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An almost existence theorem for non-contractible periodic orbits in cotangent bundles

Pedro A. S. Salomão

Instituto de Matemática e Estatística, Universidade de São Paulo
Rua do Matão 1010, Cidade Universitária, CEP 05508-090, São Paulo-SP, Brasil

Joa Weber

Instituto de Matemática, Estatística e Computação Scientífica,
Universidade Estadual de Campinas,
Rua Sérgio Buarque de Holanda 651, Cidade Universitária "Zeferino Vaz",
CEP 13083-859, Campinas-SP, Brasil

Abstract. Assume M is a closed connected smooth manifold and $H : T^*M \to \mathbb{R}$ a smooth proper function bounded from below. Suppose the sublevel set $\{H < d\}$ contains the zero section M and α is a non-trivial homotopy class of free loops in M. Then for almost every $s \in [d, \infty)$ the level set $\{H = s\}$ carries a periodic orbit z of the Hamiltonian system (T^*M, ω_0, H) representing α. Examples show that the condition $\{H < d\} \supset M$ is necessary and almost existence cannot be improved to everywhere existence.

1. Introduction and main result

Suppose M is a smooth manifold and its cotangent bundle $\pi : T^*M \to M$ is equipped with the canonical symplectic structure $\omega_0 = -d\theta$. Here θ $(= pdq)$ denotes the canonical Liouville 1-form on T^*M. We view the elements of T^*M as pairs (q, p) where $q \in M$ and $p \in T^*_qM$. Given any function H on T^*M, the identity $dH = \omega_0(X_H, \cdot)$ uniquely determines the Hamiltonian vector field X_H on T^*M. The integral curves of X_H are called (Hamiltonian) orbits. They preserve the level sets of the total energy H. Of particular interest are periodic orbits, namely orbits $\gamma : \mathbb{R} \to T^*M$ such that $\gamma(t + T) = \gamma(t)$ for some constant $T > 0$ and every $t \in \mathbb{R}$. The

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infimum[1] over such T is called the period of γ. Given a family of energy levels, the question arises which levels carry a periodic orbit.

Existence of a periodic orbit on a dense set of energy levels was proved for $T^*\mathbb{R}^n$ by Hofer and Zehnder [9] in 1987 and for T^*M by Hofer and Viterbo [8] in 1988. The result for $T^*\mathbb{R}^n$ was extended to existence almost everywhere by Struwe [15] in 1990. Existence of *non-contractible* periodic orbits was studied among others in 1997 by Cieliebak [2] on starshaped levels in T^*M, in 2000 by Gatien and Lalonde [3] employing Lagrangian submanifolds, and in 2003 by Biran, Polterovich, and Salamon [1] on T^*M for $M = \mathbb{R}^n/\mathbb{Z}^n$ or M closed and negatively curved. The dense existence theorem in [1] was generalized in 2006 to all closed Riemannian manifolds in [17]. Theorem A below is the corresponding almost existence theorem. In contrast the almost existence theorem of Macarini and Schlenk [13] requires finiteness of the π_1-sensitive Hofer-Zehnder capacity. An assumption that has been verified to the best of our knowledge only for such cotangent bundles which carry certain circle actions; see [11,12]. For further references concerning dense and almost existence results we refer to [5] and concerning non-contractible orbits to [7].

Theorem A (Almost existence)**.** *Assume M is a closed connected smooth manifold and $H : T^*M \to \mathbb{R}$ is a proper[2] smooth function bounded from below. Suppose the sublevel set $\{H < d\}$ contains M. Then for every non-trivial homotopy class α of free loops in M the following is true. For almost every $s \in [d, \infty)$ the level set $\{H = s\}$ carries a periodic Hamiltonian orbit z that represents α in the sense that $[\pi \circ z] = \alpha$ where $\pi : T^*M \to M$ is the projection map.*

