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Known Results and Open Problems on $\mathcal{C}^1$ linearization in Banach Spaces

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Dedicated to Luis T. Magalhães and Carlos Rocha, on the occasion of their 60th birthday

Abstract. The purpose of this paper is to review the results obtained by the authors on linearization of dynamical systems in infinite dimensional Banach spaces, especially in the $\mathcal{C}^1$ case, and also to present some open problems that we believe that are still important for the understanding of the theory.

The purpose of the present paper is to make a review of results obtained by the authors in the field of linearization of dynamical systems in infinite dimensional Banach spaces, with a focuss on $\mathcal{C}^1$ linearization of contractions, with the intention of stating in the right context a number of open problems we believe it would be worth to be solved.

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With this structure, the paper does not pretend to make a list of open problems in the whole field of infinite dimensional linearization or conjugation, that has many different aspects and areas, but merely to concentrate in the areas where the authors have been working, and where we feel the open questions as something that concern us more directly. Also, the present paper has its origin on a summary that, for different reasons, the authors have wanted to do on its joint work. As a kind of statement of purposes, we can say that the goal of our research in this field has been to try to discover the aspects of the problems that are truly related to the fact that the phase space is infinite dimensional.

This joint work started with results on $\mathcal{C}^1$ linearization, and we also start our presentation in this area. The first result that inspired us was the important result by P. Hartman that essentially says that every smooth contraction in finite dimensions can be conjugated in a neighborhood of its fixed point to its linear part at the fixed point, by a conjugacy that is of the class $\mathcal{C}^1$:

Theorem 1. ($\mathcal{C}^1$ linearization of contractions in finite dimensions, P. Hartman, 1960, [5]) *Let A be an $n \times n$ invertible matrix such that $\|A\| < 1$, and $\mathcal{X} \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ be such that $\mathcal{X}(0) = 0$ and $D\mathcal{X}(0) = 0$. Then, for the nonlinear map $Tx = Ax + \mathcal{X}(x)$ there exists a conjugacy map $Rx = x + \phi(x)$ in a neighborhood of $x = 0$, with $\phi \in \mathcal{C}^{1,\beta}$ $(0 < \beta < 1)$, $\phi(0) = 0$ and $D\phi(0) = 0$, and such that $RTR^{-1} = A$.*

We remark that in the same paper, P. Hartman presented an example in three dimensions, that is not a contraction but a saddle, that cannot be linearized in the class $\mathcal{C}^1$ because of a resonance relation. Nevertheless, he also proved that in two dimensions a saddle can always be linearized in the class $\mathcal{C}^1$. This result for the saddle was the motivation for one of our results, namely Theorem 4 below. There is also a well-known example due to Sternberg of a contraction in two dimensions that cannot be linearized in the class $\mathcal{C}^2$.

Another influential result for our work was that of X. Mora and the second author, that, to our knowledge, was the first to deal with some cases of smooth linearization of contractions in infinite dimensions:

Theorem 2. (X. Mora and J. S.-M., 1987, [7]) *Let Let X be a Banach space, A, $A^{-1} \in \mathcal{L}(X)$ and suppose that $\mathcal{X} \in \mathcal{C}^1(X)$ be such $\mathcal{X}(0) = 0$, $D\mathcal{X}(0) = 0$ and that the following properties are satisfied for some $\eta > 0$:*

$$\begin{cases} D\mathcal{X}(x) = o(||x||^{\eta}), \\ ||A^{-1}|| \; ||A||^{1+\eta} < 1. \end{cases} \tag{1}$$

Then there exists a $\mathcal{C}^1$ diffeomorphism R defined in a neighborhood of $x = 0$ with $R(0) = 0$ and $DR(0) = I$ such that $RT = TR$.

This Theorem is also reproduced in [8]. Observe that η is restricted to be positive, but otherwise can be large. It is easy to see that the inequality $||A^{-1}||\, ||A||^{1+\eta} < 1$ is satisfied by an equivalent norm (see [16], for example) if and only if $|\sigma(A)|$ is contained in an open interval of the form (a^-, a^+) with $(a^+)^{1+\eta} < a^-$. So, the restrictions on $\sigma(A)$ become weaker when the order of the nonlinearity near zero becomes larger. This will give rise to one of the open problems below.

