The Inverse Problem of Variational Calculus with Non-holonomic Constraints

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Dedicated to Luís Magalhães and Carlos Rocha for the occasion of their 60th birthdays.

Abstract. We will discuss some new results for the inverse problem of Variational Calculus. We will consider problems with functionals given by action forms of order greater than one and subject to non-holonomic constraints.

1. Introduction

Griffiths (see [16]) presented a new approach to variational problems in the context of exterior differential systems, and proposed mixed endpoint conditions for problems with non-holonomic constraints to obtain stationary solutions. With these non-holonomic constraints it is generally not possible to have variations of integral manifolds subject to fixed endpoint conditions. These mixed endpoint conditions will make the integral over the boundary of the first variation vanish. In [26] we generalized Griffiths's framework to variational problems given by multiple integrals, and established mixed boundary conditions for variational problems with non-holonomic constraints. The study of Variational Calculus for functionals defined by multiple integrals was developed by Caratheodory [1929], Weil-De Donder [1936], Lepage [1936-1942]. Other authors like Dedecker [1953-1977], Liesen [1967], R. Hermann [1966], H. Goldschmidt and S. Sternberg [1973], Ouzilou [1972], D. Krupka [1970-1975], I. M. Anderson [1980], P. L. Garcia and A. Pérez-Rendón [1969-1978], C. Günther [1987], Edelen [1961] and Rund [1966] contributed with their work to this field.

In 1887, Helmholtz presented the inverse problem of Variational Calculus in the following way: Given $P_i = P_i(x, u^j, u^j_x, u^j_{xx})$, is there a Lagrangian $L(x, u^j, u^j_x)$ such that $E_i(L) = \partial L/\partial u^i - D_x \partial L/\partial u^i_x = P_i$, where $D_x = \partial/\partial x + u^i_x \partial/\partial u^i + u^i_{xx} \partial/\partial u^i_x$? Necessary conditions were found for P_i to be a Euler-Lagrange system (see (3.1) (3.2) and (3.3)). These conditions were proved to be locally sufficient.

I. M. Anderson [1992], [1980], P. J. Olver [1986], F. Takens [1979], W. M. Tulczyjew [1980] and A. M. Vinagradov [1964] generalized Helmholtz's conditions both for higher order systems of partial differential equations and for multiple integrals.

In the present text we describe new results for the the inverse problem of Variational Calculus for multiple integrals in the context of exterior differential systems. We deal with non-holonomic constraints in the setting of the mixed boundary conditions defined in [26]. This work is a follow up of [29] and [30]. In section I and II we present a short review of the latter work. In section III we discuss the inverse problem of Variational Calculus, and conclude in section IV with a study of the generalized Lagrange problem with non-holonomic constraints.

1.1. **Integral manifolds and valued differential systems.** Let us consider a manifold X and two subbundles of the cotangent bundle T^*X, satisfying:

   i) $I^* \subset T^*X$,
   ii) $L^* \subset T^*X$ with $I^* \subset L^* \subset T^*X$,

with the rank $(L^*/I^*) = n$ (a natural number).

We define an integral manifold of (I^*, L^*) as an oriented connected compact n-dimensional smooth manifold N together with a smooth mapping

$$f : N \to X$$

satisfying:

$$I^*_{f(x)}{}^\perp = L^*_{f(x)}{}^\perp + f_*(TN), \tag{1.1}$$

for all $x \in N$.

N may admit a piecewise smooth boundary ∂N.

$V(I^*, L^*)$ is the collection of integral manifolds f of (I^*, L^*).

Let φ be an n-form on X. A valued differential system of (I^*, L^*) is a triple (I^*, L^*, φ).

