local diffeomorphism at t_2, that is σ maps some open interval t_2, diffeomorphically on the open interval $\sigma(\delta_2)$ containing t_1. Set $\chi = \frac{d\sigma}{dt} \circ \sigma^{-1} : \sigma(\delta_2) \to I\!R$. Then $E4$ is equivalent to

$$\chi^2(\nabla_{\dot{p}}\dot{p}) + \chi\dot{\chi}\dot{p} - D(\chi\dot{p}) + \text{grad } V(p) = \mu \qquad E5$$

where $\mu = \lambda \circ \sigma^{-1}$ is flat at t_1.

Subtracting $E1$ from $E5$ we get:

$$(\chi^2 - 1)\nabla_{\dot{p}}\dot{p} + \chi\dot{\chi}\dot{p} + D(\dot{p}) - D(\chi\dot{p}) = \mu. \qquad E6$$

$E6$ is equivalent to an infinite sequence of conditions on the ∞–jet of p, obtained by equating the sucessive covariant derivates at t_1 on both sides of $E6$. For this we need some notations. $J^k(M, I\!R)$ will denote the space of k–jets of mappings from M into $I\!R$, and $J^k(I\!R, 0; I\!R)$ will denote the space of all k–jets at 0 of mappings $I\!R \to I\!R$. Taking the n^{th} covariant derivative of $E6$ along the curve p, we get for $n \geq 0$:

$$\begin{aligned}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} &\left[\left(\frac{d^k}{dt^k}(\chi^2 - 1)\right)\nabla_{\dot{p}}^{n-k+1}\dot{p} + \left(\frac{d^k}{dt^k}(\chi\dot{\chi})\right)\nabla_{\dot{p}}^{n-k}\dot{p}\right] \\ &+ [d_v D(\dot{p}) - \chi d_v D(\chi\dot{p})]\nabla_{\dot{p}}^{n}\dot{p} \\ &+ Q_n(\dot{p}, \nabla_{\dot{p}}\dot{p}, \ldots, \nabla_{\dot{p}}^{n-1}\dot{p}, j_0^n\chi_{t_1}) = \nabla_{\dot{p}}^{n}\mu\end{aligned} \qquad E7n$$

where Q_n is a fiber-bundle mapping:

$$\underbrace{TM \times_M \ \dots \ \times_M TM}_{n \text{ times}} \times J^n(\mathbb{R}, 0; \mathbb{R}) \ \rightarrow \ TM$$

and χ_{t_1} is the translate $\chi_{t_1}(t) = \chi(t + t_1)$, where $TM \times_M \ \dots \times_M TM$ means a fiber product bundle. Deriving $E1$ covariantly n times along p we get:

$$\begin{aligned}\nabla_{\dot p}^n \dot p = &-\nabla^{n-1}\text{grad } V(\dot p, \dots, \dot p) \\ &+ R_n(\dot p, \text{grad } V(p), \nabla \text{grad } V(\dot p), \dots, \nabla^{n-2}\text{grad } V(\dot p, \dots, \dot p))\end{aligned} \qquad E8n$$

where R_n is a fiber bundle map: $\underbrace{TM \times_M \ \dots \ \times_M TM}_{n \text{ times}} \ \rightarrow \ TM$ depending also on D and its derivates and ∇^n grad $V : TM \times_M \times \dots \times_M TM \ \rightarrow \ TM$ is the nth covariant differential of grad $V : M \rightarrow TM$. $E8n$ and $E7n$, $n \geq 1$, give us the following:

$$\begin{aligned}&(\chi^2 - 1)\nabla^n \text{grad } V(\dot p, \dots, \dot p) \ + \ S_n(\dot p, \text{grad } V(p), \\ &\dots, \nabla^{n-1}\text{grad } V(\dot p, \dots, \dot p), j_0^{n+1}\chi_{t_1}) \ = \ \nabla_{\dot p}^n \mu, \quad \text{for } n \geq 0,\end{aligned} \qquad E9n$$

where S_n is a fiber bundle mapping $S_n : TM \times_M \dots \times_M TM \times J^{n+1}(\mathbb{R}, 0; \mathbb{R}) \rightarrow TM$.

Assume now that we are in the first case, that is, $\chi(t_1) = \frac{d\sigma}{dt}(t_1) \neq -1$. Evaluating $E9n$ at t_1, since $\chi(t_1)^2 \neq 1$ we have for $n \geq 0$:

$$\begin{aligned}&\nabla^n \text{grad } V(\dot p(t_1), \dots, \dot p(t_1)) + \\ &+ \frac{1}{\chi(t_1)^2 - 1} S_n(\dot p(t_1), \text{grad } V(p(t_1)), \dots, \\ &\nabla^{n-1}\text{grad } V(\dot p(t_1), \dots, \dot p(t_1)), j_{t_1}^{n+1}\chi) = 0.\end{aligned} \qquad E10n$$

Denote by J_1^{n+1} the topological subspace of $J^{n+1}(\mathbb{R}, 0; \mathbb{R})$ of all jets $j_0^{n+1}\omega$ such that $\omega(0)^2 \neq 1$. Define the subset Σ_n of $J^n(M, \mathbb{R}) \times (TM)_o \times J_1^{n+1}$ as follows:

$$\begin{aligned}\Sigma_n \ = \ \Big\{&(j_x^n W, u, j_0^{n+1}\omega) \mid u \in (T_x M)_0, j_0^{n+1}\omega \in J_1^{n+1}, \\ &\nabla^k \text{grad } W(u, \dots, u) + \\ &\frac{1}{\omega(0)^2 - 1} S_k(u, \text{grad } W(x), \dots, \nabla^{k-1}\text{grad } W(u, \dots, u), \\ &j_0^{k+1}\omega) = 0, \ \ 0 \leq k \leq n\Big\}.\end{aligned}$$

We can summarize our discussion up to now as follows: if (t_1, t_2) is an accumulation point of $C(y, z)$ at which $y(t_1) \neq 0$, $z(t_2) \neq 0$ and $y(t_1) + z(t_2) \neq 0$, then there exists a $j_0^{n+1}\omega$ in J_1^{n+1} such that the triple $(j^n_{p(t_1)}V, y(t_1), j_0^{n+1}\omega)$ belongs to $\sum_n$.

Assume now that we are in the second case. We claim that $\dot{\chi}(t_1) = \frac{d\chi}{dt}(t_1)$ is not zero. $E6$ evaluated at t_1 gives

$$-\dot{\chi}(t_1)\dot{p}(t_1) + D(\dot{p}(t_1)) - D(-\dot{p}(t_1)) = 0. \qquad E11$$

Multiplying scalarly by $\dot{p}(t_1)$ one has

$$-\dot{\chi}(t_1)\|\dot{p}(t_1)\|^2 + < D(\dot{p}(t_1)), \dot{p}(t_1) > + < D(-\dot{p}(t_1)), -\dot{p}(t_1) >= 0.$$

Since the second and third terms are negative, $\dot{\chi}(t_1)$ cannot be zero.

Evaluating $E7n$ at $t = t_1$ we get for $n \geq 1$

$$\begin{aligned}
&[-(2n+1)\dot{\chi}(t_1) + d_v D(\dot{p}(t_1)) + d_v D(-\dot{p}(t_1))](\nabla^n_{\dot{p}}\dot{p})(t_1) + \\
&+ \sum_{k=2}^{n} \frac{n!}{k!(n-k)!} \frac{d^k}{dt^k}(\chi^2 - 1)(\nabla_{\dot{p}}^{n-k+1}\dot{p})(t_1) \qquad E12n \\
&+ Q_n(\dot{p}(t_1), \ldots, \nabla_{\dot{p}}^{n-1}\dot{p}(t_1), j_0^{n+1}\chi_{t_1}) = 0.
\end{aligned}$$

Using $E8n$ we get for $n \geq 1$:

$$\begin{aligned}
&[+(2n+1)\dot{\chi}(t_1) - d_v D(\dot{p}(t_1)) - d_v D(-\dot{p}(t_1))] \times \\
&\qquad \nabla^{n-1}\text{grad } V(\dot{p}(t_1), \ldots, \dot{p}(t_1)) + \\
&\Phi_n(\text{grad } V(p(t_1)), \ldots, \nabla^{n-2}\text{grad } V(\dot{p}(t_1), \ldots, \dot{p}(t_1), j_0^{n+1}\chi_{t_1}) = 0. \qquad E13n
\end{aligned}$$

Define the subset $\sum_n(-1)(n \geq 1)$ of $J^{n-1}(M, I\!R) \times (TM)_o \times J_{11}^{n+1}$, where J_{11}^{n+1} is the subset of $J^{n+1}(I\!R, 0; I\!R)$ of all $j_0^{n+1}\omega$ such that $\omega(0) = -1$ and $\omega(0) \neq 0$, as follows: $\sum_n(-1)$ is the set of all triples $(J_x^{n-1}W, u, j_0^{n+1}\omega)$ in $J^{n-1}(M, I\!R) \times (TM)_0 \times J_{11}^{n+1}$ such that for all k, $1 \leq k \leq n$, $u \in (T_xM)_0$, one has:

$$\begin{aligned}
&[(2k+1)\omega(0) - d_v D(u) - d_v D(-u)]\nabla^{k-1}\text{grad } W(u, \ldots, u) + \\
&\quad + \Phi_n(\text{grad } W(x), \ldots, \nabla^{k-2}\text{grad } W(u, \ldots, u), j_0^{n+1}\omega) = 0.
\end{aligned}$$

Then as before (t_1, t_2) will be an accumulation point at which $y(t_1) \neq 0$, $z(t_2) \neq 0$ and $y(t_1) + z(t_2) = 0$ if and only if there exists a $j_0^{n+1}\omega \in J_{11}^{n+1}$ such that the triple $(j^{n-1}_{p(t_1)}V, \dot{p}(t_1), j_0^{n+1}\omega)$ belongs to $\sum_n(-1)$.

The case 3) happens when $\dot{z}(t_2) = 0$. By taking time translates of y and z we can assume that $t_1 = t_2 = 0$. This case is more involved than the preceding ones. For a start, we claim that $\frac{d^2\sigma}{dt^2}(0) \neq 0$. In fact, evaluating $E4$ at $t = 0$, we have $\ddot{\sigma}(0)\dot{p}(0) + \text{grad } V(p(0)) = 0$. Since grad $V(p(0))$ is not zero, $\ddot{\sigma}(0) \neq 0$. From this it follows that there exists a local diffeomorphism ψ at 0 such that $\sigma = \frac{\varepsilon\psi^2}{2}$ and ε is $+1$ if $\ddot{\sigma}(0) > 0$ and -1 if $\ddot{\sigma}(0) < 0$.

Setting $\eta = \frac{d\psi}{dt} \circ \psi^{-1}$, we see that $E4$ is equivalent to:

$$\begin{aligned} &\tau^2\eta(\tau^2)(\nabla_{\dot{p}}\dot{p})(\frac{\varepsilon\tau^2}{2}) + \varepsilon(\eta(\tau)^2 + \tau\eta(\tau)\dot{\eta}(\tau))\dot{p}(\frac{\varepsilon\tau^2}{2}) \\ &\quad - D(\varepsilon\tau\eta(\tau)\dot{p}(\frac{\varepsilon\tau^2}{2})) + \text{grad } V(p(\frac{\varepsilon\tau^2}{2})) = \nu(t) \end{aligned} \qquad E14$$

where $\nu = \lambda \circ \psi^{-1}$.

We shall proceed as in the 1st. and 2nd. cases and replace $E14$ by more manageable conditions on the jets of V and η. To do this, we need the following estimate which can be obtained easily by induction on n. Let ξ be any smooth vector field along the curve p. Then

$$\nabla_{\dot{p}}^n\xi\left(\frac{\varepsilon\tau^2}{2}\right) = \sum_{i=0}^{n_1} \varepsilon^{n-i} a_{n,i}\tau^{n-2i}\left(\nabla_{\dot{p}}^{n-i}\xi\right)\left(\frac{\varepsilon\tau^2}{2}\right) \qquad E15$$

where the coefficients $a_{n,i}$ are positive integers such that $a_{n+1,i} = a_{n,i} + (n - 2i + 2)a_{n,i-1}$ and $n_1 = \frac{n}{2}$ or $\frac{n-1}{2}$ according to n being even or odd. Setting $t = \tau^2$ in $E1$ we get from $E1$ and $E14$

$$(\tau^2\eta^2 - 1)(\nabla_{\dot{p}}\dot{p})(\frac{\varepsilon\tau^2}{2}) + \varepsilon(\eta^2 + \tau\eta\dot{\eta})\dot{p}(\frac{\varepsilon\tau^2}{2}) + D(\dot{p}(\frac{\varepsilon\tau^2}{2})) - D(\varepsilon\tau\eta\dot{p}(\frac{\varepsilon\tau^2}{2})) = \nu. \quad E16$$

Deriving $E16$ covariantly $2n$ times with respect to τ and evaluating at $\tau = 0$ we get the relations

$$-a_{2n,n}(\nabla_{\dot{p}}^{n+1}\dot{p})(0) + K_n(\dot{p}(0), \ldots, (\nabla_{\dot{p}}^n\dot{p})(0); j_0^{2n}\eta) = 0 \qquad E17n$$

where K_n is a fiber bundle mapping:

$$\underbrace{TM \times_M \ \ldots \ \times_M TM}_{n \text{ times}} \times J^{2n}(\mathbb{R}, 0; \mathbb{R}) \to TM.$$

Using $E8n$, the relations $E17n$ imply

$$\begin{aligned} 0 &= a_{2n,n}\nabla^n \text{grad } V(\dot{p}(0), \ldots, \dot{p}(0)) \\ &\quad + L_n(\dot{p}(0), \text{grad } V(p(0)), \ldots, \nabla^{n-1}\text{grad } V(\dot{p}(0), \ldots, \dot{p}(0)), j_0^{2n}\eta). \end{aligned} \qquad E18n$$

Let us denote by $\sum_n(0)$, the subset of the jet space $J^n(M, \mathbb{R}) \times (TM)_0 \times J_0^{2n}, J_0^{2n}$ being the set of all jets $j_0^{2n}\omega \in J^{2n}(\mathbb{R}, 0; \mathbb{R})$ such that $\omega(0) \neq 0$, defined as follow: $\sum_n(0)$ is the set of all triples $(j^n_x W, u, j^{2n}_0\omega)$, $u \in (T_x M)_0$, satisfying all the relations

$$a_{2\ell,\ell}\nabla^\ell \text{grad } W(u,\ldots,u) + L_n(u, \text{grad } W(x),\ldots,$$

$$\nabla^{\ell-1}\text{grad } W(u,\ldots,u), j^{2\ell}_0\omega) = 0, \quad 0 \le \ell \le n$$