The following was our joint result on $\mathcal{C}^1$ linearization of contractions:

Theorem 3. ($\mathcal{C}^1$ linearization of contractions, H.M.R. and J. S.-M., 2004, [12]) *Let X be a $\mathcal{C}^{1,1}$ Banach space and $A,\ A^{-1} \in \mathcal{L}(X)$. Let $\nu_i^-,\ \nu_i^+,\ i = 1, \ldots, n$ be such that:*

$$\begin{gathered} 0 < \nu_n^- < \nu_n^+ < \nu_{n-1}^- < \nu_{n-1}^+ < \cdots \nu_1^- < \nu_1^+ < 1 \\ |\sigma(A)| \subset \cup_{i=1}^n (\nu_i^-, \nu_i^+) \\ \nu_1^+ \nu_i^+ < \nu_i^-,\ i = 1, \ldots, n \end{gathered} \tag{2}$$

Let $\mathcal{X} = \mathcal{X}(x)$ be a $\mathcal{C}^{1,1}$-function in a neighborhood of the origin, such that $\mathcal{X}(0) = 0,\ D\mathcal{X}(0) = 0$. Then, for the map $Tx = Ax + \mathcal{X}(x)$ there exists a $\mathcal{C}^1$-map $Rx = x + \phi(x)$ satisfying $\phi(0) = 0,\ D\phi(0) = 0$, such that $RTR^{-1} = A$ in a sufficiently small neighborhood of the origin.

(We recall that a Banach space is said to be of class $\mathcal{C}^{1,1}$ if it admits a cut-off function in this class).

With similar hypotheses, very close results were proved independently by Mohamed El Bialy ([4], 2001) and Brahim Abbaci ([1], 2004). Despite of the fact that we were sorry for not having been aware of these independent works, in some sense, the coincidence of the three lines of research in that a *non-resonance condition* like the condition $\nu_1^+ \nu_i^+ < \nu_i^-,\ i = 1, \ldots, n$ of (2), or the condition 1 in Theorem 2 should be really necessary. Our interest in this point was reinforced by the following sentence written in [1] as the last phrase in the paper: *Apparently, it is not known whether every $\mathcal{C}^{1,1}$ strict contraction germ in an infinite dimensional space is $\mathcal{C}^1$-linearizable.*

It is worth to say that the non-resonance conditions (2) are automatically satisfied by every linear contraction in finite dimensions, and in this sense Theorem 3 extends the result of Theorem 1. We can also mention that in our paper [12] an application of Theorem 3 to some partial differential equations, namely semilinear damped wave equations, was also given.

The previous considerations gave us the approximate idea that the $\mathcal{C}^1$ linearization results of Hartman could be extended to infinite dimensions provided that the modulus of the elements of $|\sigma(A)|$ lie in sufficiently small intervals. This point of view lead us to prove the following theorem, that linearizes the case of at least some saddles in infinite dimensions:

Theorem 4. ($\mathcal{C}^1$ linearization of a saddle, H.M.R. and J. S.-M., 2004, [13]) *Let X be a $\mathcal{C}^{1,1}$ Banach space, A, $A^{-1} \in \mathcal{L}(X)$. Suppose there exist real numbers s^-, s^+, u^-, u^+ such that*

$$\begin{array}{l} 0 < s^- < s^+ < 1 < u^- < u^+ \\ s^+u^+ < u^-, \; s^+ < s^-u^-, \; (s^+)^2 < s^-, \; u^+ < (u^-)^2 \\ |\sigma(A)| \subset (s^-, s^+) \cup (u^-, u^+). \end{array} \tag{3}$$

Let $\mathcal{X} = \mathcal{X}(x)$ be a $C^{1,1}$-function in a neighborhood of $x = 0$ with values in X, such that $\mathcal{X}(0) = 0$, $D\mathcal{X}(0) = 0$. Then for the map $T : x \mapsto x^1$, $x^1 = Lx + \mathcal{X}(x)$, there exists a $\mathcal{C}^1$-map $R : x \mapsto u$, $u = x + \phi(x)$, satisfying $\phi(0) = 0, D\phi(0) = 0$, such that $RTR^{-1} : u \mapsto u^1$ has the form $u^1 = Au$ in a sufficiently small neighborhood of $x = 0$.