We define the functional ϕ associated with (I^*, L^*, φ) in $V(I^*, L^*)$ by:

$$\phi : V(I^*, L^*) \to R,$$

$$f \to \phi[f] = \int f^* \varphi. \tag{1.2}$$

1.2. Local embeddability. Let us define a Pfaffian differential system which is locally embeddable in $J^1(R^n, R^m)$. The differenial system for the Lagrange problem will be defined later [26]. Let $d(C^\infty(X, L^*)) \subset C^\infty(X, L^* \wedge T^*X)$ and $d' = \dim X$, $s = \operatorname{rank} I^*$. We denote $d(C^\infty(X, L^*)$ as the set of images obtained by the exterior derivative of $C^\infty(X, L^*)$. We can set for every $p \in X$ a chart coordinate system $\{u^1, ..., u^{s+n}, v^1, ..., v^{d'-s-n}\}$ so that:

   i)

$$L^* = \operatorname{span}\{du^\alpha | 1 \le \alpha \le s + n\}) \tag{1.3}$$

   ii)

$$L^{*\perp} = \operatorname{span}\{\frac{\partial}{\partial v} | 1 \le i \le d' - s - n\} \tag{1.4}$$

for an open subset U of X with $p \in U$, using the Frobenius theorem. Let δ be the map $I^* \wedge \Omega \to \Lambda^{n+1}(T^*U)/I_u^* \wedge (\Lambda^n(T*U))$ induced by:

$$d: C^\infty(U, I^* \wedge \Omega) \to C^\infty(U, \Lambda^{n+1}(T^*U))$$

in $I^* \wedge \Omega$.

Definition 1.1. *A Pfaffian differential system (I^*, L^*) with $d(C^\infty(X, L)) \subset C^\infty(X, L^* \wedge T^*X)$ is locally embeddable if for every $p \in X$ there exists an open neighborhood U of p and local coframes $CF = \{\theta_1, ..., \theta_s\}$ for I_U^* and $CF' = \{\theta_1, ..., \theta_s, du^{"s+1}, du^{"s+n}\}$ for L_U^*, satisfying:*

   (i) $\delta(I_U^* \wedge \Omega) \subset T^*U \wedge \Lambda^n(L_U^*)/T^*U \wedge I_U^* \wedge \Lambda^{n-1}(L^*))$,
   (ii) *Ker δ is a constant rank subbundle of $I^* \wedge \Omega$,*

   where $\Omega = \operatorname{span}\{du^{"s+1} \wedge ... \wedge \widehat{du^{"s+j}} \wedge ... \wedge du^{"s+n}\}$,

$\widehat{du^{"s+j}}$ - means deletion of $s + j$ factor (for $n = 1, \widehat{du^{"s+1}} = 1$).

If I^* has no Cauchy characteristics, the structure equations are locally:

$$d\theta^\alpha \equiv \pi_j^\alpha \wedge du^{"s+j} + A_{\alpha'\beta}^{\alpha j'} \pi_{j'}^{\alpha'} \wedge \theta^\beta + B_{\beta j}^\alpha \theta^\beta \wedge du^{"s+j} \bmod I \wedge I \tag{1.5}$$

$1 \le \alpha, \alpha', \beta \le s, 1 \le j, j', j'' \le n, I = C^\infty(X, I^*)$.

1.3. **The Cartan system.** We will now define the Cartan system whose solutions, when projected in X, will be candidates for extremum of ϕ for appropriate boundary conditions.

We begin by assuming that (I^*, L^*, φ) is a valued differential system on X, and that W is the total space of I^*. Let us consider χ the canonical form on T^*X, and i the inclusion map $W \overset{i}{\hookrightarrow} T^*X$.

We assume that the n-form ω is locally given by:

$$\omega = \omega^1 \wedge ... \wedge \omega^n, \tag{1.6}$$

inducing a nonzero section of $\Lambda^n(L^*/I^*)$.

$$\omega_i = (-1)^{i-1}\omega^1 \wedge ... \wedge \widehat{\omega^i}... \wedge \omega^n. \tag{1.7}$$

W^n is the n-Cartesian power of W. We define Z as a subset of W^n by $Z = \{z \in W^n : \pi'(z) \in X\}$, where π' is the projection $\pi'(z) : W^n \to X$, and ΔX^n is the diagonal submanifold of X^n.

