

Product of distributions and the zero-pressure gas dynamics system

Anupam Pal Choudhury

Centre for Applicable Mathematics
Tata Institute of Fundamental Research
Sharadanagar
Bangalore 560065, India

E-mail address: anupam@math.tifrbng.res.in

K. T. Joseph

Centre for Applicable Mathematics
Tata Institute of Fundamental Research
Sharadanagar
Bangalore 560065, India

E-mail address: ktj@math.tifrbng.res.in

Abstract. In this survey article, we discuss a few applications of three successful notions of the product of distributions (the Colombeau algebra, the method of weak asymptotics and the notion of paths) with regard to the zero-pressure gas dynamics system and the associated adhesion approximation. It is shown that the three notions are closely related atleast in the case of one dimensions.

1. Introduction

A rigorous development of the theory of distributions in mathematics turned out to be a major boost and ushered in a new outlook to the theory of partial differential equations. A more systematic approach towards a well-developed theory of differential equations was initiated. At the same time, mathematicians now knew how to make sense out of the seemingly magical results obtained by the physicists by performing the so-far non-mathematical techniques such as the usage of dirac functions (which is not a function at all !!!). A more careful investigation of the properties

2000 *Mathematics Subject Classification.* 35A20,35L50,35R05.

Key words: adhesion approximation, large scale structure.

of differential equations revealed the necessity of understanding the product of distributions and to the dismay of all *Schwartz's impossibility result* followed, ruling out the possibility of defining the notion of products in the class of distributions. The finite-time breakdown of the classical solutions of nonlinear evolution equations was a phenomenon which was quite evident now and the continuation of the solutions further often involved product of distributions which baffled the entire community. What followed, as always, were a number of spirited attempts in giving a *useful* meaning to the products. In this article, we discuss three such successful theories (the Colombeau algebra [3, 5, 6, 8, 12], the method of weak asymptotics [11, 27, 28], the notion of paths [10, 30]) from the perspective of the zero-pressure gas dynamics system

$$\begin{aligned} u_t + (u \cdot \nabla)u &= 0, \\ \rho_t + \nabla \cdot (\rho u) &= 0, \quad x \in \mathbb{R}^n, \quad t > 0, \end{aligned} \tag{1.1}$$

where u denotes the velocity of the particles and ρ is the density. For the dimensions $n = 1, 2, 3$, this system has applications in cosmology and is closely related to the Zeldovich approximation ([32]). This model describes the evolution of matter in the last stage of the expansion of the universe as cold dust moving under gravity alone and the laws are governed by the system (1.1). The fastest growing mode in the linear theory has decaying vorticity and therefore potential solutions, that is, the case when the velocity u can be represented in terms of a velocity potential ϕ , is of main interest. The fastest particles overrun the slowest ones and in finite time the density becomes infinite, thus as in the case of most nonlinear systems, rendering the existence of global smooth solutions for (1.1) even with smooth initial data

$$u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x)$$

impossible, in general. The mapping from the Lagrangian space $L(y)$ to the Eulerian space $E(x)$ given by

$$x = y + tu_0(y)$$

is bijective only for short time and

$$u(x, t) = u_0(y), \quad \rho(x, t) = \frac{\rho_0(y)}{\det\left(\frac{\partial x_i}{\partial y_j}\right)}$$

provides only short time existence result.

The adhesion model corresponding to (1.1) was introduced in [13]. The adhesion approximation describes the motion of the particles by the motion of sticky particles, wherein the velocity obeys the Burgers' equation and the density is governed by the continuity equation

$$\begin{aligned} u_t + (u \cdot \nabla)u &= \epsilon \Delta u, \quad u = \nabla \phi \\ \rho_t + \nabla \cdot (\rho u) &= 0. \end{aligned} \tag{1.2}$$

When the initial data is of the form $u(x, 0) = \nabla \phi_0(x)$, the exact formula for the velocity u^ϵ can be written down using the Hopf-Cole transformation ([20, 32]). The equation for ρ is then a linear equation with smooth coefficients and can be solved explicitly. As $\epsilon \rightarrow 0$, the limit u of u^ϵ is a locally bounded BV function while the limit ρ of ρ^ϵ is a Radon measure. The products $(u \cdot \nabla)u$ and ρu can no more be described in the sense of distributions. An additional difficulty is the nonconservative nature of the first equation. The idea therefore is to use the microscopic behaviour of the adhesion approximation to make sense of the products involved and to formulate an appropriate notion of solution.

In this survey article, we discuss a few results obtained in this direction by the authors in a series of papers [4, 16, 17, 18, 19, 20, 21] using the three theories mentioned before. The results exhibit that at least for the case $n = 1$, all the three notions are related.

In Section 2, we briefly discuss the basic ideas involved in the three theories. Section 3 discusses about the construction of some explicit solutions for the adhesion approximation of the multidimensional zero-pressure gas dynamics system. In Section 4, the corresponding one dimensional system is discussed wherein the connections between the three theories has been stressed upon. We conclude by showing the existence of solution to the Riemann problem of a related system using the method of weak asymptotics in Section 5.

2. Brief discussion on a few endeavours towards a meaningful notion of products

2.1. The Colombeau Algebra. Here we quickly recall the notion of generalized functions in the sense of Colombeau and refer to [3, 5, 6, 7, 8, 12] for further details and applications.

Let $C^\infty(\Omega)$ denote the class of infinitely differentiable functions in $\Omega = \{(x, t) : x \in \mathbb{R}, t > 0\}$. Let us consider the infinite product $\mathcal{E}(\Omega) = [C^\infty(\Omega)]^{(0,1)}$. Any element v of $\mathcal{E}(\Omega)$ is thus a map from $(0, 1)$ to $C^\infty(\Omega)$ and hereafter we denote it by $v = (v^\epsilon)_{0 < \epsilon < 1}$. We call an element $v = (v^\epsilon)_{0 < \epsilon < 1}$ *moderate* if given any compact subset $K \subset \Omega$ and non-negative integers j, l , there exists $N > 0$ such that

$$\| \partial_t^j \partial_x^l v^\epsilon \|_{L^\infty(K)} = \mathcal{O}(\epsilon^{-N}) \tag{2.1}$$

as ϵ tends to 0. An element $v = (v^\epsilon)_{0 < \epsilon < 1}$ is called *null* if for all compact subsets $K \subset \Omega$ and for all nonnegative integers j and l and for all $M > 0$,

$$\| \partial_t^j \partial_x^l v^\epsilon \|_{L^\infty(K)} = \mathcal{O}(\epsilon^M) \quad (2.2)$$

as ϵ goes to 0. Let us denote the set of all moderate elements by $\mathcal{E}_M(\Omega)$ and the set of all null elements by $\mathcal{N}(\Omega)$. Then $\mathcal{E}_M(\Omega)$ is an algebra with partial derivatives and $\mathcal{N}(\Omega)$ is an ideal closed under differentiation, the operations being defined pointwise on representatives. The quotient space

$$\mathcal{G}(\Omega) = \frac{\mathcal{E}_M(\Omega)}{\mathcal{N}(\Omega)}$$

is an algebra (called the algebra of generalized functions of Colombeau) with partial derivatives, the operations being again defined on representatives. Hence the *product* of two generalized functions $u, v \in \mathcal{G}(\Omega)$ is the class of functions containing $R_1.R_2$, where R_1, R_2 are representatives of u and v respectively. Two elements u and v in $\mathcal{G}(\Omega)$ are said to be *associated*, if for some (and hence all) representatives $(u^\epsilon)_{0 < \epsilon < 1}$ and $(v^\epsilon)_{0 < \epsilon < 1}$, of u and v , $u^\epsilon - v^\epsilon$ goes to 0 as ϵ tends to 0, in the sense of distribution and is denoted by " $u \approx v$ ". Here we remark that this notion is different from the notion of equality in $\mathcal{G}(\Omega)$, which means that $u - v \in \mathcal{N}(\Omega)$, or in other words,

$$\| \partial_t^j \partial_x^l (u^\epsilon - v^\epsilon) \|_{L^\infty(K)} = \mathcal{O}(\epsilon^M)$$

for all M , for all compact subsets K of Ω and for all non-negative integers j, l .

2.2. The Method of Weak Asymptotics. As the name already suggests, the method of weak asymptotics is an asymptotic method which has proved to be highly successful in the study of nonlinear waves. This method, speaking naively, proceeds by assigning to the nonlinear waves a function which for each time variable t , takes value in the space of distributions (with respect to the space variable x). The idea then is to construct approximate solutions (also called weak asymptotic solutions) which satisfy the differential equations upto $O(\epsilon^j)$, j being a positive number, uniformly for all $t > 0$.

The starting point in this method is the choice of regularizations of a distribution f . It would be worthwhile to recall that a family of smooth functions $f^\epsilon(x)$ is called a *regularization* of the distribution f provided $f^\epsilon(x)$ converges to f in the sense of distributions as $\epsilon \rightarrow 0$, that is,

$$\lim_{\epsilon \rightarrow 0} \langle f^\epsilon(x), \phi(x) \rangle = \langle f, \phi \rangle,$$

for every test function ϕ . Given f , the collection of all such regularizations \mathcal{F} form an asymptotic subalgebra modulo $\mathcal{O}_{\mathcal{D}'}(\epsilon)$. Once such a regularization is chosen, the next step is to construct a weak asymptotic solution

by substituting the regularizations in the equation. This step reduces the problem to that of studying a set of ordinary differential equations.