As before, a necessary condition for (t_1, t_2) to be an accumulation point of $C(y,z)$ when $y(t_1) \neq 0$ but $z(t_2) = 0$, is that there exists a jet $j^{2n}_0\omega$ such that, for all integers n, the triple $(j^n_{p(t_1)}V, \dot p(t_1), j^{2n}_0\omega)$ belongs to $\sum_n(0)$. To finish the proof, we need to consider the case when $y(t_1) = z(t_2)$. Then z is a time translate of $y : z = y_\tau$, $\tau = t_1 - t_2$, that is, $z(t) = y(t+\tau)$ for all $t \in]a_-, +\infty[$. Let $\{(t_1(n), t_2(n)) \mid n \ge 1\}$ be a sequence in $C(y,z)$ converging to (t_1,t_2). Setting, for each integer $n \ge 1$, $t'_1(n) = t_2(n) + \tau$, the sequence $t'_1(n)$ converges to t_1 and $p(t'_1(n)) = p(t_1(n))$ for all n. Since $y(t'_1(n)) = z(t_2(n)) \neq y(t_1(n))$, it follows that $t'_1(n) \neq t_1(n)$ for all n. If for an infinite sequence $\{n_j \mid j \ge 1\}$ of integers, $(t'_1(n_j) - t_1)(t_1(n_j) - t_1) > 0$, $j = 1, 2, \ldots$, then the ∞–jet of y at t_1 reduces to $O_{p(t_1)}$. Since $\text{grad } V(p(t_1)) = D(y(t_1)) - \nabla_{\dot p}\dot p(t_1) = 0$, $p(t_1)$ is a singular point of the system. Then y and z both reduce to the point $p(t_1)$ and $C(y,z)$ is empty, which is a contradiction. Hence we assume that $(t'_1(n) - t_1)(t_1(n) - t_1) < 0$ for all n. By relabeling some of the $t'_1(n), t_1(n)$, we can assume that $t'_1(n) < t_1 < t_1(n)$ for all integer n. By taking a time translate of y we can also assume that $t_1 = 0$. Then $y(0) = \dot p(0) = 0$ and $\nabla_{\dot p}\dot p(0) + \text{grad } V(p(0)) = 0$. If $\text{grad } V(p(0)) = 0$, then y is reduced to the point $y(0)$ and we get a contradiction as before. Otherwise $\nabla_{\dot p}\dot p(0) \neq 0$. This implies that there exists a local diffeomorphism $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma(0) = 0$, $\dot\sigma(0) > 0$, at 0, and a germ of smooth curve $s : (\mathbb{R}, 0) \to (M, p(0))$ such that $p(t) = s(\frac{\sigma(t)^2}{2})$ for all t in a neighbourhood of 0. In fact, taking a coordinate system $x^1, \ldots, x^m : O \to \mathbb{R}$ in a neighbourhood O of $p(0)$, $x^i(p(0)) = 0$, $1 \le i \le m$, for some i, say $i = 1$, the coordinate function $p^1(t) = x^1(p(t))$ will have a non zero second derivative at 0. Then there exists a local diffeomorphism σ such that $p^1 = \frac{\varepsilon^1\sigma^2}{2}$ where ε^1 is $\ddot p^1(0)/|\ddot p^1(0)|$ and $\dot\sigma(0) > 0$. Since $p^1(t'_1(n)) = p^1(t_1(n))$ for all $n \ge 1$, $\sigma((t'_1(n)) = -\sigma(t_1(n))$ for all $n \ge 1$. Let σ^{-1} denote the inverse of σ. Denote by s_1 the composition $p \circ \sigma^{-1}$ i.e., $p = s_1 \circ \sigma$. Then for n big enough, setting $\tau_n = \sigma(t_1(n)), s_1(\tau_n) = s_1(-\tau_n)$. This shows that all the derivatives of s_1 of odd order at 0 are zero. So there exists a germ of smooth curve $s : (\mathbb{R}, 0) \to (M, p(0))$ such that s_1 and the curve $t \to s(\frac{t^2}{2})$ have the same ∞–jet at 0, $j^\infty_0 p = j^\infty_0(s \circ \sigma^2)$. Using $E1$, we see that $(\dot\sigma = \frac{d\sigma}{dt})$:

$$(\dot\sigma\sigma)^2(\nabla_{\dot s})(\frac{\sigma^2}{2}) + (\dot\sigma^2 + \sigma\ddot\sigma)\dot s(\frac{\sigma^2}{2}) - D(\sigma\dot\sigma\dot s(\frac{\sigma^2}{2})) + \text{grad } V(s(\frac{\sigma^2}{2})) = \alpha \qquad E19$$

where σ is flat at 0.

Setting $\chi = \frac{d\sigma}{dt} \circ \sigma^{-1}$, we have:

$$\begin{aligned}(t\chi(t))^2.\nabla_{\dot{s}}(\frac{t^2}{2}) + (\chi^2(t) + t\chi(t)\dot{\chi}(t))\dot{s}(\frac{t^2}{2}) - D(t\chi(t)\dot{s}(\frac{t^2}{2}))+ \\ + \text{grad } V(s(\frac{t^2}{2})) = \alpha \circ \sigma^{-1} \quad \text{for all } t \text{ in a neighbourhood of } 0.\end{aligned} \qquad E20$$

Deriving $E20$ covariantly $2n$ times, $n \geq 1$, along the curve $t \to s(\frac{t^2}{2})$, evaluating at $t = 0$ we get for $n \geq 1$:

$$\begin{aligned}\chi^2(0)c_n\nabla_{\dot{s}}^n\dot{s}(0) + G_n(\dot{s}(0), \ldots, \nabla_{\dot{s}}^{n-1}(0), j_0^{2n}\chi)+ \\ + a_{2n,n}\nabla^n \text{grad } V(\dot{s}(0), \ldots, \dot{s}(0) = 0,\end{aligned} \qquad E21n$$

where $G_n : \underbrace{TM \times_M \ldots \times_M TM}_{n \text{ times}} \times J^{2n}(\mathbb{R}, 0; \mathbb{R}) \to TM$ is some polynomial fiber bundle mapping, $c_n = 2n(2n-1)a_{2n-2,n-1} + 2_{2n,n}, a_{n,i}$ being the positive integers appearing in formula $E15$. Deriving $E20$ $2n+1$ times, $n \geq 0$, and evaluating at $t = 0$, we get:

$$\begin{aligned}[(2n+1)c_n\frac{d\chi^2}{dt}(0) - \chi(0)d_vD(0)]\nabla_{\dot{s}}^n\dot{s}(0)+ \\ + H_n(s(0), \ldots, \nabla_{\dot{s}}^{n-i}\dot{s}(0), j_0^{2n+1}\chi) = 0,\end{aligned} \qquad E22n$$

where H_n is a polynomial fiber bundle mapping:

$$\underbrace{TM \times_M \ldots \times_M TM}_{\text{n times}} \times J^{2n+1}(\mathbb{R}, 0; \mathbb{R}) \to TM.$$

Since $c_n \neq 0$ for all $n \geq 1$ and $\chi^2(0) \neq 0$, we can solve the equations $E21n$ successively for the $\nabla_{\dot{s}}^n\dot{s}(0)$, in terms of

$$gradV(s(0)), \ldots, \nabla^n \text{grad } V(\dot{s}(0), \ldots, \dot{s}(0)).$$

Carrying these values into the relations $E22n$ we shall get the following relations, $n \geq 1$:

$$\begin{aligned}[(2n+1)c_n\frac{d\chi^2}{dt}(0) - \chi(0)d_vD(0)]\nabla^n \text{grad } V(\dot{s}(0), \ldots, \dot{s}(0))+ \\ + E_n(\dot{s}(0), \text{grad } V(s(0)), \ldots, \nabla^{n-1}\text{grad } V(\dot{s}(0)), \ldots, \dot{s}(0)), j_0^{2n+1}\chi) = 0\end{aligned} \qquad E23n$$

where E_n is a rational fiber-bundle mapping:

$$\underbrace{TM \times_M \ldots \times_M TM}_{n \text{ times}} \times J_0^{2n+1} \to TM.$$

To sum up, if (t_1, t_2) is an accumulation point of $C(y,z)$ such that $y(t_1) = z(t_2)$, then, for any integer $n \geq 1$, there exists a jet $j_0^{2n+1}\chi \in J_0^{2n+1}$ such that the triple $(j_{p(t_1)}^{n+1}V, u_0, j_0^{2n+1}\chi)$, where $u_0 = \frac{\nabla_{\dot p}\dot p(0)}{\chi(0)^2}$ belongs to the subset $\sum_n(0,0)$ of $J^{n+1}(M, I\!R) \times (TM)_0 \times J_0^{2n+1}$ of all triples $(j_x^{n+1}W, u, j_0^{2n+1}w)$, $u \in (T_xM)_0$, satisfying the conditions: $1 \leq k \leq n$,

$$[(2k+1)c_k \frac{dw^2}{dt}(0) - w(0)d_vD(0_x)]\nabla^k \text{grad } W(u, \ldots, u) +$$
$$+ E_n(u, \text{grad } W(x), \ldots, \nabla^{k-1}\text{grad } W(u, \ldots, u), j_0^{2k+1}w) = 0.$$

It is clear that $\sum_n$ and $\sum_n(0)$ are submanifolds of the jet spaces

$$J^n(M; I\!R) \times (TM)_0 \times J_1^{n+1} \quad \text{and} \quad J^n(M, I\!R) \times (TM)_0 \times J_0^{2n},$$

respectively, having codimensions $(n+1)m$ and nm. Since $w(0) \neq 0$, in the sequence of endomorphisms of $T_xM : [\dot w(0) - L(u)], [3\dot w(0) - L(u)], \ldots, [(2n+1)\dot w(0) - L(u)]$, $L(u)$ being the endomorphism $d_vD(u) - d_vD(-u)$, $u \in (T_xM)_0$, at least $n-m$ are invertible. $\sum_n(-1)$ is contained in a codimension $(n-m)m$ submanifold $\widetilde{\sum}_n(-1)$ of $J^{n-1}(M, I\!R) \times (TM)_0 \times J_{11}^{n+1}$. Finally, if $w(0) \neq 0$, in the sequence of endomorphisms of $T_xM : [3c_1\frac{dw^2(0)}{dt} - w(0)d_vD(O_x)]$, $[5c_2\frac{dw^2}{dt}(0) - w(0)d_vD(O_x)], \ldots, [(2m+1)c_n\frac{dw^2(0)}{dt} - w(0)d_vD(O_x)]$, at least $n-m$ are invertible. In case $\frac{dw^2}{dt}(0) = 0$, they are all equal to $w(0)d_vD(O_x)$, which is invertible. Hence $\sum_n(0,0)$ is contained in a codimension $(n-m)m$ submanifold $\widetilde{\sum}_n(0,0)$ of the jet space $J^n(M, I\!R) \times (TM)_0 \times J_0^{2n+1}$.

To end the proof of Proposition 3.3, we will apply the transversality density Theorem 19.1 p.48 of reference [1] choosing for the $\mathcal{A}$ of that theorem the space of all Morse functions on M. The choices of the manifolds X, Y, W and of the mapping $\rho : \mathcal{A} \to C(X,Y), V \to f_V$ are indicated in the table below for each case:

Case	X	Y
Σ_n	$(TM)_0 \times J_1^{n+1}$	$J^k(M, I\!R) \times X$
$\Sigma_n(0)$	$(TM)_0 \times J_0^{2n}$	$J^n(M, I\!R) \times X$
$\Sigma_n(-1)$	$(TM)_0 \times J_{11}^{n+1}$	$J^{n-1}(M, I\!R) \times X$
$\Sigma_n(0,0)$	$(TM)_0 \times J_0^{2n+1}$	$J^n(M, I\!R) \times X$

W	f_V
Σ_n	$f_V(u, j_0^{n+1}w) = j_x^nV, x = \pi(u)$
$\Sigma_n(0)$	$f_V(u, j_0^{2n}w) = j_x^nV, x = \pi(u)$
$\widetilde{\Sigma}_n(-1)$	$f_V(u, j_0^{n+1}w) = j_x^{n-1}V, x = \pi(u)$
$\widetilde{\Sigma}(0,0)$	$f_V(u, j_0^{2n+1}w) = j_x^nV, x = \pi(u)$

For proposition 3.3(i) n has to be chosen greater than $4m+5$; for Proposition 3.3(ii), greater than $3m+3$. ■

An unstable (stable) manifold of a singular point of $X \in SDMS$ will be called simply an unstable (stable) manifold of X.

Proposition 3.4. *Given any pair (X_0, x_0) in $GIW(D) \times TM$ (resp. $SDMS(V) \times TM$) there exist open neighbourhoods N_0 of x_0 in TM, $\mathcal{U}_0$ of X_0 in $SDMS(D)$ (resp. $SDMS(V)$) such that, if N is the number of singular points of X_0:*

(i) *There is a continuous mapping*

$$\mathcal{U}_0 \ni X \longrightarrow (O_{1,X}, \ldots, O_{N,X}) \in M^N = \underbrace{M \times M \times \ldots \times M}_{\text{N times}}$$

such that for each X in $\mathcal{U}_0$, $\{O_{1,X}, \ldots, O_{N,X}\}$ is the set of all singular points of X;

(ii) *If x_0 does not lie on any unstable manifold of X_0, for any $X \in \mathcal{U}_0$ no unstable manifold of X meets N_0;*

(iii) *If x_0 lies on an unstable manifold of X_0, $W^u_{X_0}(O_{1X_0})$ say, then the set of all X in $\mathcal{U}_0$ such that $W^u_X(O_{1X}) \cap N_0$ is transversal to all the stable manifolds of X is a Baire subset (residual) of $\mathcal{U}_0$.*

For the proof of Proposition 3.4, we use Proposition 3.5 below, to be proved later. Given a trajectory $z :]a_-, +\infty) \to TM$ of (V, D) with projection q, we say that an interval $I \subset]a_-, +\infty)$ is free of multiple points if, for any $t \in I$, $q^{-1}(q(t)) = \{t\}$.

Proposition 3.5. *Let $X_0 = (V_0, D_0)$ be a system in $SDMS$ and x_0 a non singular point of X_0 lying on an unstable manifold $W^u_{X_0}(O_{X_0})$ of X_0. Let $z_0 : \mathbb{R} \to TM$ be the trajectory of X_0 passing through x_0 at time 0. Assume that z_0 satisfies the property: any open subset $\mathcal{O}$ in $\mathbb{R}$ contains an open interval In free of multiple points for z_0 and such that $z_0(In)$ does not intersect the zero section of TM. Then there exist neighbourhoods $\mathcal{U}_0$ of X_0, in $SDMS(D_0)$ (resp. $SDMS(V_0)$), N_0 of x_0 in TM, Θ of O in $\mathbb{R}^c$ where c is the codimension of $W^u_{X_0}(O_{X_0})$ in TM and a continuous mapping $(X, \theta) \in \mathcal{U}_0 \times \Theta \longrightarrow V_{X,\theta} \in C^\infty(M; \mathbb{R})$ (resp. $D_{X,\theta} \in \mathcal{D}$) having the following properties:*

(i) *There exists a continuous mapping $X \in \mathcal{U}_0 \to (O_{1,X}, \ldots, O_{N,X}) \in M^N$ such that the set $\{O_{1,X}, \ldots, O_{N,X})$ is the set of all singular points of X and $O_{1,X_0} = O_{X_0}$.*

(ii) *For any $X = (V_X, D_0)$ in $\mathcal{U}_0$ and $\theta \in \Theta$:*

$$V_{X,\theta} = V_X + \sum_{i=1}^{c} \theta^i V_i, \quad \theta = (\theta^1, \ldots, \theta^c) \in \mathbb{R}^c$$

where the functions V_i have their supports contained in a compact subset Q of M (resp.: For all $\theta \in \Theta$, $D_{X,\theta} - D_X$ has its support contained in a fixed compact subset Q in TM).

(iii) *For any $X \in \mathcal{U}_0$, the fields $X_\theta = (V_{X,\theta}, D_0)$ have the same singular set $(O_{1,X}, \ldots, O_{N,X})$ as X and they coincide with X in a neighbourhood of this singular set.*

(iv) *The set $T^s(N_0, X)$ of the projections on M of all positive semi trajectories starting in N_0 (resp. the set $T^s(N_0, X)$ of all positive semi-trajectories starting in N_0) does not meet Q. Hence $T^s(N_0, X)$ is identical with the analogous set $T^s(N_0, X_\theta)$ for X_θ.*

(v) *For any X in $\mathcal{U}_0$, there exist an open subset P_X of $W^u_X(O_{1,X})$ and a diffeomorphism $e_X : P_X \times \Theta \longrightarrow TM$ such that:*
   1) *the open subset $e_X(P_X \times \Theta)$ of TM contains N_0.*
   2) *$e_{X,0} : P_X \longrightarrow TM$, $x \longrightarrow e_X(x, 0)$ is just the injection of P_X in TM.*
   3) *For any θ in Θ we have the inclusions $W^u_{X_\theta}(O_{1,X}) \cap N_0 \subset e_X(P_X \times \{\Theta\}) \subset W^u_{X_\theta}(O_{1,X})$.*

Proof of Proposition 3.4. We can easily find an open neighbourhood $\mathcal{U}_1$ of X_0 such that (i) is satisfied. If x_0 does not lie on an unstable manifold of X_0, then the negative semi-trajectory of X_0 ending at x_0 cuts any energy level surface $\{E_{V_0} = A\}$, V_0 being the potential of X. Choose A so big that all the unstable manifolds of X_0 lie in $\{E_{V_0} \leq A\}$. There will exist a compact neighbourhood N_0 of x_0 in TM such that all the negative semi-trajectories of X_0 ending in N_0 cut the level surface $\{E_{V_0} = 2A\}$. Then it is easy to find an open neighbourhood $\mathcal{U}_0 \subset \mathcal{U}_1$ of X_0 such that: 1) for any X in $\mathcal{U}_0$ all the unstable manifolds of X lie in $\{E_{V_0} \leq \frac{3A}{2}\}$; 2) all the negative semi-trajectories of X ending in N_0 cut the level surface $\{E_{V_0} = 2A\}$. Obviously for any X in $\mathcal{U}_0$ no unstable manifold of X cuts N_0. This ends the proof of Proposition 3.4 when x_0 does not lie on an unstable manifold of X_0.