Z is a vector subbundle over X and $dimZ = d + sn$. Let ψ be

$$\psi = \pi^*\varphi + (\pi^j o i')^*[i^*(\chi)] \wedge \pi^*\omega_j. \tag{1.8}$$

π^j is the natural projection into the jth component $\pi^j : W^n \to W$, i' is the inclusion map $Z \to W^n$, π is the natural projection $\pi : Z \to X$ and

$$\Psi = d\psi. \tag{1.9}$$

Locally, $(\pi^j o i')^*[i^*(\chi)] \wedge \pi^*\omega_j = \lambda^i_j \theta_i{}^j$ with $\theta_i{}^j = \theta^j \wedge \omega_j$.

Definition 1.2. *The Cartan system $C(\Psi)$ is the ideal generated by the set of n-forms*

$$\{v \lrcorner \Psi \quad where \quad v \in C^\infty(Z, TZ)\}.$$

Integral manifolds of $(C(\Psi), \omega)$ are oriented connected compact n-dimensional smooth manifolds N (possibly with a piecewise smooth boundary ∂N) together with a smooth mapping $f : N \to X$, satisfying:

$$f^*\theta = 0 \quad for \ every \quad \theta \in C(\Psi) \tag{1.10}$$

and

$$f^*(\omega) \neq 0. \tag{1.11}$$

We can now express the first variation of ϕ by:

$$\delta\phi = \int_{f(N)} v \lrcorner d\psi + d(v \lrcorner \psi). \tag{1.12}$$

1.4. **The momentum space.** Let us assume that we have on Z:

  (i) a closed $(n+1)$-form Ψ with the associated Cartan system $C(\Psi)$,
  (ii) $\pi'^*\omega$ is the pull-back to Z of ω, which is the n-form inducing a nonzero section on $\wedge^n(L^*/I^*)$.

Definition 1.3. *Let $(C(\Psi), \pi'^*\omega)_n$ be the ideal generated by $(C(\Psi), \pi'^*\omega)$ in $C^\infty(Z, \wedge^n T^* Z)$. We say that $[z_0, E_0^p]$, with $z_0 \in Z$ and E_0^p a p-dimensional subspace of the tangent space T_{z_0}, is a p-dimensional integral element of $(C(\Psi), \pi'^*\omega)_n$ if*

  (i) $< E_0^p, \alpha >= 0$ *for all* $(C(\Psi), \pi'^*\omega)_n$,
  (ii) $< E_0^p \lrcorner \omega > \neq 0$.

$V_n(C(\Psi), \pi^*\omega))$, the set of integral elements $[z_0, E_0^p]$, is a subset of $G_n(Z)$. Let π'' be the projection $G_n(Z) \to Z$. Let us assume that:

$$Z_1 = \pi''(V_n(C(\Psi), \pi^*\omega)), V_n'(C(\Psi), \pi^*\omega))) =$$

$$\{E \in V_n(C(\Psi), \pi^*\omega) : E \text{ tangent to } Z_1\}, \qquad (1.13)$$

$$Z_2 = \pi''(V_n'(C(\Psi), \pi^*\omega)), V''_n(C(\Psi), \pi^*\omega))) =$$

$$\{E \in V_n'(C(\Psi, \pi^*\omega) : E \text{ tangent to } Z_2\}... \qquad (1.14)$$

are subbundles of Z.

Definition 1.4. *Let (I^*, L^*, φ) be a locally embeddable valued differential system, and $\omega = \omega^1 \wedge ... \wedge \omega^n$. If there exists a $k_0 \in \mathbb{N}$, such that in the above construction $Z_{k_0} = Z_{k_0+1} = ... = Z_{k_0+n'}(n' \in N)$, with*

  (i) Z_{k_0} *a manifold of dimension $(n+1)m + n$ for $m \in \mathbb{N}$, and*
  (ii) $(C(\Psi), \pi^*\omega)_{Z_{k_0}}$ *being a differential system in Z_{k_0} with $r_n = 0$ (Cartan number in Cartan-Kähler Theorem) for all $V_{n-1}(C(\Psi), \pi^*\omega))$,*

then (I^*, L^*, φ) is a non-degenerate valued differential system. We will rename Z_{k_0} the momentum space Y.