The next and the trickiest part of the analysis then is the choice of a meaningful definition of the weak solution and proving that the passage to the limit respects this formulation. One of the main advantages of using the weak asymptotics method is that the interaction of nonlinear waves can be illustrated analytically even when exact integration methods are unavailable, a phenomenon that is made possible due to the asymptotic subalgebras.

The roots of this method can be traced back to the works of Maslov ([27]). We refer to the articles [12] and [11] for illustrated discussions on this topic and many interesting examples. For an application in the study of singular shocks for conservation laws, we would refer to [1] and the references mentioned therein.

2.3. The notion of Paths due to Volpert and Dal Maso et al. In a seminal work on the functions of bounded variation, Volpert [30] introduced the idea of *averaged superposition* of a BV function $u : (a, b) \rightarrow \mathbb{R}^n$ by a bounded Borel function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as the function $\hat{g}(u) : (a, b) \rightarrow \mathbb{R}^n$ defined by

$$\hat{g}(u)(x) = \int_0^1 g(u(x-) + s(u(x+) - u(x-))) ds$$

for each $x \in (a, b)$. The function $\hat{g}(u)$ is bounded and measurable on (a, b) . The *Volpert product* of $g(u)$ and $\frac{dv}{dx}$ can then be defined for a BV function $v : (a, b) \rightarrow \mathbb{R}^n$ as the bounded Borel measure $\hat{g}(u) \frac{dv}{dx}$.

A closer look at the definition of $\hat{g}(u)$ would reveal the use of a *straight line* path joining the left and right states across a discontinuity of the function u . This very idea was later generalised in [10] wherein the straight line path was now replaced by a locally Lipschitz map $\phi : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the following consistency and regularity properties:

- (A) For any $u_L, u_R \in \mathbb{R}^n$, $\phi(0; u_L, u_R) = u_L$ and $\phi(1; u_L, u_R) = u_R$,
- (B) For any $u \in \mathbb{R}^n$ and $s \in [0, 1]$, $\phi(s; u, u) = u$,
- (C) For every bounded set $\Omega \subset \mathbb{R}^n$, there exists $k \geq 1$ such that for every $u_L, u_R, v_L, v_R \in \mathbb{R}^n$ and for a.e. $s \in [0, 1]$,

$$\left| \frac{\partial \phi}{\partial s}(s; u_L, u_R) - \frac{\partial \phi}{\partial s}(s; v_L, v_R) \right| \leq k |(u_L - v_L) - (u_R - v_R)|.$$

Considering a fixed family of such paths ϕ , given $u, v \in BV((a, b), \mathbb{R}^n)$ and a locally bounded Borel function $g : \mathbb{R}^n \times (a, b) \rightarrow \mathbb{R}^n$, one can then define the *nonconservative product* of $g(u, \cdot)$ by $\frac{dv}{dx}$ as the unique measure μ satisfying the following properties:

- (I) $\mu(B) = \int_B g(u(x), x) \left(\frac{dv}{dx}\right)(x)$ for any Borel set $B \subset (a, b)$ consisting

entirely of the continuity points of u ,

(II) For $x_0 \in (a, b)$,

$$\mu(\{x_0\}) = \int_0^1 g(\phi(s; u(x_0-), u(x_0+)), x_0) \frac{\partial \phi}{\partial s}(s; v(x_0-), v(x_0+)) ds.$$

It is easy to see then that when ϕ is taken to be the canonical family of straight lines, the measure μ coincides with the Volpert product.

We would like to refer to [26] for the connection between traveling wave profiles and the choice of paths. The article [25] probably is the first one to deal with the construction of solutions in the class of functions of bounded variation for systems in nonconservative form using Glimm's scheme. The solution to the Riemann problem there was dealt with using the notion of paths.

Remark 2.1. *We would like to conclude this section by referring to the article [12] for an illuminating discussion on the theory of generalized functions and the article [7] wherein the connection between the notion of paths and the generalized functions in the sense of Colombeau has been exhibited.*

3. The Multidimensional zero-pressure gas dynamics system and the adhesion approximation

In this section, we briefly discuss the construction of explicit weak asymptotic solution of the system (1.1) using the adhesion approximation (1.2), under an additional condition that $u = \nabla_x \phi$. We refer to the article [4] for a more detailed discussion on the topic. To begin with let us recollect the notion of weak asymptotic solutions in the context of the inviscid system (1.1) (see [1, 28]).

Definition: A family of smooth functions $(u^\epsilon, \rho^\epsilon)_{\epsilon > 0}$ is called a *weak asymptotic solution* of the system (1.1) with initial conditions $u(x, 0) = u_0(x)$, $\rho(x, 0) = \rho_0(x)$ provided

$$\begin{aligned} u_t^\epsilon + (u^\epsilon \cdot \nabla) u^\epsilon &= o_{\mathfrak{D}'}(\mathbb{R}^n)(1), \\ \rho_t^\epsilon + \nabla \cdot (\rho^\epsilon u^\epsilon) &= o_{\mathfrak{D}'}(\mathbb{R}^n)(1), \\ u^\epsilon(x, 0) - u_0(x) &= o_{\mathfrak{D}'}(\mathbb{R}^n)(1), \\ \rho^\epsilon(x, 0) - \rho_0(x) &= o_{\mathfrak{D}'}(\mathbb{R}^n)(1), \quad \epsilon \rightarrow 0. \end{aligned} \tag{3.1}$$

The above relations are required to hold uniformly in $t > 0$.

The following theorem describes the construction of explicit weak asymptotic solution of the system (1.1) with appropriate initial conditions.

Theorem 3.1. *(see [4]) Assume $u_0(x) = \nabla_x \phi_0$ where $\phi_0 \in W^{1,\infty}(\mathbb{R}^n)$ and $\rho_0 \in L^\infty(\mathbb{R}^n)$. Let $\phi_0^\epsilon = \phi_0 * \eta^\epsilon$, $\nabla_x \phi_0^\epsilon = \nabla_x(\phi_0 * \eta^\epsilon)$ and $\rho_0 = \rho_0 * \eta^\epsilon$, where η^ϵ is the usual Friedrichs mollifier in the space variable $x \in \mathbb{R}^n$. Further let*

$$u^\epsilon(x, t) = \frac{\int_{\mathbb{R}^n} (\nabla_y \phi_0^\epsilon(y)) e^{-\frac{1}{\epsilon} \left[\frac{|x-y|^2}{2t} + \phi_0^\epsilon(y) \right]} dy}{\int_{\mathbb{R}^n} e^{-\frac{1}{\epsilon} \left[\frac{|x-y|^2}{2t} + \phi_0^\epsilon(y) \right]} dy}, \quad (3.2)$$

$$\rho^\epsilon(x, t) = \rho_0^\epsilon(X^\epsilon(x, t, 0)) J^\epsilon(x, t, 0),$$

where $X^\epsilon(x, t, s)$ is the solution of $\frac{dX(s)}{ds} = u^\epsilon(X, s)$ with $X(s=t) = x$ and $J^\epsilon(x, t, 0)$ is the Jacobian matrix of $X^\epsilon(x, t, 0)$ w.r.t. x .

Then $(u^\epsilon, \rho^\epsilon)$ is a weak asymptotic solution to (1.1) with the described initial conditions $u(x, 0) = u_0(x)$, $\rho(x, 0) = \rho_0(x)$.

Proof. We include a proof of the theorem here for the sake of completion. The main idea used in the construction is the usage of Hopf-Cole transformation for the velocity component u^ϵ and then using the method of characteristics, one can solve for ρ^ϵ in the continuity equation. Let us begin by noting that (see [15, 20, 31]) if ϕ^ϵ solves the equation

$$\phi_t + \frac{|\nabla_x \phi|^2}{2} = \frac{\epsilon}{2} \Delta \phi$$

$$\phi(x, 0) = \phi_0^\epsilon(x), \quad (3.3)$$

then $u^\epsilon = \nabla_x \phi^\epsilon$ solves (1.2) with the initial condition $u^\epsilon(x, 0) = \nabla_x \phi_0^\epsilon(x)$.