If x_0 lies on $W^u_{X_0}(O_1 X_0)$, we can find neighbourhoods $\mathcal{U}_0$ of X_0, N_0 of x_0 satisfying all the properties of Proposition 3.5. Since the stable manifolds are submanifolds of TM, it is clear that the set $\mathcal{G}(\mathcal{U}_0)$ of all X in $\mathcal{U}_0$ such that $W^u_X(O_1 X) \cap N_0$ is transversal to all the stable manifolds of X is a G_δ (countable intersection of open subsets of $\mathcal{U}_0$).

If we prove that $\mathcal{G}(\mathcal{U}_0)$ is dense in $\mathcal{U}_0$, it will follow that it is a Baire subset of $\mathcal{U}_0$.

Take any X in $\mathcal{U}_0$. Using the notations of Proposition 3.5, denote by $\mathrm{pr}_2 : P_X \times \Theta \to \Theta$ the second canonical projection. Sard's theorem tells us that in any neighbourhood of O in Θ, there exists a $\overline{\theta}$ which is a regular value for the restriction of pr_2 to the family $\{e_X^{-1}(W^s_X(O_i X)) \mid 1 \leq i \leq N\}$ of submanifolds of $P_X \times \Theta$ and such that $X_{\overline{\theta}}$ lies in $\mathcal{U}_0$. Since the positive semi-trajectories of $X_{\overline{\theta}}$ starting in N_0, do not meet the support Q of the deformation $X_{\overline{\theta}}$, for any j, $1 \leq j \leq N$, $W^s_{X_{\overline{\theta}}}(O_j X_{\overline{\theta}}) \cap N_0 = W^s_X(O_j X) \cap N_0$ and the choice of $\overline{\theta}$ ensures that the manifold $e_X(P_X \times \{\overline{\theta}\})$ is transversal to the family $\{W^s_X(O_iX) \mid 1 \leq i \leq N\}$. Since $e_X(P_X \times \{\overline{\theta}\})$ contains $W^u_{X_{\overline{\theta}}} \cap N_0$, we get the statement (iii) of Proposition 3.4. ■

Proof of Proposition 3.5. We shall discuss only the case where the dissipative force is kept fixed. This case is much harder to handle that the one where the potential is kept fixed because even if we use local perturbation of the potential V (i.e. with small compact support) the corresponding perturbations of the system will not be local anymore, since they will affect all the points in the tangent bundle located above the support of the perturbation of V. Hence more sophisticated tools are needed to treat this case than the case where the dissipative force is perturbed, which can be treated by standard methods.

To prove the Proposition, it is sufficient to construct an open neighborhood $\mathcal{V}$ of X_0 in $SDMS(D_0)$, times $\tau_u < t_1 < t_2$, compact neighbourhoods N_u, N_s of $z_0(\tau_u)$, $z_0(0)$ respectively, in TM, Q of $q_0([t_1, t_2])$, $q_0 = \pi_M \circ z_0$, in M, and c smooth functions $V_i : M \to I\!R$, $1 \leq i \leq c$, with supports contained in Q such that:

0) There exists a continuous mapping $X \in \mathcal{V} \to (O_{1,X}, \ldots, O_{N,X}) \in M^N$ such that $\{O_{1,X}, \ldots, O_{N,X}\}$ is the singular set of X and $O_{1X_0} = O_{X_0}$. Also $Q \cap \{O_{1,X}, \ldots, O_{N,X}\} = \emptyset$ for X in $\mathcal{V}$.
1) Let $T^s(N_s, X)$ denote the set of the projections on M of all positive semi-trajectories starting in N_s. Let $T^u(N_u, X)$ denote the set of all negative semi-trajectories tending to a singular point as t tends to $-\infty$ and ending in N_u for $t = 0$. $T^s(N_s, X) \cap Q$ and $T^u(N_u, X) \cap Q$ are both empty for any X in $\mathcal{V}$.
2) The mapping $f_{X_0} : [N_u \cap W^u_{X_0}(O_{X_0})] \times I\!R^c \to TM$, defined as: $f_{X_0}(x, \theta)$ is the position at time 0 of the trajectory of the system $X_{0\theta} = (V_0 + \sum_{i=1}^c \theta^i V_i, D_0)$ passing through x at time τ_u, is infinitesimally inversible at x_0.

In fact, if we have properties 1-2 above, f_{X_0} is a local diffeomorphism at x_0. Since $f_{X_0}(z_0(\tau_u), 0) = x_0$, we can restrict both N_u and N_s and choose a neighbourhood Θ of O in $I\!R^c$ such that f_{X_0} maps $[\overset{o}{N}_u \cap W^u_{X_0}(O_{X_0})] \times \Theta$, $\overset{o}{N}_u =$ interior of N_u, diffeomorphically onto a subset of TM containing N_s. Then we can find an open subneighbourhood $\mathcal{U}_0$ of X_0 in $\mathcal{V}$ such that this last assertion is true for the mapping f_X constructed in the same way as f_{X_0}, but starting with X instead of $X_0 : f_X$ maps $\overset{o}{N}_u \cap W^u_X(O_{1X}) \times \Theta$ diffeomorphically onto a set in TM containing N_s.

Then we define P_X and e_X as follows:

$$P_X = f_X(\overset{o}{N}_u \cap W^u_X(O_{1,X}), 0)$$
$$e_X(f_X(x, 0), \theta) = f_X(x, \theta).$$

As N_0 we take N_s. Then all the conditions (i), (ii), (iii), (iv), (v)-1, (v)-2 of Proposition 3.5 are obviously satisfied. To check (v)-3 note that by property

1 of N_u, the intersection $W^u_{X_\theta}(O_{1,X}) \cap N_u$ coincides with the intersection $W^u_X(O_{1,X}) \cap N_u$. Hence $e_X(P_X \times \{\theta\}) = f_X(\overset{o}{N}_u \cap W^u_{X_\theta}(O_{1,X}), \theta)$ is contained in $W^u_{X_\theta}(O_{1,X})$. If y is a point in $W^u_{X_\theta} \cap N_0$ then it is the image $f_X(x,\theta)$ of a point x in $\overset{o}{N}_u \cap W^u_X(O_{1,X})$ which is the same as $\overset{o}{N}_u \cap W^u_{X_\theta}(O_{1,X})$. Hence $y = e_X(f_X(x,0),\theta)$. This proves the second inequality of (v)-3. It remains to construct $\mathcal{V}$, N_u, N_s, Q, and the V_i's so as to satisfy 0)-1)-2).

To check that f_{X_0} is infinitesimally inversible at x_0 it is necessary and sufficient to show that the vectors $\frac{\partial f_{X_0}}{\partial \theta'}(z_0(\tau_u), 0)$, $1 \le u \le c$, in $T_{x_0}TM$, are linearly independent modulo the subspace $T_{x_0}W^u_{X_0}(O_{X_0})$ of $T_{x_0}TM$. Now these vectors are the values at $t = 0$ of vector fields along z_0 which represent the infinitesimal deformations of the trajectories when X_0 undergoes the deformation X_θ. These vector fields are solutions of the linearized flow equation along z_0.

To study this linearized equation we need a good representation of it and more generally of the double tangent bundle TTM. In our opinion the best is to use the Levi Civita connection of the Riemannian manifold M. At a great expense in calculations and symbols one could avoid the connection and use coordinate charts. But the computation would be very messy and the results would not be intrinsic.

A) To proceed we have to recall some more or less well known results about the double tangent space TTM. It can be considered as a vector space bundle in two ways: first it is the tangent bundle of the tangent bundle TM of M. As such it has a projection $\pi_{TM} : TTM \longrightarrow TM$.

Second, the canonical projection $\pi_M : TM \longrightarrow M$ of the tangent bundle TM of M induces a tangent mapping $T\pi_M : TTM \longrightarrow TM$. This is a vector bundle projection and we have the relation:

$$\pi_M \circ \pi_{TM} = \pi_M \circ T\pi_M.$$

The kernel of $T\pi_M$ is a subbundle Ver_M of the bundle $(TTM,\ \pi_{TM},\ TM)$ called the vertical bundle. Ver_M is isomorphic to the fiber product $TM \times_M TM$ endowed with the first canonical projection $\mathrm{pr}_1 : TM \times_M TM \longrightarrow TM$, $(u,v) \longrightarrow u$. A canonical isomorphism $j : TM \times_M TM \to \mathrm{Ver}_M$ is defined as follows: if $(u,v) \in TM \times_M TM$, $j(u,v)$ is the tangent vector at $\lambda = 0$ of the curve $\lambda \in I\!R \longrightarrow u + \lambda v \in TM$.

The Levi-Civita connection defines another subbundle $\mathcal{H}_M$ of the bundle (TTM, π_{TM}, TM) called the horizontal bundle as follows. Define a smooth mapping $C : TM \times_M TM \longrightarrow TTM$: for any pair $(u,v) \in TM \times_M TM$, choose any smooth curve $\sigma :]-\varepsilon,\varepsilon[\longrightarrow M$, $t \longrightarrow \sigma(t)$ such that its tangent vector at 0, $T\sigma(0)$, is u. Let $\tau_\sigma(t) : T_qM \to T_{\sigma(t)}M$ $(q = \sigma(0) = \pi_M u = \pi_M v)$ be the parallel transport along σ defined by the Levi-Civita connection. Then the tangent vector at 0 to the smooth curve $t \in]-\varepsilon,\varepsilon[\longrightarrow \tau_\sigma(t)v$ is independent of the choice of σ but depends only on the pair (u,v). We denote it by $C(u,v)$.

C defines a vector bundle injection of the bundle $(TM \times_M TM,\ \mathrm{pr}_2,\ TM)$ [pr_2 is the second canonical projection $TM \times_M TM \longrightarrow TM$, $(u, v) \longrightarrow v$] into the bundle (TTM, π_{TM}, TM). Its image $\mathcal{H}_M$ is the horizontal bundle.

The following formulas are useful:

$$\begin{array}{ll} T\pi_M\ C(u,v) = u & T\pi_M(j(u,v)) = o \\ \pi_{TM}C(u,v) = v & \pi_{TM}\ j(u,v) = u \end{array}$$

The vector bundle (TTM, π_{TM}, TM) is the direct sum $\mathcal{H}_M \oplus \mathrm{Ver}_M$ of its horizontal and vertical subbundles. This direct sum, in turn, is isomorphic to the fiber product $(TM \times_M TM) \times_{\mathrm{pr}_2,\mathrm{pr}_1} (TM \times_M TM)$ which, in turn, is isomorphic to the triple fiber product $TM \times_M TM \times_M TM$. The isomorphism $\Delta : TM \times_M TM \times_M TM \longrightarrow TTM$ defined in this way is:

$$\Delta(u, v, w) = C(u, v) + j(v, w).$$

The triple (u, v, w) corresponds to the element $[(u, v), (v, w)]$ of the fiber product

$$(TM \times_M TM)\ \times_{\mathrm{pr}_2,\mathrm{pr}_1}\ (TM \times_M TM).$$

The inverse Δ^{-1} of Δ can be expressed as follows:

$$\Delta^{-1} :\ TTM \longrightarrow TM \times_M TM \times_M TM,$$

$$\Delta^{-1}(\tau) = (T\pi_M(\tau), \pi_{TM}(\tau),\ K(\tau))$$

where the mapping $K : TTM \longrightarrow TM$ is the unique smooth mapping satisfying the relation:

$$j(\pi_{TM}(\tau), K(\tau)) = \tau - C(T\pi_M(\tau), \pi_{TM}(\tau)).$$

The last element belongs obviously to the vertical bundle.

The following considerations will be useful for the future. Let $z :]a, b[\longrightarrow TM$ be any smooth curve and let $q :]a, b[\longrightarrow M$ be its projection on M, then the image $\Delta^{-1}(\frac{dz}{dt})$ of the tangent vector field $\frac{dz}{dt} \in TTM$ along z is:

(1) $$\Delta^{-1}(\frac{dz}{dt}) = (\frac{dq}{dt},\ z,\ \nabla_t z).$$

where $\frac{dq}{dt}$ is the tangent vector field to q and $\nabla_t z$ is the covariant derivative of the vector field along q.

We also have the formula:

(2) $$\frac{dz}{dt} = C(\frac{dq}{dt}, z) + j(z, \nabla_t z).$$

B) The preceeding remarks in **A**, allow us to avoid the consideration of the double tangent bundle TTM and work with objects in M and TM. In particular we can give the following nice representation of the flow of a system $X = (V, D)$. The projections on M of the trajectories of X are the curves $q :]a_-, +\infty[\longrightarrow M$ satisfying the second order equation:

$$\nabla_{\dot q}\dot q - D(\dot q) + \text{grad } V(q) = 0$$

where $\dot q$ denotes the tangent vector field $\frac{dq}{dt}$ and $\nabla_{\dot q}$ the covariant derivative in the $\dot q$ direction. The trajectory of X whose projection is q, is simply the tangent vector field $\frac{dq}{dt}$.

Let us now study the linearized flow along a trajectory. Let $\theta \in \Theta$ (open set in $\mathbb{R}^c$) $\longrightarrow X_\theta = (V_\theta, D_\theta)$ denote a smooth deformation of the field X_0 and let $\theta \in \Theta \longrightarrow z_\theta :]a_-(\theta), +\infty[\longrightarrow TM$ be a smooth family of curves such that z_θ is a trajectory of X_θ. The vector field $\frac{\partial z_\theta}{\partial \theta} |_{\theta=0}$ is the infinitesimal deformation of the family along z_0. Let χ be the vector field $T\pi_M\left(\frac{\partial z_\theta}{\partial \theta} |_{\theta=0}\right)$ along q_0, projection of z_0 on M.

Lemma 3.6. *A vector field χ along q_0 is the projection on M of an infinitesimal deformation of z_0 corresponding to the deformation X_θ of X if and only if:*

$$P_0\chi = \frac{\partial D_\theta}{\partial \theta} |_{\theta=0} (\dot q_0) - \text{grad } \frac{\partial V}{\partial \theta} |_{\theta=0} (q_0)$$

where P_0 is the second order operator along q_0:

$$P_\xi = \nabla_t^2 \xi - R(\dot q_0)\nabla_t \xi + S(\dot q_0)\xi.$$

The tensor fields R, S are defined in the proof of the Lemma. The relation between χ and $\frac{\partial z}{\partial \theta} |_{\theta=0}$ is as follows:

$$\frac{\partial z}{\partial \theta} |_{\theta=0} = \Delta\ (\chi, z_0, \nabla_t \chi).$$

C) For any interval $I \subset]a_-(0), +\infty[$ denote by $\Gamma(I, TM)$ the space of all smooth vector fields along the curve restriction to q_0 of I.

P_0 defines a linear operator $\Gamma(I, TM) \longrightarrow \Gamma(I, TM)$ with respect to the L_2 scalar product defined by the Riemannian metric on M. P_0 has an adjoint $P_0^* : \Gamma(I, TM) \longrightarrow \Gamma(I, TM)$

$$P_0^*\psi = \nabla_{\dot q_0}^2 \psi + \nabla_{\dot q_0}(R(\dot q_0)^*\psi) + S(\dot q_0)^*\psi$$

where R^*, S^* are the adjoints of the tensors R, S with respect to the Riemannian scalar product.

We have a **Green's formula**: let I be closed, $I=[a,b]$, then:

$$\int_a^b [< P_0\psi_1, \psi_2 > - < \psi_1, P_0^*\psi_2 >]dt = B(\dot{q}_0)[(\nabla_t\psi_1, \psi_1), \nabla_t\psi_2, \psi_2)] \mid_a^b$$

for all $\psi_1, \psi_2 \in \Gamma(I, TM)$, where for each $u \in T_qM$, $B(u)$ is the multilinear form $T_qM \times T_qM \times T_qM \times T_qM \longrightarrow I\!R$:

$$B(u)[(u_1, v_1), (u_2, v_2)] = < u_1, v_2 > - < u_2, v_1 > - < R(u)v_1, v_2 > .$$

It is clear that $B(u)$ is non-degenerate for each u.