For $n = 1$ we follow [16] and replace condition (ii) by $\psi \wedge \Psi^n \neq 0$ on Z_{k_0}.

We call $(C(\Psi), \pi^*\omega)_Y$ the prolongation of $(C(\Psi), \pi^*\omega)$ in the momentum space. In this construction, the differential system $(C(\Psi), \pi^*\omega)_Y$ satisfies:

  (i) the projection $(C(\Psi), \pi^*\omega) \to Y$ is surjective,
  (ii) and the integral manifolds of $(C(\Psi), \pi'^*\omega)$ on Z coincide with those of $(C(\Psi), \pi^*\omega)$ on Y.

1.5. **Well-posed valued differential systems.** Let us assume that

(a) we have the following subbundles of T^*X

$$
\begin{array}{ccccc}
I^* & \subset & L^* & \subset & T^*X \\
\cup & & \cup & & \\
P^* & \subset & M^* & &
\end{array}, \tag{1.15}
$$

(b) the locally given n-form ω also induces a nonzero section on $\Lambda^n(M^*/P^*)$, and

(c) $Y \subset (P^*)^n|_{\Delta X^n}$, with Y being a subbundle of $(P^*)^n|_{\Delta X^n}$.

Definition 1.5. $(I^*, L^*, \varphi, P^*, M^*)$ *is a well-posed valued differential system if we have the following conditions fulfilled:*

(i) (I^*, L^*, φ) *is a non-degenerate valued differential system (with $\dim Y = (n+1)m + n$) and $\varphi = L\omega$ for a smooth function L on X,*

(ii) *there exists a subbundle P^* of I^* of rank m and a subbundle M^* of L^* of rank $m + n$ as in (1.15),*

(iii) $\pi''^*M^* = span\{\pi^*\theta|\theta \in C^\infty(X, M^*)\}$ *is completely integrable on Y, where $\pi'' = \pi \circ i$, with i once more denoting the inclusion mapping $Y \to Z$ and π the projection $Z \to X$.*

$CF = \{\theta^\alpha, du^{s+j}, \pi_{j'}^{\alpha'}, \pi_j^{\alpha''}|1 \le \alpha \le s, 1 \le \alpha' \le s_l, j' \in L_{\alpha'}, s_{l+1} \le \alpha'' \le s, 1 \le j \le n\}$ for T^*X with $L_{\alpha'} \subset \{k \in N, 1 \le k \le n\}$:

(i)

$$
I^* = \text{ span}\{\theta^\alpha|1 \le \alpha \le s\}, \tag{1.16}
$$

(ii)

$$
L^* = \text{ span}\{\theta^\alpha, du^{s+j}|1 \le \alpha \le s, 1 \le j \le n\}, \tag{1.17}
$$

(iii) $T^*X = L^* \oplus R^*$ ($\oplus$ denotes a direct sum) with $R^* = span\{\pi_{j'}^{\alpha'}, \pi_j^{\alpha''}|1 \le \alpha' \le s_l, j' \in L_{\alpha'}, s_{l+1} \le \alpha'' \le s, 1 \le j \le n\}$,

(iv)

$$
d\theta_{j''}^{\alpha'} \equiv 0 \bmod I, \text{ for } j'' \notin L_{\alpha'}\{\theta_{j''}^{\alpha'} = \theta^{\alpha'} \wedge \omega_{j''}\}, \tag{1.18}
$$

(v)

$$
d\theta_{j'}^{\alpha'} \equiv \pi_{j'}^{\alpha'} \wedge \omega \bmod I, \text{ for } j' \in L_{\alpha'}, \tag{1.19}
$$

(vi)

$$
d\theta_j^{\alpha''} \equiv \pi_j^{\alpha''} \wedge \omega \bmod I, \text{ when } 1 \le j \le n, \tag{1.20}
$$

(vii) $\pi_{j'}^{\alpha'}, \pi_j^{\alpha''}$ are linearly independent mod L.