In the same paper we also applied this theorem to abstract semilinear damped wave equations.

Our next result dealt with an example of an infinite dimensional smooth contraction that cannot be linearized in the class $\mathcal{C}^1$. We published two of these examples ([14], [15]), quite similar in its structure. The second was simpler, and it allowed to be shown to be the time one map of a smooth evolution ordinary differential equation in the Hilbert space ℓ^2. We present here only the second example.

We introduce some notations in order to state the result. We define the infinite matrix J and the nonlinear function $f(y)$ for $y \in \mathbb{R}$ as:

$$J := \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad f(y) := \begin{pmatrix} y^2 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{4}$$

Let a, δ and ε be positive real numbers. Consider the following infinite matrix A and the nonlinear function $\mathcal{X}$, acting in the Hilbert space $\ell^2 \ni x = (y, z) = (y, z_1, z_2, \dots)$ defined as:

$$A := \begin{pmatrix} a & 0 \\ 0 & \delta J + aI \end{pmatrix}, \quad \mathcal{X}(x) := \begin{pmatrix} 0 \\ \varepsilon f(y) \end{pmatrix}, \quad x := \begin{pmatrix} y \\ z \end{pmatrix} \tag{5}$$

And with these notations, we state the next result:

Theorem 5. (A first counter-example on $\mathcal{C}^1$ linearization of contractions, H.M.R. and J. S.-M., 2006, [15]) *Let $\varepsilon \neq 0$ and $0 < a < 1$. Under the hypothesis,*

$$a - a^2 \leq \delta < \min\{1 - a, a\} \tag{6}$$

the operator A is an invertible contraction on ℓ^2, its spectrum $\sigma(A)$ is the closed disk of center a and radius δ and the local analytic diffeomorphism defined in ℓ^2 by,

$$x_1 = Tx := Ax + F(x)$$

does not conjugate, even locally, with its linear part A, through a conjugation R, with R and R^{-1} differentiable at $x = 0$.

Also, this map T is the time-one map of an ordinary differential equation in ℓ^2 of the form

$$\dot{x} = Lx + G(x) \tag{7}$$

where $L := \log A$ is a bounded linear operator and $G : \ell^2 \to \ell^2$ is defined by

$$G(x) := \begin{pmatrix} 0 \\ y^2\beta \end{pmatrix}$$

for some $\beta \in \ell^2$.

The idea of the proof is very simple. When we write $x \in \ell^2$ as $x = (y, z_1, z_2, \dots)$ one easily sees that if this smooth linearization exists then also a one-dimensional invariant manifold tangent to the y-axis at $y = 0$ should also exist. And the nice thing is that if this manifold is written as $\{(y, \phi_1(y), \phi_2(y), \dots)\}$ then the functions $\phi_i(y)$ can be recursively calculated, and turn out necessarily to be

$$\phi_i(y) = \frac{\delta^{i-1}}{(a^2 - a)^i} \varepsilon y^2$$

for $i = 1, 2 \dots$ and this obviously does not define an element of ℓ^2 if $\delta > |a^2 - a|$.

Another result of [15] is worth to be explained now, and for the clarity of the presentatation will be stated as a new theorem:

Theorem 6. (A second counter-example on $\mathcal{C}^1$ linearization of contractions, H.M.R. and J. S.-M., 2006, [15]) *For each $b \in (0, 1)$ and each sufficiently small $r > 0$ the example of Theorem 5 above can be modified to $T_{b,r} = A_{b,r} + \mathcal{X}_{b,r}$ instead of $T = A + \mathcal{X}$ in such a way that $[b^2, b] \subset |\sigma(A_{b,r})| \subset [b^2, b+r]$ and $T_{b,r}$ is still not linearizable in the class $\mathcal{C}^1$. The map $T_{b,r}$ can also be seen to be the time-one map of a regular ordinary differential equation in ℓ^2.*

The proof follows the same ideas as before, but it is slightly more involved. It requires functional calculus of bounded operators, the Spectral Theorem and the Open Mapping Theorem.