Proof. There are three main ingredients in the proof. The main player is the Biran-Polterovich-Salamon (BPS) [1] capacity $\mathrm{c}_{\mathrm{BPS}}$ whose monotonicity axiom Proposition 2.3 naturally leads to the monotone function $c_\alpha : [d, \infty) \to [0, \infty]$ defined by

$$c_\alpha(s) := \mathrm{c}_{\mathrm{BPS}}(\{H < s\}, M; \alpha). \tag{1}$$

Secondly, the existence result [17, Thm. A] concerning periodic orbits enters as follows: A priori the range of $\mathrm{c}_{\mathrm{BPS}}$ includes ∞ (by Definition 2.2 this is the case if no 1-periodic orbit representing α exists). To prove finiteness of the function c_α pick as an auxiliary quantity a Riemannian metric g on M. Then using [17, Thm. A] one readily calculates that the BPS capacity of the open unit disk cotangent bundle relative to its zero section is equal to the smallest length ℓ_α among all closed geodesics representing α; see [17, Thm. 4.3]. The rescaling argument in Lemma 2.4 shows that

[1] Here and thoughout we use the convention $\inf \emptyset = \infty$.

[2] A map is called proper if preimages of compact sets are compact.

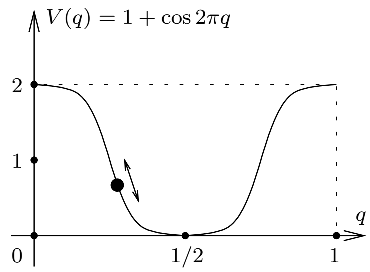

FIGURE 1. Potential energy V

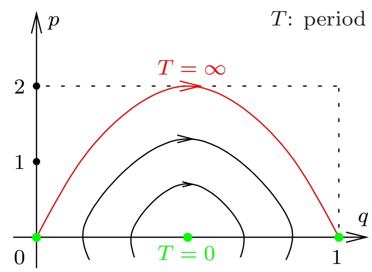

FIGURE 2. Pendulum phase portrait

the capacity of the open radius r disk cotangent bundle $D_r T^*M$ is $r\ell_\alpha$. Observe that $\{H \le s\}$ is compact since H is proper and bounded below. Hence the set $\{H < s\}$ is bounded and therefore contained in $D_r T^*M$ for some sufficiently large radius $r = r(s)$. Thus $c_\alpha(s) \le r(s)\ell_\alpha$ by the monotonicity axiom and this proves finiteness of c_α.

Thirdly, by Lebesgue's last theorem, see e.g. [14], it is well known, yet amazing, that monotonicity of the map $c_\alpha : [d,\infty) \to [0,\infty)$ implies differentiability, thus Lipschitz continuity, at almost every point s in the sense of measure theory. Now the key input is Theorem 3.1 whose proof is by an analogue of the Hofer-Zehnder method [10, Sec. 4.2] and which detects for each such s a periodic orbit on the corresponding level set $\{H = s\}$. □

Example 1.1 (Necessary condition)**.** The condition $\{H < d\} \supset M$ cannot be dropped in Theorem A. First of all, together with H being proper and bounded below, it guarantees that each level set $\{H = s\}$ is actually nonempty whenever $s \in [d,\infty)$. Now consider a pendulum. It moves on $M = S^1 = \mathbb{R}/\mathbb{Z}$ in a potential of the form $V(q) = 1 + \cos 2\pi q$; see Figure 1. The Hamiltonian $H : T^*M = S^1 \times \mathbb{R} \to \mathbb{R}$ is given by $H(q,p) = \frac{1}{2}p^2 + V(q)$; see Figure 2 for the phase portrait. Energies below the maximum value 2 of the potential V do not allow for full rotations. For such low energies the pendulum can just swing hence and forth. Observe that $\{H < 1\} \not\supset M$. On the other hand, for any energy $s \in [1,2)$ the level set $\{H = s\}$ consists of a periodic orbit which is contractible onto the stable (lower) equilibrium point $(x,y) \equiv (1/2,0)$. So none of these orbits represents a homotopy class $\alpha \neq 0$. (For $s > 2$ the sets $\{H = s\}$ represent classes $\alpha \neq 0$. The set $\{H = 2\}$ consists of the unstable (upper) equilibrium point and two homoclinic orbits one of them indicated red in Figure 2.)