The Hopf-Cole transformation $\theta = e^{\frac{\phi}{\epsilon}}$ then transforms (3.3) to the system

$$\theta_t = \frac{\epsilon}{2} \Delta \theta,$$

$$\theta(x, 0) = e^{-\frac{\phi_0^\epsilon(x)}{\epsilon}}. \quad (3.4)$$

The equation (3.4) is the heat equation and therefore θ^ϵ is given by

$$\theta^\epsilon(x, t) = \frac{1}{(2\pi t \epsilon)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{1}{\epsilon} \left[\frac{|x-y|^2}{2t} + \phi_0^\epsilon(y) \right]} dy. \quad (3.5)$$

Integrating by parts in (3.5) with respect to y , we get

$$\theta^\epsilon(x, t)_{x_j} = \frac{1}{(2\pi t \epsilon)^{\frac{n}{2}}} \int_{\mathbb{R}^n} -\frac{1}{\epsilon} (\partial_{y_j} (\phi_0^\epsilon)(y)) e^{-\frac{1}{\epsilon} \left[\frac{|x-y|^2}{2t} + \phi_0^\epsilon(y) \right]} dy,$$

whence it follows that

$$u^\epsilon(x, t) = \frac{\int_{\mathbb{R}^n} (\nabla_y \phi_0^\epsilon(y)) e^{-\frac{1}{\epsilon} \left[\frac{|x-y|^2}{2t} + \phi_0^\epsilon(y) \right]} dy}{\int_{\mathbb{R}^n} e^{-\frac{1}{\epsilon} \left[\frac{|x-y|^2}{2t} + \phi_0^\epsilon(y) \right]} dy}. \quad (3.6)$$

The following properties of u^ϵ can then be derived easily:

(A) $\|u^\epsilon\|_{L^\infty(\mathbb{R}^n \times [0, \infty))} \leq \|\nabla_x \phi_0^\epsilon\|_{L^\infty(\mathbb{R}^n)},$

(B) $|\partial_x^\alpha \partial_t^j u^\epsilon| \leq \frac{C_{\alpha,j}}{\epsilon^{|\alpha|+j}}$, where $C_{\alpha,j}$ depends only on $\|\nabla_x \phi_0^\epsilon\|_{L^\infty(\mathbb{R}^n)}$.

Now since u^ϵ is smooth, the method of characteristics can be implemented to find a solution ρ^ϵ of the continuity equation

$$\rho_t + \nabla \cdot (u^\epsilon \rho) = 0, \quad (3.7)$$

with the regularized initial condition

$$\rho(x, 0) = \rho_0^\epsilon(x).$$

Using the properties (A) and (B) satisfied by u^ϵ , it follows by the existence and uniqueness theory for ODEs, that

$$\begin{aligned} \frac{dX(s)}{ds} &= u^\epsilon(X, s), \\ X(s=t) &= x \end{aligned}$$

admits a unique solution $X^\epsilon(x, t, s)$ for $s \in [0, t]$.

Then

$$\rho^\epsilon(x, t) = \rho_0^\epsilon(X^\epsilon(x, t, 0))J(X^\epsilon(x, t, 0)) \quad (3.8)$$

is a solution of (3.7) (here $J(X^\epsilon(x, t, s))$ denotes the Jacobian determinant of $X^\epsilon(x, t, s)$ with respect to x), and satisfies the estimate (the right-hand side in this case is identically 0) for it to be a weak asymptotic solution.

The property (A) satisfied by u^ϵ shows that it is bounded independent of ϵ and (x, t) . Therefore we have

$$\epsilon \int_{\mathbb{R}^n} \Delta u^\epsilon(x, t) \eta(x) dx = \epsilon \int_{\mathbb{R}^n} u^\epsilon(x, t) \Delta \eta(x) dx = O(1)\epsilon$$

uniformly in t for every $\eta \in C_0^\infty(\mathbb{R}^n)$, whence it follows that u^ϵ satisfies the estimate for the weak asymptotic solution as well.

That $(u^\epsilon, \rho^\epsilon)_{\epsilon>0}$ satisfies the required estimates on the initial conditions can be seen easily from the identities

$$\begin{aligned} u^\epsilon(x, 0) - \nabla_x \phi_0(x) &= u^\epsilon(x, 0) - \nabla_x \phi_0^\epsilon(x) + \nabla_x \phi_0^\epsilon(x) - \nabla_x \phi_0(x) \\ &= \nabla_x \phi_0^\epsilon(x) - \nabla_x \phi_0(x), \\ \rho^\epsilon(x, 0) - \rho_0(x) &= \rho^\epsilon(x, 0) - \rho_0^\epsilon(x) + \rho_0^\epsilon(x) - \rho_0(x) \\ &= \rho_0^\epsilon(x) - \rho_0(x), \end{aligned}$$

since as $\epsilon \rightarrow 0$, $\nabla_x \phi_0^\epsilon(x) - \nabla_x \phi_0(x)$ and $\rho_0^\epsilon(x) - \rho_0(x)$ tend to 0 in the sense of distributions.

□

Remark 3.2. *The velocity component in the vanishing viscosity limit is well understood. For the general data $u(x, 0) = \nabla_x \phi_0(x)$, it was shown in*

[20] that formula (3.2) implies

$$u(x, t) = \lim_{\epsilon \rightarrow 0} u^\epsilon(x, t) = \frac{x - y(x, t)}{t}$$

where $y(x, t)$ is a minimizer in

$$\min_{y \in \mathbb{R}^n} \left\{ \phi_0(y) + \frac{|x - y|^2}{2t} \right\}.$$

For almost every (x, t) , this minimizer is unique and so $u(x, t)$ is well defined a.e. However the passage to the limit in the density component is still open.

4. The one-dimensional case

When considered in one space dimension, the Zeldovich approximation is of the form

$$\begin{aligned} u_t + (u^2/2)_x &= 0, \\ \rho_t + (\rho u)_x &= 0. \end{aligned} \tag{4.1}$$

It is also worth remarking that we arrive at the same system (4.1) when studying the radial components of velocity and density (see [4]). Now when the initial data

$$u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x) \tag{4.2}$$

are smooth, (4.1) has a finite-in-time smooth solution which can be constructed considering the fact that u is constant along the characteristics and hence we can proceed as already described in the introduction.

Our aim in this section is twofold. First, we discuss (following [22]) the construction of explicit solutions of (4.1) and (4.2) using the modified adhesion model (see [14])

$$\begin{aligned} u_t + (u^2/2)_x &= \frac{\epsilon}{2} u_{xx}, \\ \rho_t + (\rho u)_x &= \frac{\epsilon}{2} \rho_{xx}. \end{aligned} \tag{4.3}$$

It turns out that with initial data in $L^\infty \cap BV_{\text{loc}}$ the explicit solutions thus obtained coincide with that obtained in [24] and [29] using different methods.

Our second aim is then to compare the solutions for (4.1) and (4.2) as obtained in the class of generalized functions of Colombeau (see [22]), the solutions constructed using Volpert's product in [24] and that obtained in [1] using the method of weak asymptotics.

4.1. Deriving explicit formula for the modified adhesion model.

We begin by deriving explicit formula for the solutions of the modified adhesion model (4.3) with initial conditions (4.2). The idea is to use a generalized Hopf-Cole type transformation (see [17]), namely

$$u = -\epsilon \frac{a_x}{a}, \quad \rho = \left(\frac{b}{a}\right)_x \quad (4.4)$$

to reduce the problem (4.3) and (4.2) to the linear problem

$$\begin{aligned} a_t &= \frac{\epsilon}{2} a_{xx}, \quad a(x, 0) = e^{-\frac{U_0(x)}{\epsilon}} \\ b_t &= \frac{\epsilon}{2} b_{xx}, \quad b(x, 0) = R_0(x) e^{-\frac{U_0(x)}{\epsilon}} \\ R_0(x) &= \int_0^x \rho_0(y) dy, \quad U_0(x) = \int_0^x u_0(y) dy \end{aligned} \quad (4.5)$$

Substituting the solutions of (4.5) in (4.4), we then obtain

Theorem 4.1. (see [22]) *Assume u_0 and ρ_0 are bounded measurable and integrable. Then*

$$\begin{aligned} u^\epsilon(x, t) &= \int_{\mathbb{R}} \frac{(x-y)}{t} d\mu_{(x,t)}^\epsilon(y) \\ \rho^\epsilon(x, t) &= \partial_x R^\epsilon(x, t) \\ R^\epsilon(x, t) &= \int_{\mathbb{R}} R_0(y) d\mu_{(x,t)}^\epsilon(y). \end{aligned} \quad (4.6)$$

where for each (x, t) , and $\epsilon > 0$, the probability measure $d\mu_{(x,t)}^\epsilon(y)$ defined by

$$\begin{aligned} d\mu_{(x,t)}^\epsilon(y) &= \frac{e^{-\frac{\theta(x,y,t)dy}{\epsilon}}}{\int_{\mathbb{R}} e^{-\frac{\theta(x,y,t)}{\epsilon}} dy} \\ \theta(x, y, t) &= U_0(y) + \frac{(x-y)^2}{2t}. \end{aligned} \quad (4.7)$$

is a solution to (4.3) and (4.2).

4.2. Analysing the vanishing viscosity limit. Next we move on to analyse the vanishing viscosity limit of the solutions of (4.3) with initial data (4.2) in $L^\infty \cap BV_{loc}$. In turn, this gives rise to explicit formula for global solutions of (4.1) with initial data (4.2). The idea here is to follow the analysis in [15] and [23] and to use the properties of the minimizers of

$$\min_{-\infty < y < \infty} \theta(x, y, t) \quad (4.8)$$

which were discussed in those papers. We therefore obtain the following result.

Theorem 4.2. ([22]) *Assume $u_0, \rho_0 \in L^\infty \cap BV_{loc}$. For each fixed $t > 0$, except for a countable x , there exists a unique minimizer $y(x, t)$ for (4.8) and at these points*

$$u(x, t) = \lim_{\epsilon \rightarrow 0} u^\epsilon(x, t) = \frac{(x - y(x, t))}{t}. \quad (4.9)$$

$$R(x, t) = \lim_{\epsilon \rightarrow 0} R^\epsilon(x, t) = \int_0^{y(x, t)} \rho_0(z) dz. \quad (4.10)$$

The functions $u(x, t)$ and $R(x, t)$ are well defined a.e and are functions of bounded variation.