In [29], we presented a set of boundary conditions for different types of well-posed valued differential systems. For these boundary conditions, solutions of the Cartan system are solutions of the Euler-Lagrange system. (These have null first variations.)

2. Generalized Lagrange problem

The framework for this Lagrange problem with or without constrains represents a set of problems that is highly relevant to the study of Calculus of Variations.

The generalized Lagrange problem is defined on $X = J^1(\mathbb{R}^n, \mathbb{R}^m)$ (the 1 jet manifold), with the canonical system I^* defined on X (i.e. $I^* = \text{span}\{\theta^\alpha = dy^\alpha - y^\alpha_{x^i} dx^i\}$) and $\varphi = L\omega$, with $\omega = dx^1 \wedge ... \wedge dx^n$. We choose $x^1, ..., x^n$ to be coordinates for $\mathbb{R}^n$, and $y^1, ..., y^m$ coordinates for $\mathbb{R}^m$.

Definition 2.1. *Let f be a solution to the canonical differential system I^*, with the independence condition given by $L^* = span\{I^*, dx^1, ..., dx^n\}$. A family $F(x, t_1, ..., t_k)$ of integral manifolds of (I^*, L^*) is a k-parameter variation of f $k \in \mathbb{N}$ if:*

  (i) *$F(x, t_1, ..., t_k)$ is smooth with $(t_1, ..., t_k) \in [0, \epsilon_1] \times ... \times [0, \epsilon_k]$, for $\epsilon_i > 0, 1 \le i \le k$,*
  (ii) *$F_{(t_1,...,t_k)} \doteq F(x, t_1, ..., t_k) \in V(I^*, L^*)$ for all $(t_1, ..., t_k) \in [0, \epsilon_1] \times ... \times [0, \epsilon_k]$,*
  (iii) *$F(x, 0) = f(x)$ for all $x \in N, N \subset \mathbb{R}$.*

$F_*(\frac{\partial}{\partial t_i})$ is an infinitesimal variation of F.

We will consider variations satisfying the condition $\pi"(F(x, t)) = \pi"(f(x))$ for all $x \in \partial N$ and $t \in [0, \epsilon]$ ($\pi"$ is the projection $J^1(R^n, R^m) \to R^n$).

Without loss of generality we can choose v so that $v \lrcorner dx^i = 0$, thus replacing a one parameter variation of f by another that has the same first and second variation while satisfying:

$$\pi"(F(x, t))_N = id_N \tag{2.1}$$

for all $t \in [0, \epsilon]$).

3. Inverse problem for calculus of variations

3.1. **First example.** In 1887, Helmholtz solved the following problem:

Example 1. *Given $P_\alpha = P_\alpha(x, u^\beta, u^\beta_x, u^\beta_{xx})$. Is there a Lagrangian $L(x, u^\beta, u^\beta_x)$ such that $E_\alpha(L) = \partial L/\partial u^\alpha - D_x \partial L/\partial u^\alpha_x = P_\alpha$, where $D_x = \partial/\partial x + u^\alpha_x \partial/\partial u^\alpha + u^\alpha_{xx} \partial/\partial u^\alpha_x$? He found the necessary conditions for P_α:*

$$\partial P_\alpha / \partial u_{xx}^\beta = \partial P_\beta / \partial u_{xx}^\alpha, \tag{3.1}$$

$$\partial P_\alpha / \partial u_x^\beta = \partial P_\beta / \partial u_x^\alpha + 2 D_x \partial P_\beta / \partial u_{xx}^\alpha, \tag{3.2}$$

$$\partial P_\alpha / \partial u^\beta = \partial P_\beta / \partial u^\alpha - D_x \partial P_\beta / \partial u_x^\alpha + D_{xx} \partial P_\beta / \partial u_{xx}^\alpha. \tag{3.3}$$