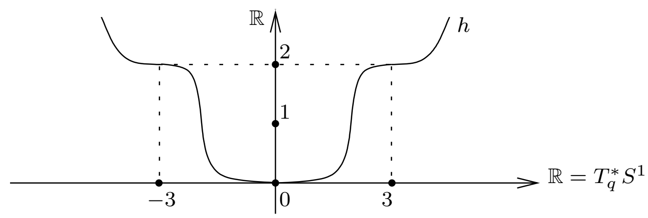

FIGURE 3. Hamiltonian $H = h(p)$ without non-constant orbits on $\{H = 2\}$

Example 1.2 (Existence everywhere not true)**.** To see that *almost* existence in Theorem A cannot be improved to *everywhere* existence consider the case $M = S^1$ and a Hamiltonian $H : S^1 \times \mathbb{R} \to \mathbb{R}$ of the form $H(q,p) = h(p)$. More precisely, pick a proper smooth function $h \geq 0$ with $h(0) = 0$ and $h(\pm 3) = 2$ and where the points $0, \pm 3$ are the only points of slope zero; see Figure 3. Then $\{H < 1\}$ contains $M = S^1$. Moreover, the whole level set $\{H = 2\}$ consists of critical points of H. Therefore on $\{H = 2\}$ the Hamiltonian vector field X_H vanishes identically and so all orbits are necessarily constant.

In contrast to this critical level counterexample it should be interesting to find a regular level of a smooth Hamiltonian H as in Theorem A without a periodic orbit in a given homotopy class $\alpha \neq 0$. One possible way to achieve this is to start with an energy level with finitely many periodic orbits representing α, then destroy them using the symplectic plugs constructed in [4].

For general symplectic manifolds existence may fail completely; see [18] and [16] for examples of closed symplectic manifolds admitting Hamiltonians with no non-constant periodic orbits.

2. Symplectic capacities

To fix notation consider $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Associate to each symplectic manifold (N, ω), of fixed dimension $2n > 0$ and possibly with boundary, a number $c(N, \omega) \in [0, \infty]$ that satisfies the axioms:

- **Monotonicity:** $c(N_1, \omega_1) \leq c(N_2, \omega)$ whenever there is a symplectic embedding $\psi : (N_1, \omega_1) \to (N_2, \omega_2)$.
- **Conformality:** $c(N, \lambda\omega) = |\lambda|\, c(N, \omega), \forall \lambda \in \mathbb{R} \setminus \{0\}$.
- **Non-Triviality:** $c(B(1), \omega_0) = c(Z(1), \omega_0) = \pi$.

Here $B(r) = \{(x, y) \in \mathbb{R}^{2n} : |x|^2 + |y|^2 < r^2\}$ is the ball of radius $r > 0$ and $Z(r) = \{(x, y) \in \mathbb{R}^{2n} : x_1^2 + y_1^2 < r^2\}$ is the symplectic cylinder of radius $r > 0$. On $(\mathbb{R}^{2n}, \omega_0)$, one checks the following re-scaling property:

$$U \subset \mathbb{R}^{2n} \text{ open} \quad \Rightarrow \quad c(\lambda U, \omega_0) = \lambda^2 c(U, \omega_0), \forall \lambda \in \mathbb{R} \setminus \{0\}.$$

A map $(N, \omega) \mapsto c(N, \omega)$ satisfying the three axioms above is called a symplectic capacity. Gromov introduced this notion in [6] and showed that

$$c_0(N, \omega) := \sup \left\{\pi r^2 : \exists \text{ symplectic embedding } \psi : (B(r), \omega_0) \to (N, \omega)\right\}$$

is a symplectic capacity, called Gromov's width. It satisfies $c_0(N, \omega) \leq c(N, \omega)$ for any other symplectic capacity c. One of the consequences of the existence of a symplectic capacity is the non-squeezing theorem which asserts that

$$\exists \text{ symplectic embedding } \psi : (B(r), \omega_0) \to (Z(R), \omega_0) \quad \Leftrightarrow \quad r \leq R.$$