D) Assume now that z_0 is a trajectory of $X_0 = (V_0, D_0)$ contained in an unstable manifold $W^u_{X_0}(\alpha(z_0))$. The tangent bundle $TW^u_{X_0}(\alpha(z_0)) \mid z_0$ of $W^u_{X_0}(\alpha(z_0))$ along z_0 is a subbundle of the tangent bundle $TTM \mid z_0$ along z_0. Its image E^u by the mapping $T\pi_M \times \pi_{TM}$ is a subbundle of $q_0^*TM \underset{q_0}{\times} q_0^*TM$. Since $TW^u_{X_0}(\alpha(z_0)) \mid z_0$ is invariant by the linearized flow along z_0, E^u is invariant by P_0, that is, if (u,v) belongs to E^u and φ is a solution of $P \circ \varphi = 0$ such that $(\nabla_t\varphi(t_0), \varphi(t_0)) = (u,v)$ for some t_0 then $(\nabla_t\varphi(t), \varphi(t)) \in E^u_t$ for all t.

Let E^* be the subbundle of $q_0^*TM \times q_0^*TM$, which is the right orthogonal complement of E^u with respect to B. Its fiber E^*_t at $t \in I\!R$ is:

$$E^*_t = \{(u_2, v_2) \in T_{q_0(t)}M \times T_{q_0(t)}M \mid B(q_0(t))[(u_1, v_1), (u_2, v_2)] = 0,$$
$$\forall (u_2, v_2) \in E^u_t\}.$$

The bundle E^* is invariant by P_0^*. In fact, take any solution ψ of $P_0^*\psi = 0$ such that $(\nabla_t\psi(t_0), \psi(t_0)) \in E^*_{t_0}$ for some t_0. Then for any solution φ of $P_0\varphi = 0$ contained in E^u, using Green's formula:

$$B(q_0(t))[(\nabla_t\varphi, \varphi), (\nabla_t\psi, \psi)] \mid_{t_0}^{t_1} = \int_{t_0}^{t_1} [< P_0\varphi, \psi > - < \varphi, P_0^*\psi >]dt = 0$$

for any t_1 in $]a_-, +\infty[$.

This relation shows that for any such t_2:

$$B(q_0(t_1))[(\nabla_t\varphi(t_1), \varphi(t_1)), (\nabla_t\psi(t_1), \psi(t_1))] = 0.$$

Since $(\nabla_t\varphi(t_1), \varphi(t_1))$ takes all possible values in E^u_t as φ varies, we get

$$(\nabla_t\psi(t_1), \psi(t_1)) \in E^*_t.$$

Since the bilinear form is non-degenerate, the dimension of the fibers of E^* is the codimension c of $W^u_{X_0}(\alpha(z_0))$.

In order to construct V_θ we need the following Lemmas:

Lemma 3.7. *Let X_0 be any system in $GIW(D)$.*

(i) For any non singular trajectory z_0: $]a_-,+\infty[\longrightarrow TM$ of X_0 and any open subset Ω of $]a_-,+\infty[$, there exists an open interval In contained in Ω free of any multiple points of $q_0 = \pi_M \circ z_0$ and not contaning any time t such that $\frac{dq_0}{dt}(t) = z_0(t) = O_M$.

(ii) Let In be any open interval in $]a_-,+\infty[$ having the properties stated in i. Let $\tau_u < t_1 < t_2 < \tau_s$ be four times such that $[\tau_u,\tau_s] \subset]a_-,+\infty[$ and $[\tau_u,t_2] \subset In$. Then there exists an open neighbourhood $\mathcal{U}_0$ of X_0 in $SDMS$, a compact neighbourhood Q of $q_0([t_2,t_2])$, compact neighbourhoods N_u, N_s of $z_0(\tau_u)$ and $z_0(\tau_s)$ respectively in TM, such that the sets $\bigcup_{X\in\mathcal{U}_0} T^s(N_s,X)$ and $\bigcup_{X\in\mathcal{U}_0} T^u(N_u,X)$ do not meet Q. $T^s(N_s,X)$ is the set of projections on M of all the positive semi-trajectories of X starting in N_s and $T^u(N_u,X)$ the set of projections on M of all the negative semi trajectories of X ending in N_u and tending to a singular point of X when t goes to $-\infty$.

Lemma 3.8. *Let X be a system in $SDMS$ and let $z_0 :]a_-,+\infty[\rightarrow TM$ be a trajectory of X and q_0 its projection on M. Then there exists a real number τ_+ such that $P_0^*\dot{q}_0 \neq 0$ for all $t \geq \tau_+$. If $\alpha(z_0)$ exists then the same is true for all $t \leq \tau_-$, τ_- an appropriate number.*

As a consequence $\dot{q}_0$ is linearly independent from the space of solutions of $P_0^*\psi = 0$ on any interval contained in $[\tau_+,+\infty[$ (resp. $]-\infty,\tau_-]$).

To start the construction of the V_θ we choose an interval In as in Lemma 3.7 and contained in $]-\infty,\tau_-]$ where τ_- is the number defined in Lemma 3.8. Take now three times τ_u, t_1, t_2 such that $\tau_u < t_1 < t_2 < 0$ and $[\tau_u,t_2]$ is contained in In. Then choose neighbourhoods $\mathcal{V}$ of X_0 in $SDMS(D_0)$, N_u of $z_0(\tau_u)$, N_s of $z_0(0) = x_0$ $(\tau_s = 0)$, Q of $q_0([t_1,t_2])$ as in Lemma 3.7-(ii).

Restricting $\mathcal{V}$, N_u further we can assume that there exists a continuous mapping $X \in \mathcal{V} \longrightarrow (O_{1,X},\dots,O_{N,X}) \in M^N$ such that $\{O_{1,X},\dots,O_{N,X}\}$ is the singular set of X and $O_{1,X_0} = O_{X_0}$. Moreover, we can assume that the correspondence $X \in \mathcal{V} \longrightarrow N_u \cap W^u_X(O_{1,X})$ is continuous in the following sense: there is a continuous mapping $X \in \mathcal{U} \longrightarrow \varepsilon_X \in \mathcal{E}$, $\mathcal{E}$ being the space of all embeddings of $N_u \cap W^u_{X_0}(O_{X_0})$ into TM with the usual topology, such that for any X in $\mathcal{V}$:

$$\varepsilon_X(N_u \cap W^u_{X_0}(O_{X_0})) = N_u \cap W^u_X(O_{1X}).$$

In order to construct the functions V_i with support in Q, we will construct vector fields F_i, $1 \leq i \leq c$, along q_0 such that for all $t \in \mathbb{R}$, the value grad $V_i(q_0(t))$ of the gradient of V_i at $q_0(t)$ will be $F_i(t)$. To do this, let

$\underline{E}^*$ denote the vector space of all vector fields ψ along $q_o \mid [t_1, t_2]$, which are contained in the fiber bundle E^* along the curve $q_0 \mid [t_1, t_2]$ and which are solutions of the equation $P_0^* \psi = 0$. This space $\underline{E}^*$ is a finite dimensional space of dimension c. Choose c vector fields F_i, $1 \le i \le c$, along $q_0 \mid [t_1, t_2]$ having compact supports contained in $]t_1, t_2[$ such that the linear forms ℓ_i on $\underline{E}^*$, $\ell_i(\psi) = \int_{t_1}^{t_2} < F_i(t), \psi(t) > dt$ form a basis of the dual of $\underline{E}^*$ and such that:

$$\int_{t_1}^{t_2} < F_i(t), \frac{dq_0(t)}{dt} > dt = 0, \quad 1 \le i \le c.$$

This is possible since given the choice of In, $\frac{dq_0}{dt}$ is linearly independent from $\underline{E}^*$ on $[t_1, t_2]$.

Now we can define the V_i. Let B_ε be the subset of $q_0^* TM \mid [t_1, t_2]$ of all vectors $v \in T_{q_0(t)}M$, $t_1 \le t \le t_2$, such that $\|v\| \le \varepsilon$ and v is orthogonal to $\frac{dq_0}{dt}(t)$. Then there exists an $\varepsilon > 0$ such that the exponential mapping exp: $B_\varepsilon \longrightarrow TM$, $v \longrightarrow \exp v$ associated to the Riemannian metric of M is a diffeomorphism and such that
$\exp B_\varepsilon \subset Q$.

Finally, let $\rho : I\!R \longrightarrow [0,1]$ be a C^∞ function such that ρ is 1 on the interval $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and 0 outside the interval $[-\frac{3\varepsilon}{4}, \frac{3\varepsilon}{4}]$.

V_i is defined as follows: on the image $\exp(B_\varepsilon)$, if $v \in T_{q_0(t)}M$, $t_1 \le t \le t_2$,

$$V_i(\exp\, v) = \rho(\|v\|) \left[< F_i(t), v > + \int_{t_1}^{t} < F_i(s), \frac{dq_0}{ds}(s) > ds \right]$$

and outside $\exp(B_\varepsilon)$, $V_i = 0$.

V_i is smooth. To check this we have to show that $V_i(\exp\, v) = 0$ when v lies in a neighbourhood of the boundary of B_ε. This happen when either $\|v\|$ is near ε, but then $V_i(\exp\, v) = 0$ since $\rho(\|v\|) = 0$ if $\|v\| \ge \frac{3\varepsilon}{4}$ or when $v \in T_{q(t)}M$ and t is near t_1 or t_2. But then t will lie outside the support of F_i and also

$$\int_{t_1}^{t} < F_i(s), \frac{dq_0}{ds}(s) > ds = \begin{cases} 0 & \text{if } t \text{ is near } t_1 \\ \displaystyle\int_{t_1}^{t_2} < F_i(s), \frac{dq_0}{ds}(s) > ds & \text{if } t \text{ is near } t_2 \end{cases}$$

But by construction this last integral is zero.

To define V_θ and more generally $V_{X,\theta}$, $X \in \mathcal{V}$, we set:

$$V_{X,\theta} = V_X + \sum_{i=1}^{c} \theta^i V_i, \quad V_\theta = V_{X_0,\theta}.$$

The deformation X_θ of $X = (V_X, D)$ is defined as the system $(V_{X,\theta}, D)$.

Finally, we can define a mapping

$$f_X : \left[N_u \cap W^u_X(O_{1X})\right] \times \mathbb{R}^c \longrightarrow TM$$

as in the beginning of the proof of Proposition 3.5: $f_X(x,\theta)$ is the position at time 0 of the trajectory of X_θ passing through x a time τ_u.

It is clear that the conditions 0-1 stated at the beginning of this proof are satisfied by our choice of $\mathcal{V}$, N_u, N_s, Q, V_i $1 \le i \le c$. All we have to do is to check the last condition 2). As we have seen, this is equivalent to proving that the vectors $\frac{\partial f_{X_0}}{\partial \theta^i}(z_0(\tau_u),0)$ in $T_{x_0}TM$ are linearly independent modulo the space $T_{x_0}TM$.

By lemma 3.6, the projection $d\pi_M\left[\frac{\partial f_{X_0}}{\partial \theta^i}(z_0(\tau_u),0)\right]$ is equal to $Y_i(0)$, where Y_i, $1 \le i \le c$ is the vector field along q_0, solution of the Cauchy problem:

$$\begin{cases} P_0 Y_i = -F_i \\ Y_i(\tau_u) = \nabla_t Y_i(\tau_u) = 0 \end{cases}$$

If the vectors $\frac{\partial f_{X_0}}{\partial \theta^i}(z_0(\tau_u),0)$, $1 \le i \le c$, were not linearly independent modulo $T_{x_0}W^u_{X_0}(\alpha(z_0))$, then the vectors $d\pi_M \times \pi_{TM}\left[\frac{\partial f_{X_0}}{\partial \theta^i}({}_0(\tau_u),0)\right]$, $1 \le i \le c$, would be linearly dependent modulo E^u_0. Now $d\pi_M \times \pi_{TM}\left(\frac{\partial f_{X_0}}{\partial \theta}(z_0(\tau_u),0)\right) = (\nabla_t Y_i(0), Y_i(0))$. We claim that for any $t > t_1$ the c vectors

$$(\nabla_t Y_1(t), Y_1(t)), \ldots, (\nabla_t Y_c(t), Y_c(t))$$

in $T_{q_0(t)}M \times T_{q_0(t)}M$ are independent modulo E^u_0. Were they not, there would exist a linear combination $Y = \sum_{i=1}^c \lambda^i Y_i$, $\lambda^1, \ldots, \lambda^c \in \mathbb{R}$, such that $(\nabla_t Y(t), Y(t))$ belongs to E^u_0. But $\begin{cases} P_0 Y = -F \\ Y(\tau_u) = \nabla_t Y(\tau_u) = 0 \end{cases}$, where $F = \sum_{i=1}^c \lambda^i F_i$.
For any $\psi \in \underline{E}^*_t$

$$\begin{aligned} -\int_{t_1}^t < F(s), \psi(s) > ds &= \int_{t_1}^t [< P_0 Y(s), \psi(s) > - < Y(s), P_0^* \psi(s) >] ds \\ &= B(\dot q_0(t))[(\nabla_t Y(t), Y(t)), (\nabla_t \psi(t), \psi(t))] \\ &= 0 \end{aligned}$$

By the choice of the F_i, this implies that $F = 0$.
Hence $\begin{cases} P_0 Y = 0 \\ Y(\tau_u) = \nabla_t Y(\tau_u) = 0 \end{cases}$.
By the uniqueness property in Cauchy's existence theorem, this implies that $Y = 0$ and proves our claim.

Proof of Lemma 3.6: We start with the relation:

$$\nabla_{\dot q_\theta} \dot q_\theta - D(\dot q_\theta) + \text{grad } V_\theta(q_\theta) = 0.$$

Let us introduce the mapping $q : \bigcup_{\theta\in\Theta}]a_-(\theta), +\infty[\longrightarrow M$, $(t,\theta) \longrightarrow q_\theta(t)$ and denote by ∇_t (resp. ∇_θ) the covariant derivative in the direction $\frac{\partial q}{\partial t}$ (resp. $\frac{\partial q}{\partial \theta}$). Hence:

$$\nabla_t \frac{\partial q}{\partial t} - D_\theta(\frac{\partial q}{\partial t}) + \mathrm{grad} V_\theta(q) = 0.$$

Deriving covariantly in the direction of $\frac{\partial q}{\partial \theta}$:

$$\nabla_\theta \nabla_t \frac{\partial q}{\partial t} - \nabla_\theta[D_\theta(\frac{\partial q}{\partial t})] + \nabla_\theta[\mathrm{grad} V_\theta(q)] = 0.$$

Now:

$$\nabla_\theta \nabla_t \frac{\partial q}{\partial t} = \nabla_t \nabla_\theta \frac{\partial q}{\partial t} + \mathrm{Curv}\ (\frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial t})\frac{\partial q}{\partial t}$$

$$= \nabla_t^2 \frac{\partial q}{\partial \theta} \quad + \quad \mathrm{Curv}\ (\frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial t})\frac{\partial q}{\partial t}$$

where Curv is the curvature tensor, since the Levi Civita connection has no torsion.

We have:

$$\nabla_\theta(\mathrm{grad}\ V_\theta)[q] = \mathrm{grad}\ \frac{\partial V}{\partial \theta}(q) + \big(\nabla_{\frac{\partial q}{\partial \theta}} \mathrm{grad}\ V_\theta\big)(q)\ .$$

The last term we have to compute is $\nabla_\theta\big[D_\theta(\frac{\partial q}{\partial t})\big]$. This case is more involved. For each fixed t, the mapping $\theta \in \Theta \longrightarrow D_\theta\big(\frac{\partial q}{\partial t}(t,\theta)\big)$ is a vector field along the curve $\theta \in \Theta \longrightarrow q(t,\theta)$. For simplicity denote by δ this field. Then the vector field $\frac{\partial \delta}{\partial \theta}$ in TTM along the curve $\theta \in \Theta \longrightarrow z(t,\theta)$ (t fixed) is given by the formula:

$$\frac{\partial \delta}{\partial \theta} = C(\frac{\partial q}{\partial \theta}, \delta) + j(\delta, \nabla_\theta \delta).$$

On the other hand we have the relation in TTM:

$$\frac{\partial \delta}{\partial \theta} = TD_\theta(\frac{\partial q}{\partial t})\frac{\partial^2 q}{\partial \theta \partial t} + j(\frac{\partial q}{\partial t}, \frac{\partial D_\theta}{\partial \theta}(\frac{\partial q}{\partial t}))$$

where $\frac{\partial^2 q}{\partial \theta \partial t}$ is the second derivative of q, $\frac{\partial^2 q}{\partial \theta \partial t} : I\!R \times \Theta \longrightarrow TTM$ and $TD_\theta(u)$ is the tangent mapping $T_u TM \longrightarrow T_u TM$ of D_θ.