Further for each $t > 0$, and $x \in \mathbb{R}$, the limits $u(x+, t)$, $u(x-, t)$, $R(x-, t)$ and $R(x+, t)$ exist.

Also $u(x, t)$ satisfies the entropy condition $u(x-, t) \geq u(x+, t)$.

Finally

$$\rho(x, t) = \lim_{\epsilon \rightarrow 0} \rho^\epsilon(x, t) = \partial_x \left(\int_0^{y(x, t)} \rho_0(z) dz \right) \quad (4.11)$$

in the sense of distributions. Further, (u, ρ) satisfies (4.1) in the sense of distributions and satisfies the initial conditions (4.2).

Proof. We include a proof of the theorem here. For initial data u_0 in the class of integrable or bounded measurable functions, it follows that for each fixed (x, t) , $\theta(x, y, t)$ has a global minima as a function of y . Let $y(x, t)$ be such a point of global minima. This point of minima need not be unique but the analysis in [15] and [23] show that the largest and the smallest of these minimizers $y^+(x, t)$ and $y^-(x, t)$ are increasing functions of x and hence the point of discontinuities are at most countable and except at these points they are equal $y(x, t) = y^-(x, t) = y^+(x, t)$. At the points where $y(x, t)$ is unique, as $\epsilon \rightarrow 0$, the measure $d\mu_{(x, t)}^\epsilon(y) \rightarrow \delta_{y(x, t)}$ in the sense that for any continuous function $g(y)$ on \mathbb{R} , we have

$$\int g(y) d\mu_{(x, t)}^\epsilon(y) \rightarrow \langle \delta_{y(x, t)}, g(y) \rangle$$

This immediately gives us the limits (4.9) and (4.10).

The next step is to show that (u, ρ) thus constructed satisfies (4.1). The challenge here lies in the way we interpret the product $u\rho$ as we only know that u and R are functions of bounded variation and hence $\rho = R_x$ is a Radon measure. In this regard we follow Volpert [30].

Since $u \in BV$, it induces the decomposition of the domain $\mathbb{R} \times [0, \infty)$ as

$$\mathbb{R} \times [0, \infty) = S_c \cup S_j \cup S_0,$$

where S_c and S_j are points of approximate continuity of u and points of approximate jump of u respectively and S_0 is a set of one dimensional Hausdorff-measure zero. Let at any point $(x, t) \in S_j$, $u(x-0, t)$ and $u(x+0, t)$ denote the left and right values of $u(x, t)$. For any continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, as already discussed in Section 2, the Volpert product $g(u)\rho = g(u)R_x$ is defined as a Borel measure in the following manner. Consider the averaged superposition of $g(u)$

$$\hat{g}(u)(x, t) = \begin{cases} g(u(x, t)), & \text{if } (x, t) \in S_c, \\ \int_0^1 g((1-\alpha)u(x-, t) + \alpha u(x+, t))d\alpha, & \text{if } (x, t) \in S_j. \end{cases} \quad (4.12)$$

Volpert [30] then proves that $\hat{g}(u)$ is measurable and locally integrable with respect to the Borel measure R_x , so that the nonconservative product $\hat{g}(u)R_x$ can be interpreted as a locally finite Borel measure. Indeed for each Borel measurable subset A of S_c ,

$$[g(u)R_x](A) = \int_A \hat{g}(u)(x, t)R_x$$

and

$$[g(u)R_x](\{(x, t)\}) = \hat{g}(u)(x, t)(R(x+0, t) - R(x-0, t))$$

if $(x, t) \in S_j$.

Thus to show that (u, ρ) is a solution of (4.1), we need to prove that the identities

$$(u, \phi_t) + \left(\frac{u^2}{2}, \phi_x\right) = 0, \quad (\rho, \phi_t) + (\hat{u}\rho, \phi_x) = 0 \quad (4.13)$$

hold for all test functions ϕ . The first identity follows in a standard manner by passing to the limit as ϵ goes to zero in (4.3), using the dominated convergence theorem and integration by parts.

To prove that ρ satisfies the second equation, we show that

$$\mu = R_t + \hat{u}R_x = 0 \quad (4.14)$$

in the sense of measures.

The arguments follow that in [24]. Nevertheless here we furnish the details adapted to our case.

Let $(x, t) \in S_c$ and $u = \frac{x-y(x, t)}{t}$. Since u satisfies (4.1), we have

$$-\frac{(x-y(x, t))}{t^2} - \frac{\partial_t y(x, t)}{t} + u(x, t) \frac{(1-\partial_x y(x, t))}{t} = 0.$$

It follows that

$$\partial_t y(x, t) + u \partial_x y(x, t) = 0.$$

Now

$$\partial_t R(x, t) + u \partial_x R(x, t) = \left(\frac{dv_0}{dx}\right)(y(x, t)) \{ \partial_t y(x, t) + u \partial_x y(x, t) \}$$

and therefore we get

$$\partial_t R(x, t) + u \partial_x R(x, t) = 0.$$

Now we consider a point $(s(t), t) \in S_j$, then

$$\frac{ds(t)}{dt} = \frac{u(s(t)+, t) - u(s(t)-, t)}{2}$$

is the speed of propagation of the discontinuity at this point.

$$\begin{aligned} & \mu\{(s(t), t)\} \\ &= -\frac{ds(t)}{dt} (R(s(t)+, t) - R(s(t)-, t)) \\ &+ \int_0^1 (u(s(t)-, t) + \alpha(u(s(t)+, t) - u(s(t)-, t))) d\alpha (R(s(t)+, t) - R(s(t)-, t)) \\ &= \left[-\frac{ds(t)}{dt} + \frac{(u(s(t)+, t) + (u(s(t)-, t)))}{2} \right] (R(s(t)+, t) - R(s(t)-, t)) \\ &= 0. \end{aligned}$$

Hence we have proved (4.14). Since $R_{tx} = R_{xt}$ in the sense of distributions and $\rho = R_x$, differentiating (4.14) w.r.t. x gives

$$\rho_t + (\bar{u}\rho)_x = 0$$

in the sense of distributions and (4.13) follows.

Next to show that the solution satisfies the initial conditions, we observe that Lax's argument in [23] gives $\lim_{t \rightarrow 0} u(x, t) = u_0(x)$, a.e. $x \geq \epsilon$. Now since

$$y(x, t) - x = u(x, t)t,$$

it follows that $y(x, t) \rightarrow x$ as $t \rightarrow 0$ a.e. x . So we get

$$\int_0^{y(x, t)} v_0(z) dz \rightarrow \int_0^x v_0(z) dz$$

as $t \rightarrow 0$ for a.e. x .

The fact that u satisfies the entropy condition $u(x-0, y) \geq u(x+0, t)$ can be inferred from the increasing nature of $y^+(x, t)$ and $y^-(x, t)$ as a function of x , for each $t > 0$ and from the formula (4.9) for u . This completes the proof of the stated theorem.

□

4.3. Formula for some special initial data. As a first example, let us consider Riemann type initial data, namely

$$\begin{aligned} u_0(x) &= \begin{cases} u_l, & \text{if } x < 0, \\ u_r, & \text{if } x > 0 \end{cases} \\ \rho_0(x) &= \begin{cases} \rho_l, & \text{if } x < 0, \\ \rho_r, & \text{if } x > 0 \end{cases} \end{aligned} \quad (4.15)$$

In this case the vanishing viscosity limit is as described in the theorem below.

Theorem 4.3. ([22]) *Let u^ϵ and ρ^ϵ be the solutions given by (4.6) with initial data of Riemann type (4.15) and $u(x, t) = \lim_{\epsilon \rightarrow 0} u^\epsilon(x, t)$ and $\rho(x, t) = \lim_{\epsilon \rightarrow 0} \rho^\epsilon(x, t)$. Then (u, ρ) are as follows.*

Case 1 $u_l = u_r = u_0$:

$$\begin{aligned} u(x, t) &= u_0 \\ \rho(x, t) &= \begin{cases} \rho_l, & \text{if } x < u_0 t, \\ \rho_r, & \text{if } x > u_0 t \end{cases} \end{aligned}$$

Case 2 $u_l < u_r$: (Figure 1)

$$u(x, t) = \begin{cases} u_l, & \text{if } x < u_l t, \\ \frac{x}{t}, & \text{if } u_l t < x < u_r t, \\ u_r, & \text{if } x > u_r t \end{cases}$$

and

$$\rho(x, t) = \begin{cases} \rho_l, & \text{if } x < u_l t, \\ 0, & \text{if } u_l t < x < u_r t, \\ \rho_r, & \text{if } x > u_r t \end{cases}$$