2.1. **Hofer-Zehnder capacity.** Hofer and Zehnder introduced in [10] a symplectic capacity defined in terms of the Hamiltonian dynamics on the underlying symplectic manifold (N, ω). Recall that a smooth function $H : N \to \mathbb{R}$ determines the Hamiltonian vector field X_H by $i_{X_H}\omega = dH$. We say that a periodic orbit of $\dot{x} = X_H(x)$ is fast if its period is < 1. A function $H : N \to \mathbb{R}$ is called admissible if it admits a maximum and the following conditions hold:

- $0 \leq H \leq \max H < \infty$.
- $\exists K \subset N \setminus \partial N$ compact, such that $H|_{N \setminus K} = \max H$.
- $\exists U \subset N$ open and non-empty, such that $H|_U = 0$.
- $\dot{x} = X_H \circ x$ admits no non-constant fast periodic orbits.

The set of admissible Hamiltonians is denoted by $\mathcal{H}_a(N, \omega)$. Let

$$\mathrm{c_{HZ}}(N, \omega) := \sup\{\max H \mid H \in \mathcal{H}_a(N, \omega)\}.$$

Theorem 2.1 (Hofer-Zehnder)**.** $\mathrm{c_{HZ}}$ *is a symplectic capacity.*

We should remark that the hard part of proving Theorem 2.1 is to show that $\mathrm{c_{HZ}}$ satisfies the non-triviality axiom.

2.2. **BPS relative capacity.** Fix a closed manifold M. The components $\mathcal{L}_\alpha M$ of the free loop space $\mathcal{L}M := C^\infty(S^1, M)$ are labelled by the elements $\alpha = [\gamma]$ of the set $\tilde{\pi}_1(M)$ of homotopy classes of free loops γ in M. Here and throughout we identify S^1 with $\mathbb{R}/\mathbb{Z}$ and think of γ as a smooth map $\gamma : \mathbb{R} \to M$ that satisfies $\gamma(t + 1) = \gamma(t)$ for every $t \in \mathbb{R}$. A function $H \in C_0^\infty(S^1 \times T^*M)$ determines a 1-periodic family of compactly supported vector fields X_{H_t} on T^*M by $dH_t = \omega_0(X_{H_t}, \cdot)$. Let

$$\mathcal{P}_1(H; \alpha) := \{z : S^1 \to T^*M \mid \dot{z}(t) = X_{H_t}(z(t))\, \forall t \in S^1, [\pi \circ z] = \alpha\}$$

be the set of 1-periodic orbits of X_{H_t} whose projections to M represent α.

Definition 2.2. Following [1] assume $W \subset T^*M$ is an open subset which contains the zero section M. For any constant $b > 0$ consider the set

$$\mathcal{H}_b(W) := \left\{ H \in C_0^\infty(S^1 \times W) \,\middle|\, m_0(H) := \max_{S^1 \times M} H \leq -b \right\}.$$

The BPS capacity of W relative M and with respect to $\alpha \in \tilde{\pi}_1(M)$ is defined by

$$\mathrm{c}_{\mathrm{BPS}}(W, M; \alpha) := \inf \{ b > 0 \mid \mathcal{P}_1(H; \alpha) \neq \emptyset \text{ for every } H \in \mathcal{H}_b(W) \}. \quad (2)$$

Note that $\mathrm{c}_{\mathrm{BPS}}$ takes values in $[0, \infty]$ since we use the convention $\inf \emptyset = \infty$. Furthermore, the BPS capacity is a relative symplectic capacity.

Proposition 2.3 (Monotonicity [1, Prop. 3.3.1])**.** *If $W_1 \subset W_2 \subset T^*M$ are open subsets containing M and $\alpha \in \tilde{\pi}_1(M)$, then $\mathrm{c}_{\mathrm{BPS}}(W_1, M; \alpha) \leq \mathrm{c}_{\mathrm{BPS}}(W_2, M; \alpha)$.*

Fix a Riemannian metric on M and constants $r, b > 0$. Denote by DT^*M the open unit disk cotangent bundle and by D_rT^*M the one of radius r. Observe that

$$H \in \mathcal{H}_b(DT^*M) \quad \Longleftrightarrow \quad H_r \in \mathcal{H}_{rb}(D_rT^*M) \quad (3)$$

whenever the Hamiltonians H and H_r are related by $H_r(t, q, p) = r \cdot H(t, q, \frac{p}{r})$. In addition, pick $\alpha \in \tilde{\pi}_1(M)$. Then there is the crucial bijection

$$\mathcal{P}_1(H; \alpha) \to \mathcal{P}_1(H_r; \alpha) : (x, y) \mapsto (x, ry) \quad (4)$$

asserting that the 1-periodic orbits of H correspond naturally with those of H_r.