We also have the equation (see formula (2), section A, after the proof of Proposition 3.5):

$$\frac{\partial^2 q}{\partial \theta \partial t} = C(\frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial t}) + j(\frac{\partial q}{\partial t}, \nabla_\theta \frac{\partial q}{\partial t}).$$

$\nabla_\theta \frac{\partial q}{\partial t} = \nabla_t \frac{\partial q}{\partial \theta}$ since the Levi Civita connection has no torsion. Hence:

$$\frac{\partial \delta}{\partial \theta} = TD_\theta(\frac{\partial q}{\partial t})C(\frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial t}) + TD_\theta(\frac{\partial q}{\partial \theta})j(\frac{\partial q}{\partial t}, \nabla_t \frac{\partial q}{\partial \theta}) + + j(\frac{\partial q}{\partial t}, \frac{\partial D_\theta}{\partial \theta}(\frac{\partial q}{\partial t}))$$

Since D_θ is a fiber mapping $TM \longrightarrow TM$, for any $(u,v) \in TM \times_M TM$ $TD_\theta(u)j(u,v)$ is a vertical vector in T_uTM.

Hence there exists a unique smooth mapping $d_vD_\theta : TM \times_M TM \longrightarrow TM$ called the vertical differential of D_θ such that:

$$TD_\theta(u)j(u,v) = j(u, d_vD_\theta(u)v).$$

d_vD_θ is intrinsic (i.e. independent of the connection):

$$d_vD_\theta(u)v = \frac{d}{d\lambda}[D_\theta(u+\lambda v)] \mid_{\lambda=0} .$$

It is easy to check that for any $(u,v) \in TM \times_M TM$ the vector

$$TD_\theta(u)C(u,v) - C(D_\theta(u),v)$$

is a vertical vector. Hence there is a unique mapping $\nabla_H D_\theta : TM \times_M TM \longrightarrow TM$ such that:

$$TD_\theta(u)C(u,v) - C(D_\theta(u),v) = j(\nabla_H D_\theta(u)v, D_\theta(u)).$$

Now we can define the tensors $R : TM \times_M TM \longrightarrow TM$ and $S : TM \times_M TM \longrightarrow TM$ as follows:

$$R(u)v = d_vD_0(u)v$$

$$S(u)v = \mathrm{Curv}(v,u)u - \nabla_H D_0(u)v + (\nabla_v \mathrm{grad}\ V_0)(\pi v)$$

and we get the equation

$$\nabla_t^2\chi - R(\dot{q}_0)\nabla_t\chi + S(\dot{q}_0)\chi = \frac{\partial D}{\partial\theta} \mid_{\theta=0} (\dot{q}_0) - \mathrm{grad}\ \frac{\partial V_\theta}{\partial \theta} \mid_{\theta=0} (q_0)$$

$$\dot{q}_0 = \frac{dq_0}{dt}.$$

Proof of Lemma 3.7: In the case where X_0 satisfies GI, the statement (i) is obvious. Hence we shall assume that X_0 satisfies the weaker property GIW only.

Let us denote by C_1 the projection of the set $C(z_0,z_0)$ on the first axis and by $\overline{C}_1$ its closure. We are going to study the structure of $\overline{C}_1$. Let τ_1 be an accumulation point of C_1. Then there exists a sequence $\{(t_1(n),t_2(n)) \mid n \in \mathbb{N}\}$ in $C(z_0,z_0)$ such that: (α) the sequence $\{t_1(n) \mid n \in \mathbb{N}\}$ converges to τ_1; (β) the sequence $\{t_2(n) \mid n \in \mathbb{N}\}$ either converges to a number τ_2 or it tends to $\pm\infty$.

In the first case of (β), the property GIW implies that either $z_0(\tau_1) = 0$ or $z_0(\tau_2) = 0$.

In the second case of (β), if we denote the projection $\pi \circ z_0$ of z_0 by q_0

$$q_0(\tau_1) = \lim_n q_0(t_1(n)) = \lim_n q_0(t_2(n)) = \begin{cases} \omega(z_0) & \text{if } t_2(n) \to +\infty \\ \alpha(z_0) & \text{if } t_2(n) \to -\infty. \end{cases}$$

This shows that the set of accumulation points of C_1 is contained in the subset B of all t in $\mathbb{R}$ such that

$$\frac{dq_0}{dt}(t) = 0 \quad \text{or} \quad q_0(t) = \begin{cases} w(z_0) \\ \text{or} \\ \alpha(z_0). \end{cases}$$

If we show that the set B_1 of all accumulation points of B is discrete, it will follow that $\overline{C}_1$ will be nowhere dense.

Let $(\tau(n) \mid n \in \mathbb{N})$ be a sequence in B converging to a number τ. Then there exists a subsequence $\{\tau(n_i) \mid i \in \mathbb{N}\}$ such that:

$$\text{either} \qquad \frac{dq_0}{dt}(\tau(n_i)) = 0 \ \text{ for all } i, \quad \text{or} \quad q_0(\tau(n_i)) = \begin{cases} \omega(z_0) & \text{for all } i \\ & \text{or} \\ \alpha(z_0) & \text{for all } i. \end{cases}$$

In all these cases the ∞–jet of q_0 at τ is the ∞–jet of the constant mapping: $t \in \mathbb{R} \longrightarrow q_0(\tau) \in M$. This implies that $O_{q_0(\tau)}$ is a singular point of the system and hence cannot be reached by the trajectory q_0 in finite time with zero end speed. We have a contradiction.

This finishes the proof of statement (i).

Proof of (ii). We shall present the proof for N_u. The case of N_s is similar.

If $\mathcal{V}$ and N_u did not exist one could find a sequence $\{(X_n, z_n) \mid n \in \mathbb{N}\}$ of fields X_n in $SDMS$ and trajectories z_n of X_n such that: $\alpha(z_n)$ exists; X_n converges to X_0; $z_n(\tau_u)$ converges to $z_0(\tau_u)$; the distance δ_n between the sets $q_n(]-\infty, \tau_u])$ and $q_n([t_1, t_2])$ tends to 0 as n goes to ∞.

Take a compact neighbourhood A of the singular points of X_0 in M such that $A \cap q_0([\tau_u, t_2]) = \emptyset$. For n sufficiently large, $n \geq n_0$ say, Sing $(X_n) \subset A$ and hence there will exist a $T > 0$ such that $q_n(]-\infty, T]) \subset A$ and $T < \tau_u$. This implies that the distance δ'_n between $q_n([T, \tau_u])$ and $q_n([t_1, t_2])$ tends to 0. Since the restrictions of q_n to $[T, \tau_u]$ and $[t_1, t_2]$ tend uniformly to the restrictions of q_0 to the same intervals respectively, then $q_0([T, \tau_u]) \cap q_0([t_1, t_2])$ is not empty. This contradicts the choice of $[t_1, t_2]$ to be without multiple points.

Proof of Lemma 3.8: Let X be any vector field along an arc of the trajectory q_0 such that $P_0 X = P_0^* X = 0$.

This means:
$$\begin{cases} \nabla_t^2 X - R(\dot{q}_0)\nabla_t X + S(\dot{q}_0)X = 0 \\ \nabla_t^2 X + \nabla_t R(\dot{q}_0)^* X + S(\dot{q}_0)^* X = 0 \end{cases}$$

Multiplying scalarly by X:

$$< \nabla_t^2 X, X > - < R(\dot{q}_0)\nabla_t X, X > + < S(\dot{q}_0)X, X >= 0$$
$$< \nabla_t^2 X, X > + < \nabla_t R(\dot{q}_0)^* X, X > + < S(\dot{q}_0)^* X, X >= 0$$

Subtracting the second relation from the first:

$$< R(\dot{q}_0)\nabla_t X, X > + < X, \nabla_t R(\dot{q}_0)^* X >= 0$$

or
$$\frac{d}{dt} < R(\dot{q}_0)X, X >= 0.$$

Assume now that $P_0^* \dot{q}_0 = 0$ on an arc $[\tau, +\infty[$. Since $P_0\dot{q}_0 = 0$, we get:

$$\frac{d}{dt} < R(\dot{q}_0)\dot{q}_0, \dot{q}_0 >= 0 \quad \text{on} \quad [\tau, +\infty[\,.$$

Integrating between t and $+\infty$

$$< R(\dot{q}_0(t))\dot{q}_0(t), \dot{q}_0(t) >= \lim_{s \to +\infty} < R(\dot{q}_0(s))\dot{q}_0(s), \dot{q}_0(s) > \ .$$

Now as s tends to $+\infty$, $\dot{q}_0(s)$ tends to $O_{\omega(z_0)}$ and since $R(u) = d_v D_0(u)$, $R(\dot{q}_0(s))$ tends to $d_v D_0(O_{\omega(z_0)})$. This means that the limit above is zero and

$$< R(\dot{q}_0(t))\dot{q}_0(t), \dot{q}_0(t) >= 0 \quad \text{for all} \quad t \geq \tau.$$

Now for any $v \in TM$

$$< d_v D(O_{\pi(v)})v, v > \leq -\alpha\|v\|^2.$$

Hence by continuity there is a positive number δ such that if $(u, v) \in TM \times_M TM$ $\|u\| \leq \delta$:
$$< R(u)v, v > \leq -\frac{\alpha}{2}\|v\|^2.$$

The relation above shows that

$$\dot{q}_0(t) = 0 \quad \text{for} \quad t \geq \tau.$$

This is a contradiction.

The same line of reasoning can be applied to intervals of the form $]-\infty, \tau]$ when $\alpha(z_0)$ exists.

We will prove now the first main openness theorem of the section:

Theorem 1.5. *The set of all systems X in $SDMS$ such that their stable and unstable manifolds are pairwise transversal is open in $SDMS$.*

The proof of this theorem will result from the Lemma below which we shall state now and prove later. For any field X in $SDMS$, let us call chain of X an ordered sequence $(\gamma_0, \gamma_1, \dots, \gamma_N)$ of trajectories of X such that $\omega(\gamma_i) = \alpha(\gamma_{i+1})$, $0 \leq i \leq N-1$ and $\alpha(\gamma_0)$ exists. The support of the chain will be the curve $\overline{\gamma_0} * \overline{\gamma_1} * \dots * \overline{\gamma_N}$ concatenation of the closures $\overline{\gamma_i} = \gamma_i \cup \{\alpha(\gamma_i), \omega(\gamma_i)\}$ of the γ_i's.

Lemma 3.9. *(i) Let $\{(X_n, \gamma^n) \mid n \in I\!N\}$ be a sequence of fields X_n in $SDMS$ and of trajectories γ^n of X_n such that the $\alpha(\gamma^n)$ all exist and the sequence X_n converges to a field X_∞ in $SDMS$. Then any limit set of the sequence of compact curves $\overline{\gamma^n}$ in the Hausdorff topology is the support of a chain of X_∞.*

(ii) The sequence (X_n, γ^n) being as in (i), assume that 1) The sets $\overline{\gamma^n}$ converge to the support of a chain $(\gamma_0, \dots, \gamma_N)$ of X_∞; 2) All the invariant manifolds of X_∞ are pairwise transversal.

Then, given any sequence of points (z_n), such that $z_n \in \gamma^n$, converging to a z_∞ which is not a singular point of X_∞, any limit plane L^u (resp. L^s) of the sequence $T_{z_n} W^u_{X_n}(\alpha(\gamma^n))$ (resp. $T_{z_n} W^s_{X_n}(\omega(\gamma^n))$) contains $T_{z_\infty} W^u_{X_\infty}(\alpha(\gamma_i))$ (resp. $T_{z_\infty} W^s_{X_\infty}(\omega(\gamma_i))$), where γ_i is the trajectory of the chain on which z_∞ lies.

Proof of the theorem: Were the theorem not true, there would exist a sequence $\{(X_n, \gamma^n) \mid n \in I\!N\}$ of fields X_n in $SDMS$ and of trajectories γ^n of X_n such that:

1) sequence (X_n) converges to a field X_∞ in $SDMS$ such that its stable and unstable manifolds are pairwise transversal.

2) $\alpha(\gamma^n)$ exist for all n and at any point z on γ^n $T_z W^u_{X_n}(\alpha(\gamma^n))$ and $T_z W^s_{X_n}(\omega(\gamma^n))$ are not transversal (they are either transversal at all points on γ^n or not transversal at all points on γ^n).

The union $\bigcup_n \overline{\gamma^n}$ is relatively compact in TM. Then by taking a subsequence of (X_n, γ^n) we can assume, using the compactness of the Hausdorff space of a compact metric space, that the compact sets $\overline{\gamma^n}$ converge in the Hausdorff metric. The limit will be the support of a chain $(\gamma_0, \dots, \gamma_N)$ of X_∞ by the statement (i) of Lemma 3.9.

Now, taking another subsequence of the sequence (X_n, γ^n), we can assume that each γ^n carries a point z_n such that the sequence (z_n) converges to a point z_∞ non singular for X_∞ and the sequences of spaces $(T_{z_n} W^u_{X_n}(\alpha(\gamma^n)))$

and $(T_{z_n}W^s_{X_n}(\omega(\gamma^n)))$ converge to the subspaces L^s and L^u of T_zTM respectively. By the statement (ii) of Lemma 3.9, $L^u \supset T_{z_\infty}W^u_{X_\infty}(\alpha(\gamma_i))$, $L^s \supset T_{z_\infty}W^s_{X_\infty}(\omega(\gamma_i))$ where γ_i is the trajectory of the chain containing z_∞. Since $W^u_{X_\infty}(\alpha(\gamma_i))$ and $W^s_{X_\infty}(\omega(\gamma_i))$ are transversal, so are L^u and L^s. This means that the canonical projection $\pi : L^u \longrightarrow T_{z_\infty}TM/L^s$ is onto. But the canonical projections $\pi_n : T_{z_n}W^u_{X_n}(\alpha(\gamma^n)) \longrightarrow T_{z_n}TM/T_{z_n}W^s_{X_n}(\omega(\gamma^n))$ converge to π. Hence for n big enough π_n will be surjective. This contradicts the fact that $T_{z_n}W^u_{X_n}\alpha(\gamma^n)$ and $T_{z_n}W^s_{X_n}(\omega(\gamma^n))$ are not transversal.

Proof of Lemma 3.9: (i) Assume that the sequence of compact sets $\overline{\gamma^n}$ converges to a compact set K_∞ in the Hausdorff metric. K_∞ will be a union of closures of trajectories of X_∞. As the limit of the compact connected sets $\overline{\gamma^n}$ it will also be connected. To show that K_∞ is the support of a chain it is sufficient to show that it cuts every energy level surface $\{E_{X_\infty} = h\}$ in at most one point.

Let R denote the slice $\{h - \eta \leq E_{X_\infty} \leq h + \eta\}$ of M, $\eta > 0$ being chosen sufficiently small so that the interval $[h-\eta, h+\eta]$ does not contain any critical value of the energy. Then there exists a neighbourhood $\mathcal{U}_\infty$ of X_∞ such that any trajectory γ of X either does not meet R or the intersection $R \cap \gamma$ is an arc $\widehat{\gamma}$ meeting all the level surfaces $\Sigma_t = \{E_{X_h} = h + t\}$, $-\eta \leq t \leq \eta$, transversally in one point. We can also choose $\mathcal{U}_\infty$ sufficiently small so that there exists a constant C such that for any X in $\mathcal{U}_\infty$, any trajectory γ of X meeting R, any z in Σ_0,

$$d_M(z, z(\gamma)) \leq C\ d_M(z, \widehat{\gamma})$$

where $z(\gamma)$ is the intersection point of γ with Σ_0. It is also clear that if δ denotes the distance between Σ_0 and the boundary of R, as soon as $d(z, \gamma) < \delta$,

$$d(z, \gamma) = d(z, \widehat{\gamma}).$$

We can assume that the γ^n meet Σ_0, otherwise $K_\infty \cap \Sigma_0$ is empty.