The last result we want to present deals with $\mathcal{C}^0$ linearization, so it lies somehow out of the main purpose of this paper, namely $\mathcal{C}^1$ linearization.

But we present it because one subject is not so far from the other. We came into that because we have perhaps the idea, influenced by the papers of K. Lu [6] and P. Bates and K. Lu [3] on linearization of parabolic equations, that some infinite dimensional dynamical systems could be often seen as limits or families of simpler systems, for example finite-dimensional. We do not claim that our result achieves this goal, but it goes in the direction of understanding linearizations inside a family of dynamical systems. In the context of the Hartman-Grobman Theorem, this is well-known when the parameter appears only in the nonlinear part, but in our result it appear both in the linear and the nonlinear parts.

Theorem 7. (A Hartman-Grobman theorem with parameters, H.M.R. and J. S.-M., 2010, [17]) *Suppose X a Banach space, Θ a metric space, $\theta \mapsto L_\theta$ is continuous from Θ to $\mathcal{L}(X)$ and $|L_\theta^{-1}| \leq N$. Suppose that*

$$\begin{cases} |L_\theta^k P_\theta| \leq M r^k, \\ |L_\theta^{-k}(I - P_\theta)| \leq M r^k \end{cases}$$

for some $r \in (0,1)$ and for a family of projections P_θ that commute with L_θ.

Suppose also that $\Theta \ni \theta \mapsto F_\theta \in BUC(X,X)$ is continuous, $Lip(F_\theta) \leq \mu$ and $\lim_{|x|\to\infty} F_\theta(x) = 0$.

If μ is sufficiently small, then for every θ there exists a unique $g_\theta \in BUC(X,X)$ such that $R_\theta := I + g_\theta$ is a homeomorphism and $R_\theta(L_\theta + F_\theta) = L_\theta R_\theta$. We claim that $\Theta \ni \theta \mapsto g_\theta$ is continuous.

For the proof of this theorem we needed a result on families of contractions with distances that depend on a parameter. This is an extension of the well-known Banach-Cacciopoli theorem, that has some interest in itself, and it is worth to be reproduced here

Theorem 8. (A contraction theorem with parameters and variable distances, H.M.R. and J. S.-M., 2010, [17]) *Suppose (M,d) is a complete metric space, Θ a metric space, $T : M \times \Theta \ni (x,\theta) \mapsto T(x,\theta) \in M$ and d_θ a distance in M, depending on $\theta \in \Theta$, such that*

$$q\ d(x,y) \leq d_\theta(x,y) \leq Q\ d(x,y),$$
$$d_\theta(T(x,\theta), T(y,\theta)) \leq \rho\ d_\theta(x,y), \qquad \rho \in (0,1)$$

Let $x = g(\theta)$ be the unique fixed point of $T(\cdot,\theta)$. Suppose that for each $\theta_0 \in \Theta$ the function: $\Theta \ni \theta \mapsto T(g(\theta_0),\theta) \in M$ is continuous in $\theta = \theta_0$. We claim that the function $g(\theta)$ in continuous in Θ.

To finish this review part we want to mention also other sources of ideas and references on linearization and smooth linearization. This subjects are really wide, and we do no pretend to give a complete list. We just mention published papers that have been influential in our work, or that we believe that are outstanding in some sense. Apart from the references that we already made, we want to mention the paper by C. C. Pugh (1969) [9], that of G. Sell (1985) [18], the first author and J.G. Ruas Filho (1997) [11], V. Rayskin (1998) [10], B. Tan (2000) [19] and the more recent paper by L. Barreira and C. Valls (2007) [2].