Lemma 2.4 (Rescaling)**.** $\mathrm{c}_{\mathrm{BPS}}(D_rT^*M, M; \alpha) = r \cdot \mathrm{c}_{\mathrm{BPS}}(DT^*M, M; \alpha)$.

Proof. By definition (2) of the BPS capacity we obtain that

$$\begin{aligned} &\mathrm{c}_{\mathrm{BPS}}(D_rT^*M, M; \alpha) \\ &= \inf \{ rb > 0 \mid \mathcal{P}_1(H_r; \alpha) \neq \emptyset \text{ for every } H_r \in \mathcal{H}_{rb}(D_rT^*M) \} \\ &= r \cdot \inf \{ b > 0 \mid \mathcal{P}_1(H; \alpha) \neq \emptyset \text{ for every } H \in \mathcal{H}_b(DT^*M) \} \\ &= r \cdot \mathrm{c}_{\mathrm{BPS}}(DT^*M, M; \alpha) \end{aligned}$$

where the second step uses (3) and (4). □

Corollary 2.5. $\mathrm{c}_{\mathrm{BPS}}(D_rT^*M, M; \alpha) = r\ell_\alpha$ *where ℓ_α is the smallest length among all closed curves representing α.*

Proof. $\mathrm{c}_{\mathrm{BPS}}(DT^*M, M; \alpha) = \ell_\alpha$ by [17, Thm. 4.3]. Apply Lemma 2.4. □

3. The Hofer-Zehnder method

Assume the Hamiltonian $H : T^*M \to \mathbb{R}$ is smooth, proper, and bounded from below and a sublevel set $\{H < d\}$ contains M. Fix a non-trivial homotopy class α of free loops in M. Consider the monotone function c_α defined on the interval $[d, \infty)$ by (1). By Lebesgue's last theorem, see e.g. [14, p. 401], the function c_α is differentiable at almost every point in the sense of measure theory.

Theorem 3.1. *Assume $s_0 \in [d, \infty)$ is a regular value of H and c_α is Lipschitz continuous at s_0. Then the hypersurface $S = H^{-1}(s_0)$ carries a periodic orbit z_T of X_H that represents α and where $T > 0$ is the period.*

Proof. The proof is an adaption of the Hofer-Zehnder method [10, Sec. 4.2] to the case at hand. To emphasize this we mainly keep their notation. Fix s_0 as in the hypothesis of the theorem. Then $S_0 := H^{-1}(s_0)$ is a hypersurface[3] in T^*M by the inverse function theorem. It is compact since H is proper and it bounds the open set $\dot{B}_0 := \{H < s_0\}$ since H is bounded below. Furthermore, by the implicit function theorem and compactness of S_0 there is a constant $\mu > 0$ such that $s_0 + \varepsilon$ is a regular value of H and $S_\varepsilon := H^{-1}(s_0 + \varepsilon)$ is diffeomorphic to S_0 whenever $\varepsilon \in [-\mu, \mu]$. Note that S_ε bounds the open set $\dot{B}_\varepsilon := \{H < s_0 + \varepsilon\}$ which itself contains the zero section M of T^*M. Furthermore, since c_α is Lipschitz continuous at s_0 there is a constant $L > 0$ such that

$$c(\varepsilon) - c(0) \leq L\varepsilon, \qquad c(\varepsilon) := c_\alpha(s_0 + \varepsilon), \tag{5}$$

for every $\varepsilon \in [-\mu, \mu]$; otherwise, choose $\mu > 0$ smaller. We proceed in three steps I–III.