As soon as $d(\overline{\gamma}^n, \overline{\gamma}^m) < \delta$,

$$d(z_n, \gamma_m) = d(z_n, \widehat{\gamma}_m),$$

where for simplicity we set $z_k = z(\gamma_k)$, $k \in I\!N$.

By the inequality above,

$$d(z_n, z_m) \leq C\ d(z_n, \widehat{\gamma}_m).$$

Hence as soon as $d(\overline{\gamma}^n, \overline{\gamma}^m) < \delta$

$$d(z_n, z_m) \leq C\ d(z_n, \gamma_m) \leq C\ d(z_n, \widehat{\gamma}_m).$$

This shows that the sequence $\{z_n \mid n \in I\!N\}$ is a Cauchy sequence and hence has a unique limit point z_∞. It is clear that $K_\infty \cap \Sigma_0$ contains z_∞. But in fact $K_\infty \cap \Sigma_0 = \{z_\infty\}$. For if z' is in $K_\infty \cap \Sigma_0$, it is a limit point of a sequence $\{z'_h \mid h \in \mathbb{N}\}$, where z'_h lies on some γ_{n_h}. But then z' is the limit of the sequence $\{z_{n_h} \mid h \in \mathbb{N}\}$, which converges to z_∞.

To Prove (ii), it is sufficient to consider the unstable case. We proceed by induction on the index i of the trajectory γ_i to which z_∞ belongs. If i is 0, z_∞ belongs to γ_0. Since $\alpha(\gamma^n)$ tends to $\alpha(\gamma_0)$, $T_{z_n} W^u_{X_n}(\alpha(\gamma^n))$ tends to $T_{z_\infty} W^u_{X_\infty}(\alpha(\gamma_0))$. For an arbitrary $i \geq 1$, denote by O_∞ the singular point $w(\gamma_{i-1}) = \alpha(\gamma_i)$. We are going to choose an appropriate sequence of points (y_n) such that $y_n \in \gamma^n$, the sequence (y_n) converges to y_∞ on γ_{i-1} and the planes $T_{y_n} W^u_{X_n}(\alpha(\gamma^n))$ converge to a limit L^u containing $T_{y_\infty} W^u_{X_\infty}(\alpha(\gamma_{i-1}))$. To prove this we are going to compare the spaces $T_{z_n} W_{X_n}(\alpha(\gamma^n))$ with the spaces $T_{y_n} W^u_{X_n}(\alpha(\gamma^n))$. To do this we shall establish the following form of the λ–Lemma (see Palis [8]).

Let us denote by O_∞ the singular point $w(\gamma_{i-1}) = \alpha(\gamma_i)$ of X_∞. There exist an open neighbourhood $\mathcal{U}$ of X_∞, an open neighbourhood Ω of O_∞, and a mapping $X \in \mathcal{U} \longrightarrow \xi_X \in \mathrm{Diffeo}(\Omega, S \times U)$, where S, U are vector spaces with dim $U =$
dim $W^u_{X_\infty}(O_\infty)$, dim $S =$ dim $W^s_{X_\infty}(O_\infty)$ such that:

1) Each $X \in \mathcal{U}$ has a unique singular point O_X in Ω and $\xi_X(O_X) = 0$.
2) For any $X \in \mathcal{U}$, $\xi_X(\Omega \cap W^u_X(O_X)) = U \cap \widehat{\Omega}_X$, $\xi_X(\Omega \cap W^s_X(O_X)) = \widehat{\Omega}_X \cap S$, where $\widehat{\Omega}_X = \xi_X(\Omega)$.

Let us denote by X_u (resp. X_s) the $U-$ (resp. $S-$) component of the image field $\widehat{X} = \xi_{X*}(X)$. By condition 2) above, there exist smooth mappings $X'_u : \widehat{\Omega}_X \longrightarrow \mathrm{End}(U)$ and $X'_s : \widehat{\Omega}_X \longrightarrow \mathrm{End}(S)$, such that $X_u(x,y) = X'_u(x,y)x$ and $X_s(x,y) = X'_s(x,y)y$ for all (x,y) in $\widehat{\Omega}_X$. Since $d\widehat{X}_\infty(0)$ is hyperbolic there exist a scalar product $<|>$ on $U \times S$ and positive constants a_s, a_u, b such that by restricting $\mathcal{U}$ and Ω if necessary for all X in $\mathcal{U}$, all (x,y) in $\widehat{\Omega}_X$, all $(u,v) \in U \times S$,

$$
\text{(I)} \quad \begin{cases}
< \dfrac{\partial X_u}{\partial x}(x,y)u \mid u > \geq a_u < u \mid u > \\[2ex]
< \dfrac{\partial X_s}{\partial y}(x,y)v \mid v > \leq -a_s < v \mid v > \\[2ex]
< X'_s(x,y)v \mid v > \leq -a_s < v \mid v > \\[2ex]
a_u < u \mid u > \leq < X'_u(x,y)u \mid u > \leq b < u \mid u >
\end{cases}
$$

By condition 2) above, for all X in $\mathcal{U}$, all $x \in \widehat{\Omega}_X \cap U$, all y in $\widehat{\Omega}_X \cap S$

$$X_u(0,y) = 0, \quad X_s(x,0) = 0.$$

Hence there exists a constant C such that:

$$\text{(II)} \qquad \|\frac{\partial X_u}{\partial y}(x,y)\| \leq C\|x\|, \quad \|\frac{\partial X_s}{\partial x}(x,y)\| \leq C\|y\|$$

for all X in $\mathcal{U}$, all (x,y) in $\widehat{\Omega}_X$.

For any X denote by $\varphi_{X,t}$ the flow of $\widehat{X}$ in $\widehat{\Omega}_X$ and by $T\varphi_{X,t}$ the derived flow on $T\widehat{\Omega}_x$. If E is a subspace of $T_{(x_0,y_0)}(U \times S)$ of the same dimension as U and transversal to $T_{x_0}S \subset T_{(x_0,y_0)}(U \times S)$ then it can be represented as the graph of a linear mapping $\Gamma_0 : T_{x_0}U \longrightarrow T_{y_0}S$. If its image $T\varphi_{X,t}(E)$ at time t is still transversal to $T_{\varphi_t(x_0,y_0)}S$, let $\Gamma(t)$ denote the mapping whose graph this image is.

We have the following estimates of the norm $\|\Gamma(t)\|$ of $\Gamma(t)$ as t varies:

Lemma 3.10: *For any field X in $\mathcal{U}$, any trajectory*

$$\{\varphi_{X,t}(x_0,y_0) = (x(t),y(t)) \mid T_- < t \leq T_+\} \text{ of } \xi_{X*}(X),$$

we have the following estimates of the norm of $\Gamma(t)$, $t \geq 0$:

(i) If $\|x(t)\| \leq a_s/2C[\|\Gamma_0\| + \frac{C}{a_u}\|y_0\|]$, *then,*

$$\|\Gamma(t)\| \leq 2[\|\Gamma_0\| + \tfrac{C}{a_u}\|y_0\|]\left(\tfrac{\|x_0\|}{\|x(t)\|}\right)^{a_s/b}$$

(ii) $\|\Gamma(t)\| \leq 6[\|\Gamma_0\| + \frac{C}{a_u}\|y_0\|]\left(\frac{\|x_0\|}{\|x(t)\|}\right)^{a_s/b}$

provided that: $\|x_0\| \leq \left(\frac{a_s}{2C[\|\Gamma_0\| + \frac{C}{a_u}\|y_0\|]}\right)^{1+b/a_s}\left[\frac{1}{\|x(t)\|^{b/a_s}}\right]$.

We shall prove this Lemma below after we finish the proof of Lemma 3.9-(ii). We can always assume by deleting a finite subset of the X_n and by sliding the z_n along their trajectories γ^n that all the X_n belong to $\mathcal{U}$ and all the z_n to Ω (z_∞ included). We can find a sequence (q_n) of points on the trajectories γ^n, q_n preceeding z_n for every n, such that: $q_n \in \Omega$ for all n, q_n converges to a point q_∞ on $\Omega \cap \gamma_{i-1}$. Also by taking a subsequence of the X_n we can assume that $T_{q_n}W^u_{X_n}(\alpha_{X_n}(\gamma^n))$ converges to a limit Λ^u in $T_{q_\infty}TM$. By induction assumption, Λ^u contains the space $T_{q_\infty}W^u_{X_\infty}(\alpha(\gamma_{i-1}))$ and hence is transversal to $T_{q_\infty}W^s_{X_\infty}(\omega(\gamma_{i-1}))$.

Using the mappings ξ_{X_n}, the space $E_n = T\xi_{X_n}(T_{q_n}W^u_{X_n}(\alpha_{X_n}(\gamma^n)))$ in $T_{(x_n,y_n)}(U \times S)$, where $(x_n,y_n) = \xi_{X_n}(q_n)$, will converge to the space $E_\infty = \xi_{X_\infty}(\Lambda^u)$ in $T_{(x_\infty,y_\infty)}(U \times S)$ where $(0,y_\infty) = \xi_{X_\infty}(q_\infty)$. Since E_∞ is transversal to $T_{x_\infty}S$, for n big enough E_n will be transversal to $T_{x_n}S$. The orthogonal complement F_n of $E_n \cap T_{x_n}S$ in E_n will be the graph of a mapping $\Gamma_n : T_{x_n}U \longrightarrow T_{x_n}S$ and will converge to the orthogonal complement F_∞ of $E_\infty \cap T_{x_\infty}S$ in E_∞, graph of a mapping $\Gamma_\infty : T_{x_\infty}U \longrightarrow T_{x_\infty}S$.

Let $(x'_n, y'_n) = T\xi_{X_n}(z_n)$ and let $t_n > 0$ be the time such that $z_n = e^{t_n X_n}(q_n)$ or $(x'_n, y'_n) = \varphi_{X_n, t_n}(x_n, y_n)$. Since the space

$$G_n = T\xi_X(Tz_n W^u_{X_n}(\alpha_{X_n}(\gamma^n)))$$

is the image $T\varphi_{X_n,t_n}(E_n)$ of E_n under the flow of X_n, it contains the space $T\varphi_{X_n,t_n}(F_n)$. This space is the graph of a mapping $\Gamma_n(t_n) : T_{x'_n}U \longrightarrow T_{y'_n}S$.

By Lemma 3.10, the norm $\|\Gamma_n(t_n)\|$ of $\Gamma(t_n)$ is bounded by

$$6[\|\Gamma_n\| + \|y_n\|](\frac{\|x_n\|}{\|x'_n\|})^{a_s/a}$$

provided that $\|x_n\| \leq \left(\frac{a_s}{2c}\right)^{1+b/a_s} / \|x'_n\|[\|\Gamma_n\| + \|y_n\|]^{1+b/a_s}$.

Since as n goes to ∞, $\|\Gamma_n\|$, $\|y_n\|$, $\|x'_n\|$ converges to $\|\Gamma_\infty\|$, $\|y_\infty\|$, $\|x'_\infty\|$ $(\xi_{X_\infty}(z_\infty) = (x'_\infty, 0))$ and $\|x_n\|$ tends to 0, it follows that $\|\Gamma_n(t_n)\|$ tends to 0. Hence the sequence of spaces $T\varphi_{X_n,t_n}(F_n)$ tends to $T_{x'_\infty}U = T\xi_{X_\infty}(T_{z_\infty}W^u_{X_\infty}(O_\infty)$. Hence L^u contains $T_{z_\infty}W^u_{X_\infty}(O_\infty)$.

Proof of Lemma 3.10: The differential system associated with $\widehat{X} = (\xi_X)_*(X)$ is:

$$\frac{dx}{dt} = X_u(x,y) = X'_u(x,y)x \qquad \frac{dy}{dt} = X_s(x,y) = X'_s(x,y)y.$$

The linearized system along a trajectory $\varphi_{X,t}(x_0, y_0) = (x(t), y(t))$ is:

$$\frac{d\xi}{dt}(t) = \frac{\partial X_u}{\partial x}(x(t), y(t))\xi(t) + \frac{\partial X_u}{\partial y}(x(t), y(t))\eta(t)$$

$$\frac{d\eta}{dt}(t) = \frac{\partial X_s}{\partial x}(x(t), y(t))\xi(t) + \frac{\partial X_s}{\partial y}(x(t), y(t))\eta(t).$$

Then $\Gamma(t)$ satisfy the Ricatti equation

$$\begin{aligned}\frac{d\Gamma}{dt}(t) = {} & \frac{\partial X_s}{\partial x}(x(t), y(t)) + \frac{\partial X_s}{\partial y}(x(t), y(t))\Gamma(t) \\ & - \Gamma(t)\frac{\partial X_u}{\partial x}(x(t), y(t)) - \Gamma(t)\frac{\partial X_u}{\partial y}(x(t), y(t))\Gamma(t) \ .\end{aligned}$$

For any t_0 $0 \leq t_0 \leq t$, its solutions satisfy:

$$\begin{aligned}\Gamma(t) = {} & R_s(t, t_0)\Gamma(t_0)R_u^{-1}(t, t_0) + \int_{t_0}^{t} R_s(t, \tau)\frac{\partial X_s}{\partial x}(x(\tau), y(\tau))R_u^{-1}(t, \tau)d\tau \\ & - \int_{t_0}^{t} R_s(t, \tau)\Gamma(t)\frac{\partial X_u}{\partial y}(x(\tau), y(\tau))\Gamma(\tau)R_u^{-1}(t, \tau)d\tau\end{aligned}$$

where R_s, R_u are the resolvent mappings

$$\begin{array}{lcl} R_s : S \longrightarrow S & & R_u : U \longrightarrow U \\ \dfrac{\partial R_s}{\partial t}(t,t_0) & = & \dfrac{\partial X_s}{\partial y}(x(t),y(t))R_s(t,t_0) \\ \dfrac{\partial R_u}{\partial t}(t,t_0) & = & \dfrac{\partial X_u}{\partial x}(x(t),y(t))R_u(t,t_0) \\ R_s(t_0,t_0) = Id_s & & R_u(t_0,t_0) = Id_u \ . \end{array}$$

The inequalities (I) show that

$$\begin{array}{ll} \|R_s(t,t_0)\| \leq e^{-a_s(t-t_0)} & t \geq t_0 \\ \|\overline{R}_u^1(t,t_0)\| \leq e^{-a_u(t-t_0)} & t \geq t_0 \ . \end{array}$$

The relation (III) and the inequalities (II) imply if $a = a_s + a_u$:

$$\text{(IV)} \quad \|\Gamma(t)\| \leq e^{-a(t-t_0)}\|\Gamma(t_0)\| + \int_{t_0}^t C\|y(t)\|e^{-a(t-\tau)}d\tau + \int_{t_0}^t C\|x(\tau)\|\|\|\Gamma(\tau)\|^2 e^{-a(t-\tau)}d\tau \ .$$

The inequalities (I) imply that:

$$\text{(V)} \quad \begin{cases} e^{a_u(t-t_0)}\|x(t_0)\| \leq & \|x(t)\| \leq x(t_0)\|e^{b(t-t_0)} \\ & \|y(t)\| \leq \|y(t_0)\|e^{-a_s(t-t_0)} \ . \end{cases}$$

Hence:

$$\int_{t_0}^t \|y(\tau)\|e^{-a(t-\tau)}d\tau \leq \frac{1}{a_u}\|y(t_0)\|e^{-a_s(t-t_0)} \ .$$

By multiplying (IV) by $e^{a_s(t-t_0)}$ and setting $\gamma(t) = \|\Gamma(t)\|e^{a_s(t-t_0)}$, for simplicity we obtain:

$$\text{(VI)} \qquad \|\gamma(t)\| \leq \|\Gamma(t_0)\| + \frac{C}{a_u}\|y(t_0)\| + \int_{t_0}^t C\|x(\tau)\|e^{-a_s(t-\tau)}\gamma(\tau)^2 d\tau \ .$$

The simple Lemma 3.11 below implies that:

$$\text{(VII)} \qquad \|\gamma(t)\| \leq 2[\|\Gamma(t_0)\| + \frac{C}{a_u}\|y(t_0)\|]$$

$$\text{if} \qquad [\|\Gamma(t_0)\| + \frac{C}{a_u}\|y(t_0)\|] \sup_{t_0 \leq \tau \leq t} \|x(\tau)\| \leq \frac{a_s}{2C}.$$

But the inequality (V) implies that $x(t)$ is an increasing function. Hence:

(VIII) (VII) is valid when $\|x(t)\|[\|\Gamma_0\| + \frac{C\|y_0\|}{a_u}] \leq \frac{a_s}{2C}$.