Now we proceed to present some open problems related to the previous results.

Problem 1: Better counter-examples.

A better understanding of the non-resonance conditions (2) needs better counterexamples. It would be good to have a counterexample as in Theorem 6 but with $r = 0$. Or, even better, a counterexample with $|\sigma(A)| = [ab, b] \cup \{a\}$, with $0 < b < a < 1$.

Problem 2: Jordan blocks.

Our counterexamples have an infinite-dimensional Jordan block. Is this necessary? Can one have counterexamples with a selfadjoint linear part A? If this is not the case, can one improve (2) if A is selfadjoint, or even normal?

Problem 3: Higher order nonlinearities.

As it is seen in the statement of Theorem 3 it is considered that the nonlinearity $\mathcal{X}(x)$ vanishes together with its first derivative at $x = 0$. Since the nonlinearity is of class $\mathcal{C}^{1,1}$ this amounts to say that $\mathcal{X}(x) = O(|x|^2)$. As it happens in Theorem 2 one can expect that larger values of η if $\mathcal{X}(x) = O(|x|^{1+\eta})$ would give less restrictive conditions on $\sigma(A)$ in (2). This deserves to be studied. Odd scalar nonlinear functions with zero derivative, for example, have a Taylor expansion that starts with terms of order three, and this is a situation that can appear in many practical applications.

Problem 4: Counter-examples for higher order nonlinearities.

At the same time as one could work on solving Problem 3 one should look for counterexamples, to see if there are limits to these less restrictive conditions.

In the directions of this last Problem 4 we present here a small new result. It is just a generalization of Theorem 6 above to different order nonlinearities. The proof is also more or less the same, but we believe that it is significant enough to reproduce the main lines of the proof also here.

Theorem 9. (Examples with nonlinearities of different order) *Let $\eta > 0$, $0 < c < 1$ and $r > 0$ be given, with r small enough. In the Hilbert space ℓ^2 there exists a bounded linear operator $A_{c,r}$ such that $[c^{1+\eta}, c] \subset |\sigma(A_{c,r})| \subset [c^{1+\eta}, c+r]$ and a $\mathcal{C}^1$ nonlinear map $\mathcal{X}_{c,r}$ such that $\mathcal{X}_{c,r}(0) = 0$ and $|D\mathcal{X}_{c,r}(x)| = O(|x|^\eta)$ with the property that $T_{c,r} = A_{c,r} + \mathcal{X}_{c,r}$ is not linearizable around $x = 0$ in the class $\mathcal{C}^1$, or even in the larger class of the homeomorphisms that are differentiable at the origin.*

Observe that η can be chosen freely in $0 < \eta < \infty$. For $0 < \eta < 1$ this means that $\mathcal{X}_{c,r}$ will not be of class $\mathcal{C}^{1,1}$ as in Theorem 3 above, but merely of class $\mathcal{C}^{1,\eta}$. Also, that when $\eta > 1$ the spectrum of the linear part becomes a larger set, something that is consistent with the condition (1), that becomes less stringent when η becomes larger. But one has to be careful with the exact meaning of η since in Theorem 9 the nonlinearity to satisfies $|D\mathcal{X}_{c,r}(x)| = O(|x|^\eta)$ and in Theorem 2 it was supposed to be slightly more, namely $o(|x|^\eta)$. Observe finally that the case $\eta = 1$ was the case considered in Theorem 6.

Proof: The plan of the proof is as follows. As in [15], section 3, we first build a bounded linear operator M in ℓ^2 of the form

$$M = \begin{pmatrix} c & 0 & 0 & 0\cdots & \cdots \\ c_1 & c & 0 & 0\cdots & \cdots \\ c_2 & c_1 & c & 0\cdots & \cdots \\ c_3 & c_2 & c_1 & c\cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{8}$$

with the following properties: *i)* M is a contraction, *ii)* $[c^{1+\eta}, c] \subset |\sigma(M)| \subset [c^{1+\eta}, c+r]$, and *iii)* $(M - c^{1+\eta} I)$ is not onto. Let $\gamma \in \ell^2 \setminus (M - c^{1+\eta} I)(\ell^2)$. Then, for $\ell^2 \ni x = (y, z) = (y, z_1, z_2, \dots)$ one defines