Case 3 $u_r < u_l$: (Figure 2)

$$u(x, t) = \begin{cases} u_l, & \text{if } x < st, \\ \frac{u_l + u_r}{2}, & \text{if } x = st, \\ u_r, & \text{if } x > st \end{cases}$$

$$\rho(x, t) = \begin{cases} \rho_l, & \text{if } x < st, \\ \frac{(u_l - u_r)(\rho_l + \rho_r)}{2} t \delta_{x=st}, & \text{if } x = st, \\ \rho_r, & \text{if } x > st \end{cases}$$

where $s = \frac{u_l + u_r}{2}$

Proof. We refer to [22] for a proof of this result. □

Our next example is with initial data of finite mass and of the form

$$u_0(x) = \begin{cases} 0, & \text{if } \{-\infty < x < 0\} \cup \{1 < x < \infty\}, \\ 1, & \text{if } \{0 < x < 1\} \end{cases}$$

$$\rho_0(x) = \begin{cases} 0, & \text{if } \{-\infty < x < 0\} \cup \{1 < x < \infty\}, \\ \rho_c, & \text{if } \{0 < x < 1\} \end{cases}$$

In this case the solution is as follows: (Figure 3)

For $0 \leq t \leq 2$

$$u(x, t) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{t}, & \text{if } 0 \leq x \leq t, \\ 1, & \text{if } t \leq x \leq \frac{t}{2} + 1, \\ 0, & \text{if } x \geq \frac{t}{2} + 1 \end{cases}$$

$$\rho(x, t) = \begin{cases} 0, & \text{if } x < t, \\ \rho_c, & \text{if } t \leq x \leq \frac{t}{2} + 1, \\ \frac{\rho_c}{2} t \delta_{x=\frac{t}{2}+1}, & \text{if } x = \frac{t}{2} + 1, \\ 0, & \text{if } x > \frac{t}{2} + 1 \end{cases}$$

and for $t > 2$

$$u(x, t) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{t}, & \text{if } 0 \leq x \leq \sqrt{2t}, \\ 0, & \text{if } x \geq \sqrt{2t} \end{cases}$$

$$\rho(x, t) = \rho_c \delta_{x=\sqrt{2t}}.$$

4.4. Solutions in the sense of Colombeau. Next we show that $u = (u^\epsilon(x, t))_{0 < \epsilon < 1}$, $\rho = (\rho^\epsilon(x, t))_{0 < \epsilon < 1}$, with u^ϵ and ρ^ϵ given by (4.6) satisfy the equation (4.1) in $\Omega = \{(x, t) : x \in \mathbb{R}, t > 0\}$ in the sense of association:

$$u_t + \left(\frac{u^2}{2}\right)_x \approx 0 \tag{4.16}$$

$$v_t + (u\rho)_x \approx 0.$$

In other words, let us consider (u, v) where $u = (u^\epsilon(x, t))_{0 < \epsilon < 1}$, $\rho = (\rho^\epsilon(x, t))_{0 < \epsilon < 1}$, with u^ϵ and ρ^ϵ being solutions of equation (4.3)

$$u_t^\epsilon + \left(\frac{u^{\epsilon 2}}{2}\right)_x = \frac{\epsilon}{2} u_{xx}^\epsilon, \tag{4.17}$$

$$\rho_t^\epsilon + (u^\epsilon \rho^\epsilon)_x = \frac{\epsilon}{2} \rho_{xx}^\epsilon$$

supplemented with the initial conditions

$$u^\epsilon(x, 0) = u_0^\epsilon(x), \rho^\epsilon(x, 0) = \rho_0^\epsilon(x), \quad (4.18)$$

where $u_0 = (u_0^\epsilon(x))_{0 < \epsilon < 1}$, $\rho_0 = (\rho_0^\epsilon(x))_{0 < \epsilon < 1}$, are in $\mathcal{G}(\mathbb{R})$, the algebra of generalized functions of Colombeau. We assume that u_0^ϵ and ρ_0^ϵ are bounded C^∞ functions in x satisfying the estimates

$$\begin{aligned} \|\partial_x^j u_0^\epsilon\|_{L^\infty([0, \infty))} &= O(\epsilon^{-j}) \\ \|\partial_x^j \rho_0^\epsilon\|_{L^\infty([0, \infty))} &= O(\epsilon^{-j}) \end{aligned} \quad (4.19)$$

for $j = 0, 1, 2, \dots$ and $u_0^\epsilon(x) \rightarrow u_0(x)$, $\rho_0^\epsilon(x) \rightarrow \rho_0(x)$ point wise a.e.

Remark 4.4. *These conditions are satisfied, for example, if we start with a bounded measurable function on \mathbb{R} , and then take its convolution with the Friedrichs mollifiers with scale ϵ .*

We have the following result in this direction (see [22]).

Theorem 4.5. *Assume that $u_0 = (u_0^\epsilon(x))_{0 < \epsilon < 1}$, $\rho_0 = (\rho_0^\epsilon(x))_{0 < \epsilon < 1}$, are in $\mathcal{G}(\mathbb{R})$, with the estimates (4.19) and as described before. Let $(u^\epsilon, \rho^\epsilon)$ be given by the formula (4.6) with $(u_0(x), \rho_0(x))$ replaced by $(u_0^\epsilon(x), \rho_0^\epsilon(x))$, for $\epsilon > 0$. Then $u = (u^\epsilon)_{0 < \epsilon < 1}$ and $\rho = (\rho^\epsilon)_{0 < \epsilon < 1} = (R_x^\epsilon)_{0 < \epsilon < 1}$ are in $\mathcal{G}(\Omega)$ and (u, v) is a solution to (4.16) with initial conditions $(u_0, \rho_0) = (u_0^\epsilon(x), \rho_0^\epsilon(x))_{0 < \epsilon < 1}$.*

Proof. As already discussed in Section 2, to show that u and ρ are in $\mathcal{G}(\Omega)$ we need to prove that u^ϵ and ρ^ϵ satisfy the estimates (2.1). From the formulas (4.6) and the estimate (4.19), it can be easily seen that the estimates

$$\|u^\epsilon\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}, \|R^\epsilon\|_{L^\infty(\Omega)} \leq \|R_0\|_{L^\infty(\mathbb{R}^N)}. \quad (4.20)$$

hold. An application of the Leibnitz's rule and the estimate (4.19), gives us

$$\|\partial_x^k u^\epsilon\|_{L^\infty(\Omega)} = O(\epsilon^{-2k}), \|\partial_x^k \rho^\epsilon\|_{L^\infty(\Omega)} = O(\epsilon^{-2k-1}). \quad (4.21)$$

Now since $(u^\epsilon, \rho^\epsilon)$ is a solution of the equation (4.17), using the estimate (4.21), we obtain

$$\|\partial_t u^\epsilon\|_{L^\infty(\Omega)} = O(\epsilon^{-2}), \|\partial_t \rho^\epsilon\|_{L^\infty(\Omega)} = O(\epsilon^{-2}). \quad (4.22)$$

Applying the differential operator $\partial_t^j \partial_x^k$ on both sides of (4.17), first $k = 1, j = 0, 1, 2, \dots$ and then $k = 2, j = 0, 1, 2, \dots$ and using (4.21) and (4.22) we get,

$$\begin{aligned} \|\partial_t^j \partial_x^k u^\epsilon\|_{L^\infty(\Omega)} &= O(\epsilon^{-2(j+k)}) \\ \|\partial_t^j \partial_x^k \rho^\epsilon\|_{L^\infty(\Omega)} &= O(\epsilon^{-2(j+k)-1}) \end{aligned}$$

These estimates show that u and v are in $\mathcal{G}(\Omega)$.

Next to show that u and v satisfy the equation (4.1) in the sense of association, we multiply (4.17) by a test function $\phi \in C_0^\infty(\Omega)$ and integrate to get

$$\int_0^\infty \int_{-\infty}^\infty (u_t^\epsilon + (\frac{u^{\epsilon 2}}{2})_x) \phi \, dx \, dt = \frac{\epsilon}{2} \int_0^\infty \int_{-\infty}^\infty u^\epsilon \phi_{xx} \, dx \, dt$$

$$\int_0^\infty \int_{-\infty}^\infty (\rho_t^\epsilon + (u^\epsilon \rho^\epsilon)_x) \phi \, dx \, dt = -\frac{\epsilon}{2} \int_0^\infty \int_{-\infty}^\infty R^\epsilon \phi_{xxx} \, dx \, dt.$$

The fact that the right hand side goes to zero as ϵ goes to zero easily follows by an application of the dominated convergence theorem as u^ϵ and R^ϵ are bounded and converge pointwise almost every where. \square

4.5. Rankine-Hugoniot conditions from the notion of paths and the method of weak asymptotics. In [24], it was shown using the Volpert's product that the pair (u, v) , where

$$u(x, t) = \frac{x - y(x, t)}{t}, \quad v(x, t) = \partial_x \left(\int_0^{y(x, t)} v_0(z) \, dz \right)$$

(here $y(x, t)$ is a minimizer of $\min_y \{ \int_0^y u_0(z) \, dz + \frac{(x-y)^2}{2t} \}$) is a weak solution of (4.1). In [1] the same system was analysed using the method of weak asymptotics and generalized Rankine-Hugoniot conditions for the delta-shock wave type solutions were derived. We next show that the solution derived in [24] satisfies the generalized Rankine-Hugoniot conditions derived in [1]. In particular we have the following theorem (see [4])