Applying (VIII) with $t_0 = 0$, we get that:

$$\|\Gamma(t)\| \leq 2[\|\Gamma_0\| + \frac{C\|y_0\|}{a_u}\|].e^{-a_s t} .$$

Since by (V), $e^{-bt} \leq \frac{\|x_0\|}{\|x(t)\|}$ we get:

$$\|\Gamma(t)\| \leq 2[\|\Gamma_0\| + \frac{C}{a_u}\|y_0\|](\frac{\|x_0\|}{\|x(t)\|})^{a_s/b} .$$

This is the first inequality of Lemma 3.10.
Now if $\|x(t)\| > a_s / 2C[\|\Gamma_0\| + \frac{C}{a_u}\|y_0\|]$, let t_1 be the instant such that:

$$\|x(t_1)\| = a_s / 2CA$$

where for simplicity we set $A = \|\Gamma_0\| + \frac{C}{a_u}\|y_0\|$.

Applying the first part of Lemma 3.10 just proved, to $t_0 = 0$ and $t = t_1$ we get:

$$\|\Gamma(t_1)\| \leq 2A\|x_0\|^{a_s/b}(\frac{2CA}{a_s})^{a_s/b} . \tag{IX}$$

Using the inequalities (V) we have:

$$\|y(t_1)\| \leq \|y_0\|e^{-a_s t_1} \leq \|y_0\|(\frac{\|x_0\|}{\|x(t_1)\|})^{a_s/b} \qquad \text{, so}$$

$$\|y(t_1)\| \leq \|y_0\|(\frac{2CA}{a_s})^{a_s/b}\|x_0\|^{a_s/b} . \tag{X}$$

Now we can apply (VIII) with $t_0 = t_1$ and we get:

$$\|\Gamma(t)\| \leq 2[\|\Gamma(t_1)\| + \frac{C}{a_u}\|y(t_1)\|]e^{-a_s(t-t_1)}$$

$$\text{provided that} \quad \|x(t)\|[\|\Gamma(t_1)\| + \frac{C}{a_u}\|y(t_1)\|] \leq \frac{a_s}{2C}.$$

This last condition can be expressed as follows, using (IX) and (X):

$$[2A + \frac{C}{a_u}\|y_0\|](\frac{2CA}{a_s})^{a_s/b}\|x_0\|^{a_s/b} \leq \frac{a_s}{2C\|x(t)\|} .$$

This proves the second inequality in Lemma 3.10.

Lemma 3.11. *Let* $z : [t_0, t_1] \longrightarrow \mathbb{R}_+$ *be a continuous function satisfying for all* $t \in [t_0, t_1]$ *the inequality:*

$$z(t) \leq \alpha + \int_{t_0}^{t} b(\tau)z(\tau)^2 d\tau$$

where α *is a constant and* $b : [t_0, t_1] \longrightarrow \mathbb{R}_+$ *a positive continuous function. Then:* $z(t) \leq 2\alpha$ *for all* $t \in [t_0, t_1]$ *such that* $\int_{t_0}^{t} b(\tau)d\tau \leq \frac{1}{2\alpha}$.

Finally, we are able to prove the main density theorem of the section:

Theorem 1.6. *Assume* $\dim M > 1$ *and* $r > 3(1 + \dim M)$ *and let* $\mathcal{G}$ *be the subset of* $SDMS(D)$ *(resp.* $SDMS(V)$*) of all systems* X *such that their invariant manifolds are pairwise transversal. Then* $\mathcal{G}$ *is open dense in* $SDMS(D)$ *(resp.* $SDMS(V)$*).*

Proof. Since we know by theorem 1.5 that $\mathcal{G}$ is open, it is sufficient to prove that $\mathcal{G}$ is everywhere dense in $SDMS(D)$ (resp. $SDMS(V)$). As before we shall give the proof in the first case only. The second case is similar but easier. Since the set $GIW(D)$ is dense in $SDMS(D)$, it is sufficient to prove the following: every X_0 in $GIW(D)$ has an open neighbourhood $\mathcal{V}_0$ such that $\mathcal{V}_0 \cap \mathcal{G}$ is a Baire subset of $\mathcal{V}_0$.

To start, if $O_{1,X_0}, \ldots, O_{N,X_0}$ denote the singular points of X_0, we can find neighbourhoods $\Omega_1, \ldots, \Omega_N$ of $O_{1,X_0}, \ldots, O_{N,X_0}$ respectively and constants $\alpha_1, \ldots, \alpha_N$ such that for each i, the manifold $\Sigma_i = \Omega_i \cap \{E_{X_0} = \alpha_i\}$ satisfies the following conditions:

1) Σ_i is transversal to X_0;
2) $\Sigma_i \cap W^u_{X_0}(O_{iX_0})$ is a compact connected manifold;
3) Each trajectory of X_0 in $W^u_{X_0}(O_{iX_0})$ cuts Σ_i in one and only one point.

Then we can find an open neighbourhood $\mathcal{V}_1$ of X_0 in $SDMS(D)$ satisfying the statement (i) of Proposition 3.4 and such that for any X in $\mathcal{V}_1$ the conditions 1-2-3 are satisfied if we replace $W^u_{X_0}(O_{iX_0})$ and X_0 by $W^u_X(O_{iX})$ and X respectively, in them.

Applying Proposition 3.4 and using the compactness of the sets

$$\Sigma_i \cap W^u_{X_0}(O_{iX_0})$$

we can find, for each i, n_i pairs $(\mathcal{U}_0^{h,i}, N_0^{ki})$ of an open neighbourhood $\mathcal{U}_0^{k,i}$ of X_0, an open set $N_0^{k,i}$ satisfying the assertions of Proposition 3.4 with respect to the pair $(\mathcal{U}_0, N_0)$ and such that the $\{N_0^{k,i} \mid 1 \leq k \leq n_i\}$ cover $\Sigma_i \cap W^u_{X_0}(O_{iX_0})$. We can always restrict $\mathcal{V}_1$ so that for any $X \in \mathcal{V}_1$ and any i, $1 \leq i \leq N$,

$\Sigma_i \cap W_X^u(O_{iX})$ is contained in $\bigcup_{k=1}^{n_i} N_0^{k,i}$. Then we can restrict the $\mathcal{U}_0^{k,i}$ so that $\mathcal{U}_0^{k,i} \subset \mathcal{V}_1$ for all k, i.

Proposition 3.4 states that the subset $\mathcal{G}^{k,i}$ of $\mathcal{U}_0^{k,i}$ of all systems X such that $W_X^u(O_{iX}) \cap N_0^{k,i}$ is transversal to all the stable manifolds of X is a Baire subset of $\mathcal{U}_0^{k,i}$. The set $\mathcal{V} = \bigcap_{i=1}^{N} \bigcap_{k=1}^{n_i} \mathcal{U}_0^{k,i}$ is an open neighbourhood of X_0 in $SDMS(\mathrm{D})$ and the intersection $\bigcap_{i=1}^{N} \bigcap_{k=1}^{n_i} \mathcal{G}^{k,i}$ is a Baire subset of $\mathcal{V}$. But the condition 3) on the Σ_i (valid for all X in $\mathcal{V}_1$) implies that this intersection is $\mathcal{G} \cap \mathcal{V}$.

4 - Proof of Theorem 1.7.

As we said in the Introduction, the main arguments in the proof of Theorem 1.7 follow the lines of [8]; we include them in the paper for completeness of exposition. Throughout the proof we implicitely assume D to be complete.

The following facts are more or less standard, some of them are remarks already made and a complete proof can be found in [3]. Denote by $\mathcal{A} = \mathcal{A}(V, D)$, $(V, D) \in DMS$, the attractor of (V, D), that is, $\mathcal{A} = \{v \in TM \mid \text{the trajectory of } (V, D) \text{ through } v \text{ is bounded}\}$. Then

i) $\mathcal{A}$ is connected and is the largest compact invariant set;
ii) $\mathcal{A}$ is uniformly asymptotically stable set for the flow on TM;
iii) $\mathcal{A}(V, D)$ is an upper semicontinuous function of (V, D) in DMS;
iv) If $f = e^X$ is the time one map associated to (V, D) and

$$\mathcal{B}_a = \{v \in TM \mid E(v) < a\}$$

then, for a sufficient large $a > 0$,

$$\mathcal{A} = \bigcap_{n \geq 0} f^n(\mathcal{B}_a);$$

v) The map $\pi_M/\mathcal{A} : \mathcal{A} \to M$ is surjective;
vi) If $(V, D) \in SDMS$, that is, (V, D) is strongly dissipative, then $\mathcal{A}$ is the union of the unstable manifolds of all (finite number) singular points.

Lemma 4.1. *Let $(V, D) \in \mathcal{G}$, $P \in Sing\,(V, D)$ and $dim\, W^u(P) = n$. Fix a n–disc B_n^u centered at P contained in $W_{\mathrm{loc}}^u(P)$. Given $\varepsilon > 0$, there exist neighbourhoods U of P and W of (V, D) in $SDMS$ such that if $(\overline{V}, D) \in W$, $Q \in Sing\,(V, D)$ and $Q^* \in Sing\,(\overline{V}, D)$ is the corresponding singular point near Q and moreover, if $W^u(Q^*) \cap U \neq \emptyset$, then $W^u(Q^*) \cap U$ is fibered by n–discs ε–C^1 close to B_u^n.*

A partial order in the set Sing (V, D) of a strongly dissipative mechanical system (V, D) is the following (see [8], [14]):

$$P \leq Q \quad \text{iff} \quad \overline{W^u(Q)} \cap W^u(P) \neq \emptyset \quad \forall\, P, Q \in \mathrm{Sing}\,(V, D)$$

The **phase diagram** of (V,D) is $(\text{Sing}\,(V,D),\leq)$. If $P \leq Q$ there exists a chain $(P_1 = Q, P_2, \ldots, P_\ell = P)$ such that

$$W^u(P_j) \cap W^s(P_{j+1}) \neq \emptyset, \quad 1 \leq j \leq \ell - 1;$$

define depth $(Q \mid P)$ as maximum of the lenghts ℓ of all chains connecting Q to P; depth $(Q \mid P) = 0$ means that $W^u(Q) \cap W^s(P) = \emptyset$. Remark that if depth $(Q \mid P) = 1$ and $G^s(P)$ is a fundamental domain ($G^s(P)$ is the boundary of a cell $B_s(P)$ centered at P and contained in $W^s_{\text{loc}}(P)$) then $W^u(Q) \cap G^s(P)$ is compact. For any $Q \in \text{Sing}\,(V,D)$ there exists at least one maximal chain of lenght $n \geq 1, (P_1 = Q, \ldots, P_n)$, that is, P_n is a sink and depth $(P_j \mid P_{j+1}) = 1$, $j = 1, 2, \ldots, n-1$.

The next lemma is lemma 7.3 of [9], pg. 87:

Lemma 4.2. *Let P be a singular point of $(V,D) \in SDMS(D)$. There exists a neighbourhood $\widetilde{U}$ of P and a continuous map $\widetilde{\pi} : \widetilde{U} \to B_s$ where*

$$B_s = B_s(P) = \widetilde{U} \cap W^s_{\text{loc}}(P)$$

such that

1) *$\widetilde{\pi}^{-1}(P) = B_u = \widetilde{U} \cap W^u_{\text{loc}}(P)$ is a disc containing P;*
2) *for each $x \in B_s$, $\widetilde{\pi}^{-1}(x)$ is a C^r–submanifold of TM transversal to $W^s_{\text{loc}}(P)$ at the point x;*
3) *$\widetilde{\pi}$ is of class C^r except possibly at the points of B_u;*
4) *the fibration defined by $\widetilde{\pi}$ is invariant for the flow φ_t of the vector field defined by (V,D), that is, if $t \geq 0$ then*

$$\varphi_t(\widetilde{\pi}^{-1}(x)) \supset \widetilde{\pi}^{-1}(\varphi_t(x)), \quad \forall\, x \in B_s.$$

In proving lemmas 4.1 and 4.2 we really have an **Unstable Foliation** of $\widetilde{U}$ at $P \in \text{Sing}\,(V,D)$, $(V,D) \in \mathcal{G}$, that is, a continuous foliation

$$\mathcal{F}(P,\widetilde{U}) : x \in \widetilde{U} \to \mathcal{F}_x(P,\widetilde{U}) = \widetilde{\pi}^{-1}(\widetilde{\pi}(x)).$$

Moreover, this unstable foliation can be easily globalized through saturation by φ_t. This way we obtain a **global unstable foliation** $\mathcal{F}(P,U)$ where

$$U = \bigcup_{t \in \mathbb{R}} \varphi_t(\widetilde{U}),$$

and a projection $\pi : U \to W^s(P)$ given by $\pi \circ \varphi_t(p) = \varphi_t \circ \widetilde{\pi}(p)$, $p \in \widetilde{U}$, and such that:

a) the leaves are C^1 manifolds with tangent spaces varying continuously in the Grassmanian and

$$\mathcal{F}_P(U,P) = W^u(P);$$

b) the leaf $\mathcal{F}_x(P,U)$ containing $x \in U$ is equal to

$$\pi^{-1}(\pi(x));$$

c) $\mathcal{F}(P,U)$ is invariant for the flow φ_t of (V,D); that is,

$$\varphi_t(\mathcal{F}_x(P,U)) = \mathcal{F}_{\varphi_t(x)}(P,U), \quad t \in \mathbb{R}, \quad x \in U, \quad \text{or}$$
$$\pi \circ \varphi_t = \varphi_t \circ \pi \quad \text{in } U.$$

The same holds for $(\overline{V},D)$ near (V,D) in $\mathcal{G}$.

For any maximal chain $(P_1, P_2, \ldots, P_n)$ on the phase diagram of (V,D) we obtain, by induction, a **compatible system of global unstable foliations**,

$$(\mathcal{F}(P_1,U_1), \mathcal{F}(P_2,U_2), \ldots, \mathcal{F}(P_n,U_n))$$

and the associated projections

$$\pi_i : U_i \to W^s(P_i), \quad \pi_i \circ (\varphi_t/U_i) = \varphi_t \circ \pi_i, \quad i = 1,2,\ldots,n.$$

The compatibility means that if a leaf F of $\mathcal{F}(P_k,U_k)$ intersects a leaf $\widetilde{F}$ of $\mathcal{F}(P_\ell,U_\ell)$, $k < \ell \le n$, then $F \supset \widetilde{F}$; moreover, the restriction of $\mathcal{F}(P_\ell,U_\ell)$ to a leaf of $\mathcal{F}(P_k,U_k)$ is a C^1 foliation.

Consider again $(V,D) \in \mathcal{G}$ and fix $a > 0$, sufficiently large, such that the bounded set $\mathcal{B}_a$ contains O_M and the set $\mathcal{A}(V,D)$. We know that for any small $\varepsilon > 0$ there exists a neighbourhood W of (V,D) in G such that $\mathcal{A}(\overline{V},D)$ is contained in the ε-neighbourhood of $\mathcal{A}(V,D)$ in $\mathcal{B}_a$, for all $(\overline{V},D) \in W$. We may also assume that the vector field corresponding to $(\overline{V},D) \in W$ points inward at every point of $\partial\mathcal{B}_a$. $\mathcal{B}_a$ is a disc bundle in TM with sphere bundle $\partial\mathcal{B}_a$ and

$$\mathcal{B}_a = \bigcup_{P_i \in \mathrm{Sing}(V,D)} W^s(P_i) \cap \mathcal{B}_a.$$

From now on, in this section, we call $W^s(P) \cap \mathcal{B}_a$ the stable manifold of P which we denote simply by $W^s(P)$. Let us denote by $\overline{W}^s(P)$ the closure of $W^s(P)$ in $\mathcal{B}_a$. The topological boundary of $W^s(P)$ in $\mathcal{B}_a$ is $\partial W^s(P) = \overline{W}^s(P) - W^s(P)$. Then $x \in \partial W^s(P)$ if and only if there exists a sequence of points y_i in a fundamental domain $G^s(P)$ and $T_i \to -\infty$ as $i \to \infty$ such that

$$x = \lim_{i \to \infty} \varphi_{t_i}(y_i)$$

where φ_t denotes the flow corresponding to (V, D). Remark also that $\partial W^s(P)$ is positively invariant. If P, Q are two distinct points of Sing (VD) such that $\overline{W}^s(P) \cap W^s(Q) \neq \emptyset$, then $Q \in \overline{W}^s(P)$ and there exists $x \in W^s(P) \cap W^u(Q)$, $x \neq Q$; furthermore, by transversality condition dim $W^s(P) >$ dim $W^s(Q)$.