$$A_{c,r} = \begin{pmatrix} c & 0 \\ 0 & M \end{pmatrix}, \quad \mathcal{X}_{c,r}(x) := \begin{pmatrix} 0 \\ |y|^{1+\eta}\gamma \end{pmatrix}, \tag{9}$$

and one has to prove that the map $T_{c,r} = A_{c,r} + \mathcal{X}_{c,r}$ does not have an invariant manifold tangent to the y axis near $x = 0$.

Let us proceed with this plan. As in [15], the operator M is constructed as $M = g(J)$, where J is the operator defined in (4) and now $g(\xi)$ is the complex-analytic function

$$g(\xi) = \frac{c - c^{1+\eta}}{2 - r} N_{1-r}(\xi) + \frac{c + c^{1+\eta} - r c^{1+\eta}}{2 - r},$$

where $N_{1-r}(\xi)$ is the Möbius function $N_{1-r}(\xi) = [(1-r) - \xi]/[1 - (1-r)\xi]$. The numbers c_j appearing in (8) are computed by using the Taylor series of $g(\xi)$ and with this and some elementary inequalities, one proves that M is a contraction. The fact that $[c^{1+\eta}, c] \subset |\sigma(M)| \subset [c^{1+\eta}, c+r]$ comes from the fact that the spectrum of J is the unit disc, N_{1-r} is an automorphism of the unit disc and the Spectral Mapping Theorem. From this theorem one also sees that $c^{1+\eta} \in \sigma(M)$, but the fact that the range of $M - c^{1+\eta}I$ is not the whole of ℓ^2 comes from the fact that one can check directly that $M - c^{1+\eta}I$ is one-to-one and an application of the Closed Graph Theorem.

Now we take $\gamma = (\gamma_1, \gamma_2, \dots)$ outside the range of $M - c^{1+\eta}I$ and define $T_{c,r}$ as above. To see that it does not have an invariant manifold tangent to the y axis near $x = 0$ we suppose that it exists, and we will arrive to contradiction. Let us write this manifold as $\{(y, \phi(y))\}$ with $\phi = (\phi_1, \phi_2, \dots)$. The invariance then means that the following equations have to be satisfied:

$$\begin{aligned}
&\phi_1(cy) = c\phi_1(y) + \gamma_1|y|^{1+\eta}\\
&\phi_2(cy) = c\phi_2(y) + c_1\phi_1(y) + \gamma_2|y|^{1+\eta}\\
&\phi_3(cy) = c\phi_3(y) + c_1\phi_2(y) + c_2\phi_1(y) + \gamma_3|y|^{1+\eta}\\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\end{aligned}$$

Then we apply recursively the following

Lemma 1. (uniqueness) *If $0 < c < 1$, $\eta > 0$ and $\rho \in \mathbb{R}$ then the functional equation,*

$$\phi(cy) = c\phi(y) + \rho|y|^{1+\eta} \tag{10}$$

has a unique local solution ϕ that is differentiable at $x = 0$ such that $\phi(0) = 0$, $\phi'(0) = 0$. This solution is given by

$$\phi(y) = \frac{\rho}{c^{1+\eta} - c}|y|^{1+\eta}.$$

To prove the lemma one checks first that the given expression is a solution, and for the uniqueness one follows the same argument as in [14].

This is the key point of this theorem: the fact that this lemma holds not only for $\eta = 1$, as it was used in [14] and [15].

The recursive application of this lemma to the previous equations gives that the only possible solution would have the form $\phi_n(y) = \alpha_n|y|^{1+\eta}$ for a sequence (α_n) that must belong to ℓ^2. But substitution of this expression into the previous equations would give

$$c^{1+\eta}(\alpha_n) = M(\alpha_n) + (\gamma_n)$$

and this is not possible, since (γ_n) is outside the range of $M - c^{1+\eta}I$.

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