Theorem 4.6. *Let us assume that the solution (u, v) is smooth except along a curve $x = s(t)$ in a neighbourhood of a point $(s(t_0), t_0)$. Then v has the form*

$$v(x, t) = v_l(x, t) + v_r(x, t)H(x - s(t)) + e(t)\delta_{x=s(t)},$$

and the solution satisfies the generalized Rankine-Hugoniot conditions

$$s'(t) = \frac{[\frac{u^2}{2}]}{[u]}, \quad e'(t) = -s'(t)[v](s(t), t) + [uv](s(t), t).$$

Proof. We include a proof of the result here following [4] in order to make our discussion complete. Let $x = s(t)$ be a curve of discontinuity for u and let $(s(t_0), t_0)$ be a point on it. We further assume that there exists a neighbourhood of $(s(t_0), t_0)$ such that it is the only point of discontinuity in that neighbourhood. It then follows from the formula that

$$v(x, t) = v_l(x, t) + v_r(x, t)H(x - s(t)) + e(t)\delta_{x=s(t)}.$$

Next we recall that the second equation in (4.1) is interpreted as

$$v_t + (\hat{u}v)_x = 0$$

in the sense of distributions, where

$$\hat{u} = \int_0^1 (u(s(t)-, t) + \tau(u(s(t)+, t) - u(s(t)-, t))) d\tau$$

. Let ϕ be any test function supported in the assumed neighbourhood. Then from the first equation, it follows using an integration by parts type of argument that $s'(t) = \frac{[\frac{u^2}{2}]}{[u]}$ and from the second equation we have

$$\begin{aligned} 0 &= \int_{x < s(t)} (v_l(x, t)\phi_t + (u_l(x, t)v_l(x, t))\phi_x) dx dt \\ &+ \int_{x > s(t)} (v_r(x, t)\phi_t + (u_r(x, t)v_r(x, t))\phi_x) dx dt \quad (4.23) \\ &+ \int_0^\infty e(t)\phi_t(s(t), t) + \hat{u}e(t)\phi_x(s(t), t) dt. \end{aligned}$$

It's not very difficult to see then that

$$\begin{aligned} \int_0^1 (u(s(t)+, t) + \tau(u(s(t)+, t) - u(s(t)-, t))) d\tau &= \frac{\frac{u^2(s(t)+, t)}{2} - \frac{u^2(s(t)-, t)}{2}}{u(s(t)+, t) - u(s(t)-, t)} \\ &= s'(t). \end{aligned}$$

Using this in (4.23), we have

$$\begin{aligned} 0 &= \int_{x < s(t)} (v_l\phi_t + (u_lv_l)\phi_x) dx dt \\ &+ \int_{x > s(t)} (v_r\phi_t + (u_rv_r)\phi_x) dx dt + \int_0^\infty e(t)\frac{d\phi(s(t), t)}{dt}. \end{aligned}$$

Now integrating by parts it follows that

$$\int_0^\infty (-s'(t)[v](s(t), t) + [uv](s(t), t) - e'(t))\phi(s(t), t) dt = 0$$

and hence the generalized Rankine-Hugoniot condition follows. \square

Remark 4.7. *Therefore we have shown that the explicit solutions constructed here for the one-dimensional zero-pressure gas dynamics system belong to the class of generalized functions in the sense of Colombeau and also coincide with that derived using the notion of paths (in this case the Volpert's product) which in turn again satisfy the Rankine-Hugoniot conditions derived using the method of weak asymptotics.*

5. Solution to the Riemann problem for a related system using the method of weak asymptotics

In this section, we show the existence of a solution to the Riemann problem for the system

$$\begin{aligned}u_t + \left(\frac{u^2}{2}\right)_x &= 0, \\R_t + uR_x &= 0\end{aligned}\tag{5.1}$$

with initial data of the form

$$\begin{aligned}u(x, 0) &= u_r + [u]H(-x) \\R(x, 0) &= R_r + [R]H(-x),\end{aligned}$$

where u_r, u_l are constants and $[u] = u_l - u_r$, $[R] = R_l - R_r$ denote the jumps in u and R .

We have already encountered this system in Section 4. This system was studied using Volpert's product in [24] where existence and uniqueness (under additional conditions) were established for initial data $u(x, 0) \in BV(\mathbb{R})$ and $R(x, 0) \in W^{1,\infty}(\mathbb{R})$ (uniqueness was showed to be true even when $R(x, 0) \in BV(\mathbb{R})$). The existence of solution to the Riemann problem for (5.1) was not discussed there. Later in [9] the result was extended to the case when $R(x, 0) \in BV(\mathbb{R})$.

Here we use the method of weak asymptotics to prove the existence of solution to the Riemann problem for (5.1). The most interesting part of this analysis is that the chosen regularizations in this case give rise to the Volpert's product.

We therefore seek solution in the form

$$\begin{aligned}u(x, t) &= u_r + [u]H(-x + \phi(t)) \\R(x, t) &= R_r + [R]H(-x + \phi(t)),\end{aligned}\tag{5.2}$$

where $\phi(t)$ is the curve of discontinuity.

The first step then is to prove the existence of weak asymptotic solution of (5.1)

$$\begin{aligned}u^\epsilon(x, t) &= u_r + [u]H^\epsilon(-x + \phi(t)) \\R^\epsilon(x, t) &= R_r + [R]H^\epsilon(-x + \phi(t)).\end{aligned}\tag{5.3}$$

For this purpose we choose regularizations $H^\epsilon(x)$ in the form

$$H^\epsilon(x) = \begin{cases} 0, & x \leq -4\epsilon \\ c, & -3\epsilon \leq x \leq 3\epsilon \\ 1, & x \geq 4\epsilon \end{cases}$$

where c is a constant, and $H^\epsilon(x)$ is continued smoothly in the intervals $(-4\epsilon, -3\epsilon)$ and $(3\epsilon, 4\epsilon)$. It can be easily seen that

$$\begin{cases} H^\epsilon(x) = H(x) + o_{\mathfrak{D}'}(\epsilon) \\ H_\xi^\epsilon(x) = \delta(x) + o_{\mathfrak{D}'}(\epsilon) \\ H^\epsilon(x)H_\xi^\epsilon(x) = \frac{1}{2}\delta(x) + o_{\mathfrak{D}'}(\epsilon), \end{cases}$$

where $H_\xi^\epsilon(\cdot)$ denotes the derivative of H^ϵ . Now

$$\begin{aligned} u_t^\epsilon(x, t) &= [u]\dot{\phi}(t)H_\xi^\epsilon(-x + \phi(t)), \\ u_x^\epsilon(x, t) &= -[u]H_\xi^\epsilon(-x + \phi(t)), \\ R_t^\epsilon(x, t) &= [R]\dot{\phi}(t)H_\xi^\epsilon(-x + \phi(t)), \\ R_x^\epsilon(x, t) &= -[R]H_\xi^\epsilon(-x + \phi(t)). \end{aligned}$$

Substituting $u_t^\epsilon, u_x^\epsilon$ in the first equation in (5.1), we get

$$\begin{aligned} u_t^\epsilon + u^\epsilon u_x^\epsilon &= [u]\dot{\phi}(t)H_\xi^\epsilon(-x + \phi(t)) - u_r[u]H_\xi^\epsilon(-x + \phi(t)) \\ &\quad - [u]^2 H^\epsilon(-x + \phi(t))H_\xi^\epsilon(-x + \phi(t)) \\ &= \left\{ [u]\dot{\phi}(t) - u_r[u] - \frac{[u]^2}{2} \right\} \delta(-x + \phi(t)) + o_{\mathfrak{D}'}(1), \quad \epsilon \rightarrow 0. \end{aligned}$$

Setting the coefficients of $\delta(-x + \phi(t))$ above to zero, we get

$$\dot{\phi}(t) = u_r + \frac{[u]}{2} \quad \left(= \frac{[\frac{u^2}{2}]}{[u]} \right)$$

and hence

$$u_t^\epsilon + u^\epsilon u_x^\epsilon = o_{\mathfrak{D}'}(1), \quad \epsilon \rightarrow 0.$$

Substituting $R_x^\epsilon, R_t^\epsilon$ in (5.1) and using the expression for $\dot{\phi}(t)$ as obtained above, we have

$$\begin{aligned} R_t^\epsilon + u^\epsilon R_x^\epsilon &= [R]\dot{\phi}(t)H_\xi^\epsilon(-x + \phi(t)) - u_r[R]H_\xi^\epsilon(-x + \phi(t)) \\ &\quad - [u][R]H^\epsilon(-x + \phi(t))H_\xi^\epsilon(-x + \phi(t)) \\ &= \left\{ [R]\dot{\phi}(t) - u_r[R] - \frac{[u][R]}{2} \right\} \delta(-x + \phi(t)) + o_{\mathfrak{D}'}(1) \\ &= o_{\mathfrak{D}'}(1), \quad \epsilon \rightarrow 0. \end{aligned}$$

Thus $u^\epsilon(x, t), R^\epsilon(x, t)$ as in (5.3) with $\dot{\phi}(t) = \frac{[\frac{u^2}{2}]}{[u]}$ is a weak asymptotic solution of (5.1).

Remark 5.1. *It would be interesting to note that*

$$\begin{aligned} uR_x &:= \lim_{\epsilon \rightarrow 0} u^\epsilon R_x^\epsilon = \lim_{\epsilon \rightarrow 0} -(u_r[R]H_\xi^\epsilon + [u][R]H^\epsilon H_\xi^\epsilon) \\ &= -(u_r + \frac{[u]}{2})[R]\delta = -(\frac{[u^2]}{[u]})[R]\delta, \end{aligned}$$

which corresponds to the Volpert's product.