I. Pick $\tau \in (0, \mu)$. Then there is a Hamiltonian $K \in C_0^\infty(S^1 \times \dot{B}_0)$ whose maximum over the zero section satisfies

$$-c(0) < m_0(K) \leq -\left(c(0) - L\tau\right)$$

and which does not admit any 1-periodic orbit representing α. Indeed if no such K exists, then $c_{\mathrm{BPS}}(\dot{B}_0, M; \alpha) \leq c(0) - L\tau$ by definition (2) of the BPS capacity. But $c(0) = c_\alpha(s_0) = c_{\mathrm{BPS}}(\dot{B}_0, M; \alpha)$ and we obtain the contradiction $c(0) \leq c(0) - L\tau$. Now pick a smooth function $f : \mathbb{R} \to [-3L\tau, 0]$ such that

$$\begin{aligned} f(s) &= -3L\tau && \text{if} \quad s \leq 0 \\ f(s) &= 0 && \text{if} \quad s \geq \tfrac{\tau}{2} \\ 0 < f'(s) &\leq 7L && \text{if} \quad 0 < s < \tfrac{\tau}{2} \end{aligned}$$

[3] A hypersurface is a smooth submanifold of codimension 1.

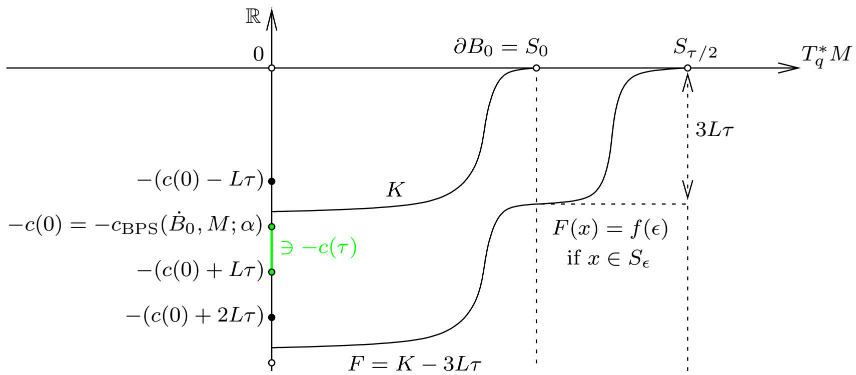

FIGURE 4. Hamiltonians $F \in \mathcal{H}_{c(\tau)}(\dot{B}_\tau)$ and K with $\mathcal{P}_1(K;\alpha) = \emptyset$

and consider the Hamiltonian $F \in C_0^\infty(S^1 \times \dot{B}_\tau)$ defined by

$$\begin{aligned}
&F(x) = K(x) - 3L\tau && \text{if} && x \in \dot{B}_0 \\
&F(x) = f(\varepsilon) && \text{if} && x \in S_\varepsilon = H^{-1}(s_0 + \varepsilon),\ 0 \le \varepsilon < \tau \\
&F(x) = 0 && \text{if} && x \notin \dot{B}_\tau
\end{aligned}$$

and illustrated by Figure 4. By (5) the Hamiltonian F satisfies the estimate

$$m_0(F) = m_0(K) - 3L\tau \le -(c(0) - L\tau) - 3L\tau < -(c(0) + L\tau) \le -c(\tau).$$

Since $m_0(F) \le -c_{\mathrm{BPS}}(\dot{B}_\tau, M; \alpha)$ the definition (2) of the BPS capacity shows that the set $\mathcal{P}_1(F;\alpha)$ is not empty. In other words, there is a 1-periodic orbit z of X_F that represents α. Observe that z cannot intersect $\dot{B}_0$: Due to compact support the open set $\dot{B}_0$ is invariant under the flow of K. But the flows of K and $K - 3L\tau = F$ coincide. Thus, if z intersects $\dot{B}_0$, then it stays completely inside. But this is impossible since $\mathcal{P}_1(K;\alpha) = \emptyset$. On the other hand, since $\alpha \ne 0$ the orbit z of X_F is non-constant and therefore it must intersect the regions foliated by the hypersurfaces S_ε where $0 < \varepsilon < \frac{\tau}{2}$. But each of them is a level set of F, hence invariant under the flow of X_F. This shows that z lies on S_ε for some $0 < \varepsilon < \frac{\tau}{2}$.