The following sequence L_i is similar to the one considered by Shashahani [13]: $L_0 = \emptyset$; L_1 is the union of all stable manifolds whose topological boundary is empty; for $i \geq 1$ one defines L_{i+1} to be the union of L_i with the union of all stable manifolds whose topological boundary is contained in L_i. It is clear that for all $i \geq 0$, L_i is closed, $L_{i+1} - L_i$ is a disjoint union of stable manifolds and $\phi = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_p = \mathcal{B}_a$.

Denote by P^* the singular point of $(\overline{V}, D)$ corresponding to $P \in$ Sing(V, D), for $(\overline{V}, D)$ near $(V, D) \in \mathcal{G}$.

We start now the construction of a homeomorphism h mapping the flow of the system (V, D) onto that of $(\overline{V}, D)$.

Take any $W^s(P_1) \in L_1$ and the corresponding $W^s(P_i^*)$. Since $W^s(P_1)$ and $W^s(P_1^*)$ are $\varepsilon - C^r$–close on compact sets (see [9], pg.75), for $(\overline{V}, D)$ near (V, D) there is a diffeomorphism

$$\widetilde{h}_1 : G^s(P_1) \to G^s(P_1^*)$$

and let us extend it to the full $W^s(P_1)$ using the flows φ_t and φ_t^* of (V, D) and $(\overline{V}, D)$. That is, if $x \in W^s(P_1)$, $x \neq P_1$, $t \in I\!R$ is the unique time t such that $\varphi_t(x) \in G^s(P_1)$, then we define $h_1(P_1) = P_1^*$ and $h_1(x) = \varphi_{-t}^* \circ \widetilde{h}_1 \circ \varphi_t(x) \in W^s(P_1^*)$. The map

$$h_1 : W^s(P_1) \to W^s(P_1^*)$$

is a homeomorphism (a diffeomorphism on $W^s(P_1) - \{P_1\}$).

Do the same for all stable manifolds of L_1.

The second step is to define a homeomorphism h_2 from

$$W^s(P_2) \in L_2 - L_1$$

onto the corresponding $W^s(P_2^*)$ in such a way that h_2 will be compatible with the defined above h_1, for the case in which $\overline{W}^s(P_2) \cap W^s(P_1) \neq \phi$. The manifolds $W^u(P_1)$ and $W^u(P_1^*)$ are $\varepsilon - C^r$–close on compact sets and we have depth $(P_1 \mid P_2) = 1$. Then the set $V_{12} = G^s(P_2) \cap W^u(P_1)$ is a compact manifold and also $W^s(P_2)$ and $W^s(P_2^*)$ are $\varepsilon - C^r$–close on compact sets. By the transversality conditions of the invariant manifolds of (V, D) and of $(\overline{V}, D)$ near (V, D) there exists a diffeomorphism h_2 from V_{12} onto $V_{12}^* = G^s(P_2^*) \cap W^u(P_1^*)$.

Let $\pi_1 : U_1 \to W^s(P_1)$ and $\pi_1^* : U_1^* \to W^s(P_1^*)$ be the projections associated to the global unstable foliations $\mathcal{F}(P_1, U_1)$ and $\mathcal{F}(P_1^*, U_1^*)$. The transversality conditions imply that we may consider $\pi_{12} = \pi_1 / TV_{12}$ and $\pi_{12}^* = \pi_1^* / TV_{12}^*$ for suitable tubular neighbourhoods

$$(TV_{12}, \sigma_2, V_{12}) \quad \text{of} \quad V_{12} \quad in \quad G^s(P_2)$$

and

$$(TV_{12}^*, \sigma_2^*, V_{12}^*) \quad \text{of} \quad V_{12}^* \quad in \quad G^s(P_2^*)$$

chosen in such a way that the open maps $h_1 \circ \pi_{12}$ and π_{12}^* have the same image in $W^s(P_1^*)$. The maps

$$(\pi_{12} \times \sigma_2) : TV_{12} \to W^s(P_1) \times V_{12}$$
$$(\pi_{12}^* \times \sigma_2^*) : TV_{12}^* \to W^s(P_1^*) \times V_{12}^*$$

and the homeomorphism

$$(h_1 \times h_2') : W^s(P_1) \times V_{12} \to W^s(P_1^*) \times V_{12}^*$$

enables us to define $h_2'' : TV_{12} \to TV_{12}^*$, uniquely, such that the diagram below is commutative:

$$\begin{array}{ccccc} & TV_{12} & \xrightarrow{h_2''} & TV_{12}^* & \\ (\pi_{12} \times \sigma_2) & \downarrow & & \downarrow & (\pi_{12}^* \times \sigma_2^*) \\ & W^s(P_1) \times V_{12} & \xrightarrow{(h_1 \times h_2')} & W^s(P_1^*) \times V_{12}^* & \end{array}$$

Note that $h_2''/(TV_{12} - V_{12})$ is a diffeomorphism.

We have to repeat the same construction of h_2'' for all Q_1 such that $W^s(Q_1) \in L_1$ and $\overline{W}^s(P_2) \cap W^s(Q_1) \neq \phi$. Using the Isotopy Extension Theorem (IET) for diffeomorphisms (see [4], pg.133 for a statement and references) we extend all the $h_2'' : TV_{12} \to TV_{12}^*$ to $G^s(P_2)$ and obtain a homeomorphism $\widetilde{h}_2 : G^s(P_2) \to G^s(P_2^*)$ which is a diffeomorphism except at the points of the compact manifolds V_{12} considered above. Finally $h_2 : W^s(P_2) \to W^s(P_2^*)$ is constructed by $h_2(z) = \varphi_{-t}^* \circ h_2 \circ \varphi_t(z)$ for $z \neq P_2$, where $t \in I\!R$ is the unique time such that $\varphi_t(z) \in G^s(P_2)$, and $h_2(P_2) = P_2^*$. The second step is finished if we do the same for all $W^s(Q_2)$ of $L_2 - L_1$. Consider the union $h_1 \cup h_2$ defined on the union of all stable manifolds of L_2.

Thus it remains to prove the continuity of $h_1 \cup h_2$. The only point where to check continuity are those $x \in \partial W^s(P_2)$ such that, say, $x \in W^s(P_1)$. We may (and will) assume that x is sufficiently close to P_1. Recall that h_2 takes leaves of $\mathcal{F}(P_1, U_1)$ near $W^u(P_1)$ to leaves of $\mathcal{F}(P_1^*, U_1^*)$. Take a sequence $x_n \in W^s(P_2)$, $x_n \to x$. The leaf through $h_2(x_n)$ converges to the leaf through $(h_1 \cup h_2)(x) = h_1(x)$. It remains to prove that $h_2(x_n)$ converges to $W^s(P_1^*)$. But this happens since the sequence of times t_n such that $\varphi_{t_n}(h_2(x_n)) \in G^s(P_2^*)$ tends to infinity.

The next (third) step is the consideration of P_3 such that $W^s(P_3) \in L_3 - L_2$ and we will construct a homeomorphism h_3 from $W^s(P_3)$ onto the corresponding $W^s(P_3^*)$ in such a way that h_3 will be compatible with h_1 and h_2. The fact that $W^s(P_3) \in L_3 - L_2$ implies that there exists at least one point $P \in \text{Sing}\ (V, D)$

such that depth $(P \mid P_3) \leq 2$. For each singular point Q_1 such that depth $(Q_1 \mid P_3) = 1$, $W^s(Q_1) \in L_1$ and h_1 is defined on $W^s(Q_1)$; we proceed as in the second step and construct germs of diffeomorphisms h_3'', defined (locally) on $G^s(P_3)$, exactly as we did before when we constructed h_2''. For points P_1 such that depth $(P_1 \mid P_3) = 2$ one considers a sequence (P_1, P_2, P_3) such that depth $(P_1 \mid P_2) =$ depth $(P_2 \mid P_3) = 1$. That implies that the manifolds $W^u(P_2)$ (resp. $W^s(P_3)$) and $W^u(P_2^*)$ (resp. $W^s(P_3^*)$) are $\varepsilon - C^r$-close on compact sets. By the transversality conditions $V_{23} = G^s(P_3) \cap W^u(P_2)$ is a compact manifold and there is a diffeomorphism h_3' from V_{23} onto $V_{23}^* = G^s(P_3^*) \cap W^u(P_2^*)$. Let $\pi_2 : U_2 \to W^s(P_2)$ and $\pi_2^* : U_2^* \to W^s(P_2^*)$ be the projections associated to $\mathcal{F}(P_2, U_2)$ and $\mathcal{F}(P_2^*, U_2^*)$. The transversality conditions imply that we may consider

$$\pi_{23} = \pi_2 / TV_{23} \quad \text{and} \quad \pi_{23}^* = \pi_2^* / TV_{23}^*$$

for suitable tubular neighbourhoods

$$(TV_{23}, \sigma_3, V_{23}) \quad \text{of} \quad V_{23} \quad \text{in} \quad G^s(P_3)$$

and

$$(TV_{23}^*, \sigma_3^*, V_{23}^*) \quad \text{of} \quad V_{23}^* \quad \text{in} \quad G^s(P_3^*),$$

such that the open maps $h_2 \circ \pi_{23}$ and π_{23}^* have the same image in $W^s(P_2^*)$. As we did before we construct h_3'' such that the following diagram is commutative:

$$\begin{array}{ccc} TV_{23} & \xrightarrow{h_3''} & TV_{23}^* \\ (\pi_{23} \times \sigma_3) \downarrow & & \downarrow (\pi_{23}^* \times \sigma_3^*) \\ W^s(P_2) \times V_{23} & \xrightarrow{(h_2 \times h_3')} & W^s(P_2^*) \times V_{23}^* \end{array}$$

The construction shows us that h_3'' takes leaves of $\mathcal{F}(P_2, U_2) \cap TV_{23}$ to leaves of $\mathcal{F}(P_2^*, U_2^*) \cap TV_{23}^*$. But moreover, since h_2 takes leaves of $\mathcal{F}(P_1, U_1)$ near $W^u(P_1)$ to leaves of $\mathcal{F}(P_1^*, U_1^*)$ and by the compatibility of the system of foliations we see that h_3'' takes leaves of $\mathcal{F}(P_1, U_1) \cap TV_{23}$, to leaves of $\mathcal{F}(P_1^*, U_1^*) \cap TV_{23}^*$.

We have to repeat the same construction of the last h_3'' for all sequences (P_1, P_2', P_3) such that

$$\text{depth } (P_1 \mid P_2') = \text{depth } (P_2' \mid P_3) = 1$$

with P_1 fixed. We assume also that we did the same for all P_1 such that depth $(P_1 \mid P_3) = 2$. Using properly the (IET) for diffeomorphisms we extend to $G^s(P_3)$ all the h_3'' constructed in the second step and obtain a homeomorphism $\widetilde{h}_3 : G^s(P_3) \to G^s(P_3^*)$. Finally we extend $\widetilde{h}_3$ to $W^s(P_3)$ using the flows φ_t and φ_t^* and obtain $h_3 : W^s(P_3) \to W^s(P_3^*)$ by $h_3(u) = \varphi_{-\tau}^* \circ \widetilde{h}_3 \circ \varphi_\tau(u)$ for

$u \neq P_3$, where $\tau \in \mathbb{R}$ is the unique time such that $\varphi_\tau(u) \in G^s(P_3)$, and $h_3(P_3) = P_3^*$.

The third step is finished if we do the same for all $W^s(Q_3)$ of $L_3 - L_2$. Consider the union $h_1 \cup h_2 \cup h_3$ defined on the union of all stable manifolds in L_3. The continuity of $h_1 \cup h_2 \cup h_3$ is proved in the same way as we did in the second step. The induction procedure is now evident. ■

We finish the section with the proof of a standard result that we needed, implicitely, for the conclusions of the theorem above:

Proposition 4.3. *The subset of all complete C^r vector fields of a manifold $\mathcal{F}$ is open in the set of all C^r vector fields with the Whitney C^r-topology.*

Proof. Let d be the distance function on the manifold $\mathcal{F}$ associated with a complete Riemannian metric. Take any complete vector field F on $\mathcal{F}$. Call $\Phi : \mathbb{R} \times \mathcal{F} \to \mathcal{F}$ the flow mapping associated to $F : \Phi(t,p) = \varphi_t^F(p)$.

To any compact subset K of $\mathcal{F}$ we associate the subset $E(K)$ of $\mathcal{F}$:

$$E(K) = \Phi([-1,+1] \times K) \cup \overline{B}(K,1)$$

where $\overline{B}(K,1) = \{x \mid d(x,K) \leq 1\}$. Then $E(K)$ is compact as union of two compact sets and $E(K) \supset K$.

We define a sequence of compact subsets K_n of $\mathcal{F}$ as follows: take any point p_0 in $\mathcal{F}$; $K_0 = \overline{B}(p_0,1)$ and $K_{n+1} = E(K_n)$. Then $K_{n+1} \supset K_n$ for all $n \geq 0$.

We claim that $\mathcal{F} = \cup_n K_n$: if $x \in \mathcal{F}$ and $q-1 \leq d(x,K_0) < q$, q integer, then, $x \in K_q$. In fact, let $\overline{x} \in K_0$ be such that $d(x,\overline{x}) = d(x,K_0)$ and let $\gamma : [0, d(x,\overline{x})] \to \mathcal{F}$ be the minimizing geodesic jorning $\overline{x}$ to x. Let $x_i = \gamma(i)$, $i < q$. Since $d(x_i, x_{i+1}) = 1$, we see by induction that $x_i \in K_i$, $i < q$. Since $d(x_{q-1}, x) < 1$ and $x_{q-1} \in K_{q-1}$, x is in K_q.

Also it is clear that K_{n+1} is a compact neighbourhood of K_n for all $n \geq 0$.

For each n there exists a constant $\varepsilon_n > 0$ such that if G is a vector field on $\mathcal{F}$ and $d_1(F, G, K_{n+1} - \overset{o}{K}_n) \leq \varepsilon_n$, $\overset{o}{K}_n$ interior of K_n, where

$$d_1(F,G,K_{n+1} - \overset{o}{K}_n) = \sup\{\|F(x) - G(x)\| +$$

$$\|\nabla F(x) - \nabla G(x)\|, \quad x \in K_{n+1} - \overset{o}{K}_n \},$$

∇ being the Levi-Civita covariant differential, then φ_t^G is defined on $K_{n+1} - \overset{o}{K}_n$ for all t, $-1 \leq t \leq +1$ and $\varphi_t^G(K_{n+1} - \overset{o}{K}_n) \subset K_{n+3}$ for all t, $-1 \leq t \leq +1$.

The set $\mathcal{U}$ of all G such that for any n, $d_1(F,G,K_{n+1} - \overset{o}{K}_n) < \varepsilon_n$, is a neighbourhood of F for the Whitney topology. We claim that every G in $\mathcal{U}$ is complete. We shall write the proof for positive times only.

Take a G in $\mathcal{U}$ and a x in $\mathcal{F}$. Then $x \in K_{n_0}$ for some n_0. By induction on q, it is easy to see that $\varphi_t^G(x) \in K_{n_0+2q}$ if $0 \leq t \leq q$: if $t \in [q, q+1]$, $\varphi_t^G(x) = \varphi_{t-q}^G \varphi_q^G(x) \in \varphi_{t-q}^G(K_{n_0+2q}) \subset K_{n_0+2q+2}$. Hence $\varphi_t^G(x)$ is defined for all $t \geq 0$. ■

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