It's not very difficult then to see that $u(x, t), R(x, t)$ where

$$u(x, t) = \lim_{\epsilon \rightarrow 0} u^\epsilon(x, t), \quad R(x, t) = \lim_{\epsilon \rightarrow 0} R^\epsilon(x, t),$$

satisfy for all $\eta(x, t) \in \mathcal{D}(\mathbb{R} \times [0, \infty))$ the integral identities

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} (u\eta_t + \frac{u^2}{2}\eta_x) dx dt + \int_{\mathbb{R}} u(x, 0)\eta(x, 0) dx &= 0, \\ \int_0^\infty \int_{\mathbb{R}} (R_t + \hat{u}R_x)\eta dx dt &= 0 \end{aligned}$$

which is exactly the formulation suggested in [24] using Volpert's product.

Acknowledgement: The first author would like to thank Deep Ray (TIFR-CAM) for introducing him to the package 'Asymptote' which helped the authors immensely in the preparation of the figures.

References

- [1] S. Albeverio and V.M. Shelkovich, On the delta-shock front problem, Analytical approaches to multidimensional balance laws, Nova Sci. Publ., New York, 2006, 45-87.
- [2] C. Bardos, A.Y. Leroux and J.C. Nedelec, First order quasi-linear equations with boundary conditions, Comm.Part. Diff. Eqn., **4** (1979), 1017-1034.
- [3] H.A.Biagioni and M.Oberguggenberger, Generalized solutions to Burgers equation, Jl.Diff.Eqn., **97** (1992) 263-287.
- [4] A.P.Choudhury, K.T.Joseph, M.R.Sahoo, Spherically symmetric solutions of multi-dimensional zero-pressure gas dynamics system, submitted.
- [5] J.F.Colombeau, New Generalized functions and Multiplication of distributions, North Holland, Amsterdam, (1984).
- [6] J.F.Colombeau and A.Heibig, Generalized Solutions to Cauchy problems, Monatsh. Math. **117** (1994) 33-49.
- [7] J.F.Colombeau and A.Heibig, Nonconservative products in bounded variation functions, SIAM J. Math. Anal. **23** (1992), no. 4, 941-949.
- [8] J.F.Colombeau, A.Heibig and M.Oberguggenberger, The Cauchy problem in a space of generalized functions 1, 2, C.R.Acad.Sci.Paris Ser.1 Math., **317** (1993) 851-855; **319** (1994) 1179-1183.
- [9] G.Crasta, P.G.Lefloch, Existence result for a class of nonconservative and non-strictly hyperbolic systems, Commun. Pure Appl. Anal. **1** (2002), no. 4, 513-530.

- [10] G.Dal Maso, P.G.LeFloch and F.Murat, Definition and weak stability of nonconservative products, *J. Math. Pures Appl.* **74** (1995) 483-548.
- [11] V.G.Danilov, G.A.Omelyanov, V.M.Shelkovich, Weak asymptotics method and interaction of nonlinear waves (Asymptotic methods for wave and quantum problems), 33-163, Amer. Math. Soc. Transl. Ser. 2, 208, Amer. Math. Soc., Providence, RI, 2003.
- [12] Y.V.Egorov, On the theory of generalized functions, *Uspekhi Mat. Nauk* 45 (1990), no. 5(275), 3-40, 222; translation in *Russian Math. Surveys* 45 (1990), no. 5, 149.
- [13] S.N.Gurbatov and A.I.Saichev, *Raiophys. Quant.Electr.* **27** (1984) 303.
- [14] S.N.Gurbatov and A.I.Saichev, New approximation in the adhesion model in the description of large scale structure of the universe, *Cosmic velocity fields ; Proc. 9th IAP Astrophysics*, ed F. Bouchet and Marc Lachieze-Rey (1993) 335-340.
- [15] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.*, **13** (1950) 201-230.
- [16] K.T.Joseph, Burgers equation in the quarter plane, a formula for the weak limit, *Comm. Pure Appl. Math.*, **41**(1988) 133-149.
- [17] K. T. Joseph, A Riemann problem whose viscosity solution contain δ - measures, *Asym. Anal.*, **7**(1993) 105-120 .
- [18] K. T. Joseph, Exact solution of a system of generalized Hopf equations, *Jl. Anal. Appl.*, **21** (2002) 669 - 680.
- [19] K.T.Joseph, Explicit generalized solutions to a system of conservation laws, *Proc. Indian Acad. Sci. (Math.Sci.)*, **109** (1999) 401-409.
- [20] K.T.Joseph and P.L. Sachdev, Initial boundary value problems for scalar and vector Burgers equations, *Studies in Appl. Math.*, **106** (2001) 481-505.
- [21] K.T.Joseph, Explicit solutions for a system of first-order partial differential equations, *Electronic Jl. Diff. Eqn.*, **2008**, (2008) no.157 1-8.
- [22] K.T.Joseph, One-dimensional adhesion model for large scale structures, *Electronic Jl. Diff. Eqn.*, **2010**(2010), no. 70, 1-15.
- [23] P.D. Lax, Hyperbolic systems of conservation laws II, *Comm.Pure Appl. Math.*, **10** (1957) 537-566.
- [24] P.G. LeFloch, An existence and uniqueness result for two nonstrictly hyperbolic systems, in *Nonlinear Evolution Equations that change type*, (eds) Barbara Lee Keyfitz and Michael Shearer, **IMA 27** (1990) 126-138.
- [25] P.G.LeFloch, Entropy weak solutions to nonlinear hyperbolic systems in nonconservative form, *Comm. Part. Diff. Equa.*, **13** (1988) 669-727.
- [26] P.G.Lefloch, Shock waves for nonlinear hyperbolic systems in nonconservative form, *IMA preprint series* 593, October 1989.
- [27] V.P.Maslov, Three algebras corresponding to unsmooth solutions of quasilinear hyperbolic equations, *Uspekhi Mat. Nauk* 35:2 (1980), 252-253.
- [28] V. M. Shelkovich, The Riemann problem admitting $\delta - \delta'$ shocks and vacuum states (the vanishing viscosity approach), *Jl. Diff. Eqn.*, **231**, 459 - 500 (2006).
- [29] D. Tan, T.Zhang and Y. Zheng, Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws, *Jl. Diff. Eqn.*, **112** (1994),1-32.
- [30] A.I. Volpert, The space BV and quasi-linear equations, *Math USSR Sb* bf 2 (1967) 225-267.
- [31] D.H.Weinberg and J.E.Gunn, Large scale structure and the adhesion approximation, *Mon. Not. R. Astr. Soc.* **247** (1990), 260-286.
- [32] Ya. Zeldovich, Gravitational instability: an approximate theory for large density perturbations, *Astron. Astrophys.*, **5** (1970) 84-89.