II. Repeat the argument for each element of a sequence $\tau_j \to 0$ to obtain sequences F_j and ε_j and a sequence z_j of 1-periodic orbits of X_{F_j} that lie on S_{ε_j} and where $\varepsilon_j \to 0$. Next we interpret each z_j as a T_j-periodic orbit of X_H by rescaling time. Most importantly, the periods T_j are uniformly bounded from above by $7L$. To see this note that on the open set

$$U := \bigcup_{\varepsilon\in(-\mu,\mu)} S_\varepsilon = \bigcup_{\varepsilon\in(-\mu,\mu)} H^{-1}(s_0+\varepsilon)$$

the Hamiltonian H is obviously given by $H(x) = s_0 + \varepsilon$ whenever $x \in S_\varepsilon$. For each τ_j and each $\varepsilon \in [0, \tau_j)$ we have

$$F_j(x) = f_j(H(x) - s_0) = f_j(\varepsilon)$$

for every $x \in S_\varepsilon$. At such x use the definition of X_{F_j} and the chain rule to get

$$\omega_0\left(X_{F_j}, \cdot\right) = dF_j = f_j'(H - s_0)dH = \omega_0\left(f_j'(\varepsilon)X_H, \cdot\right).$$

Thus, because z_j lies on S_{ε_j}, it satisfies the equation

$$\dot z_j(t) = X_{F_j} \circ z_j(t) = T_j \cdot X_H \circ z_j(t), \qquad T_j := f_j'(\varepsilon_j),$$

and the periodic boundary condition $z_j(t+1) = z_j(t)$ for every $t \in \mathbb{R}$.

III. Uniform boundedness of the periods T_j is crucial in the following proof of existence of a 1-periodic orbit z of X_H which lies on the original level hypersurface $S_0 = H^{-1}(s_0)$ and represents the given class α. Indeed note that $S_{\varepsilon_j} \subset \{H \leq s_0 + \mu\} =: B_\mu$ and that B_μ is compact since H is proper and bounded below. In other words, the sequence of loops z_j is uniformly bounded in C^0. Concerning C^1 we obtain the uniform estimate

$$|\dot z_j(t)| = |T_j| \cdot |X_H \circ z_j(t)| \leq 7L \left\|X_H\right\|_{C^0(B_\mu)}$$

for all $t \in S^1$ and $j \in \mathbb{N}$. Therefore by the Arzelà-Ascoli theorem there is a subsequence, still denoted by z_j, which converges in C^0 and by using the equation for z_j even in C^∞ to a smooth 1-periodic solution z of the equation $\dot z = T \cdot X_H(z)$ where $T = \lim_{j\to\infty} T_j$. Since $\varepsilon_j \to 0$ the orbit z takes values on the desired level hypersurface $S_0 = H^{-1}(s_0)$. To prove that $z = (x, y)$ represents the same class α as does each $z_j = (x_j, y_j)$ we need to show that $[x] = [x_j]$ for some j. To see this consider the injectivity radius $\iota > 0$ of the compact Riemannian manifold (M, g) and pick j sufficiently large such that the Riemannian distance between $x(t)$ and $x_j(t)$ is less than $\iota/2$ for every $t \in S^1$. Setting $\exp_{x(t)} \xi(t) = x_j(t)$ provides the desired homotopy $h_\lambda(t) = \exp_{x(t)} \lambda\xi(t)$ between $h_0 = x$ and $h_1 = x_j$.
Reparametrize time to obtain the T-periodic solution $z_T(t) := z(t/T)$ of

$$\dot z_T(t) = \tfrac{1}{T}\dot z(t/T) = X_H \circ z(t/T) = X_H \circ z_T(t)$$

which obviously represents the same class α as z. Since $\alpha \neq 0$ the loop z_T cannot be constant and so the period necessarily satisfies $T > 0$. This concludes the proof of Theorem 3.1. □

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Second author. Financial support: FAPESP grant 2011/01830-1. Email: joa@math.sunysb.edu

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