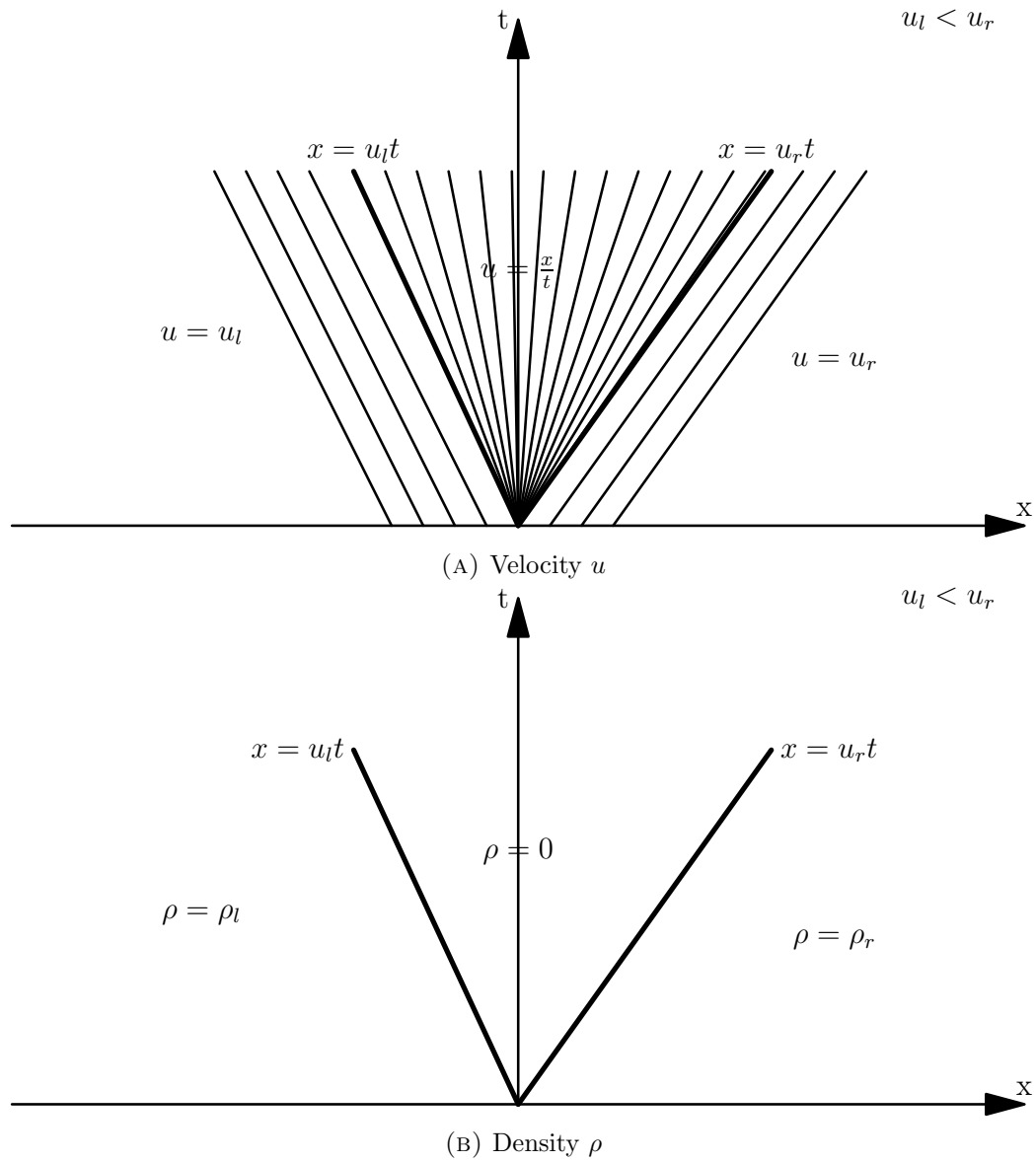


FIGURE 1. Solution for the Riemann problem for Case 2

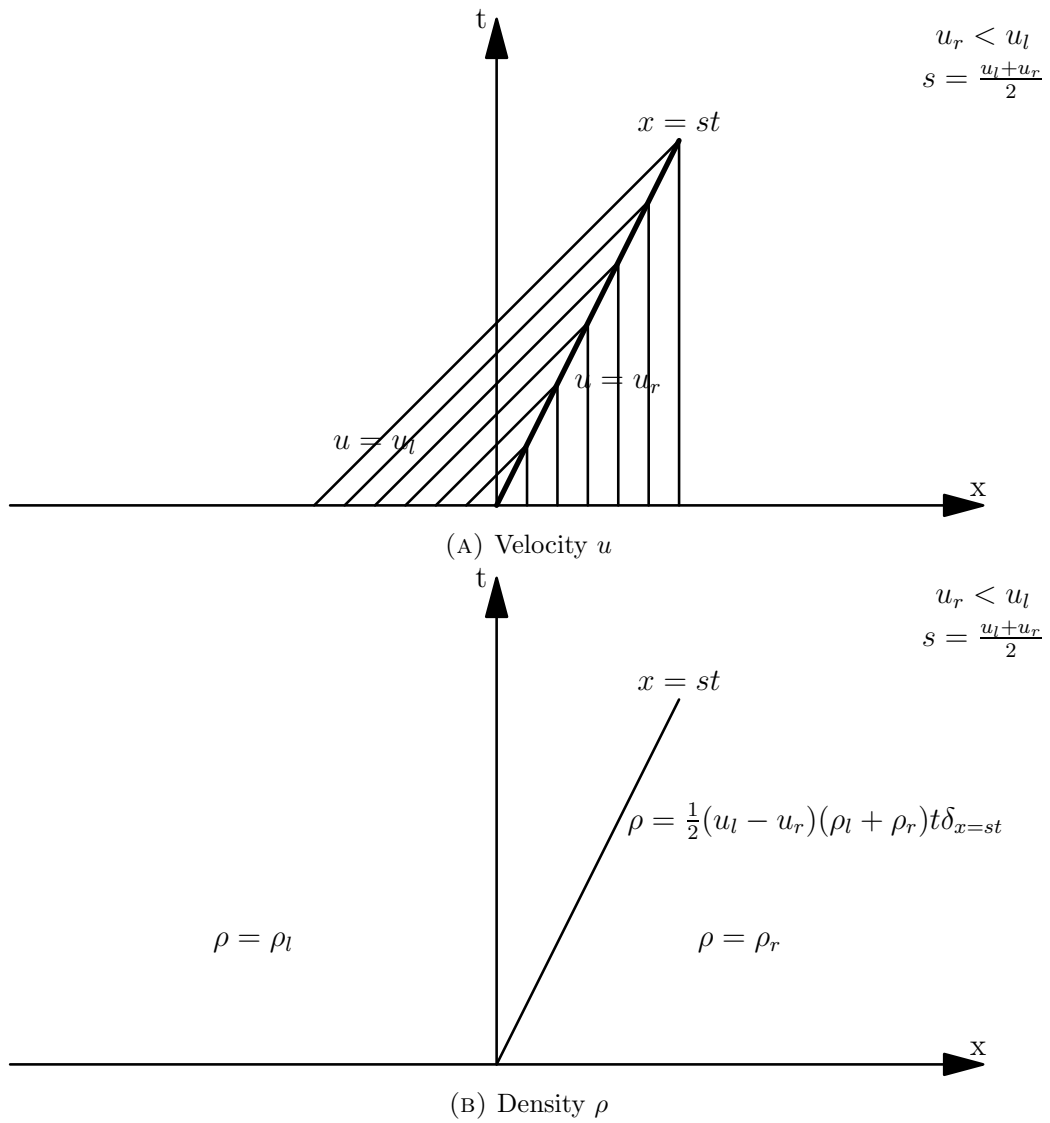
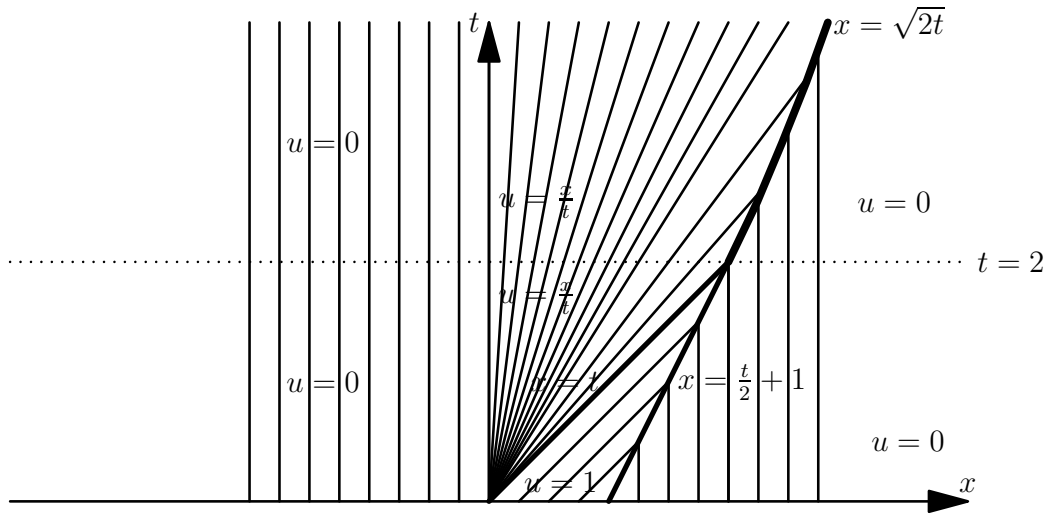
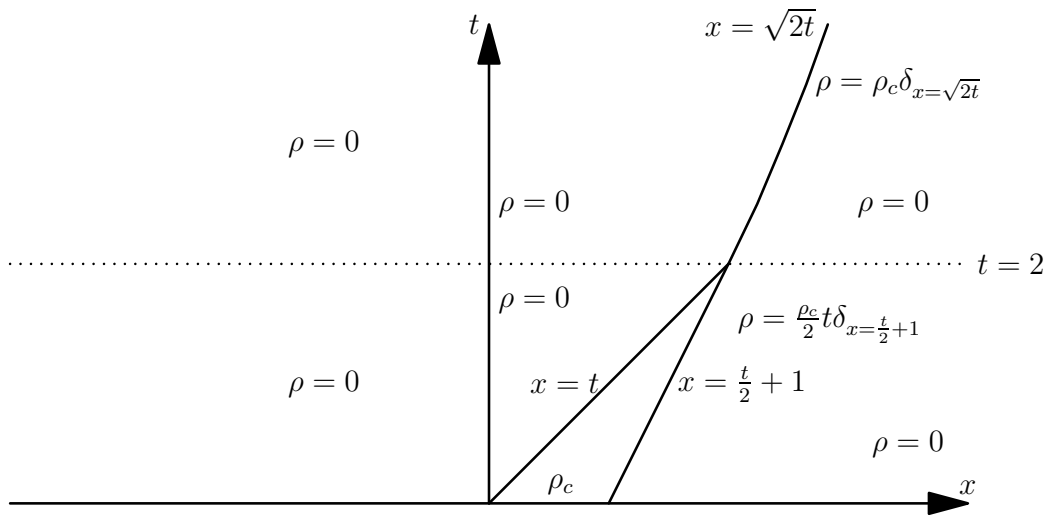


FIGURE 2. Solution for the Riemann problem for Case 3



(A) Velocity u



(B) Density ρ

FIGURE 3. Solution for the Riemann problem with finite mass