Let $E \to M$ be a fibered manifold. $J^\infty(E)$ (see [5] and [41]) is the infinite jet of E.
Let

$$\theta^i = du^\alpha - u_x^\alpha dx, \tag{3.4}$$

$$\theta_x^\alpha = du_x^\alpha - u_{xx}^\alpha dx, \tag{3.5}$$

and

$$\Omega_P = P_\alpha \theta^\alpha \wedge dx + 1/2[\partial P_\alpha / \partial u_x^\beta - D_x \partial P_\alpha / \partial u_{xx}^\alpha]\theta^\beta \wedge \theta^\beta$$

$$+ 1/2[\partial P_\alpha / \partial u_{xx}^\alpha + \partial P_\beta / \partial u_{xx}^\alpha]\theta^\alpha \wedge \theta_x^\beta. \tag{3.6}$$

If P satisfies the Helmholtz conditions (3.1), (3.2) and (3.3), then $d\Omega_P = 0$. If Ω_P is exact (equivalently, if $H^{n+1}(E)$ n+1, de Rham cohomology group of E is trivial), then P_α is globally variational.

If $\theta_L = Ldx + \partial L / \partial u_x^\alpha \theta^\alpha$, then $d\theta_L = \Omega_P$.

Volterra [51] showed that if $L = \int_N u^\alpha P_\alpha(x, tu^\beta, tu_x^\beta, tu_{xx}^\beta)dt$, where $N = [0, 1]$, then:

$$E_\alpha(L) = P_\alpha. \tag{3.7}$$

We obtain a global solution to the inverse problem in the case of one independent variable and $P_\alpha = 0$ equations of second order.

In 1964, Vaingberg [50] generalized this result. However, this Lagrangian is usually of much higher order than necessary.

From [5] one can derive the following theorem:

Theorem 3.1. *Let Δ be a differential operator of order $2k$*

$$\Delta = P_\alpha(x^i, u^\beta, u_1^\beta, ..., u_{2k}^\beta)\theta^\beta \wedge \omega. \tag{3.8}$$

Then Δ is the Euler-Lagrange operator of a kth- order Lagrangian $L(x^i, u^\beta, u_1^\beta, ..., u_k^\beta)$, if and only if Δ satifies the higher order Helmholtz conditions, and the functions

$$P_\alpha(t) = P_\alpha(x^i, u^\beta, u_1^\beta, ..., u_k^\beta, tu_{k+1}^\beta..., t^k u_{2k}^\beta) \tag{3.9}$$

are polynomials in t of degree less or equal to k.

u_k^β denote all possible k^{th}- order derivatives of u^β, $1 \le \alpha, \beta \le m$ and $1 \le i \le n$, $\theta^\beta = du^\beta - u_{x^i}^\beta dx^i$ and $\omega = dx^1 \wedge ... \wedge dx^n$.

3.2. **Variational Bicomplex.** Let us recall now a very important tool for a globalization of the inverse problem [5], [41].

Definition 3.1. *A p-form ω on $J^\infty(E)$ is said to be of type (r, s), where $r + s = p$, if at each point x of $J^\infty(E)$*

$$\omega(X_1, X_2, ..., X_p) = 0, \tag{3.10}$$

whenever either

(i) *more than s of the vectors $X_1, X_2, ..., X_p$ are π_M^∞ vertical, or*
(ii) *more than r of the vectors $X_1, X_2, ..., X_p$ annihilate all contact one forms.*

Note: $\Omega^{r,s}$ denotes the space of type (r, s) forms on $J^\infty(E)$.

(i) $\pi : E \to M$ is a fiber bundle.
(ii) There exists a set of differential equations on sections of E.

$$d = d_H + d_V$$

$$d_H : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r+1,s}(J^\infty(E)), \tag{3.11}$$

$$d_V : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r,s+1}(J^\infty(E)), \tag{3.12}$$

$$d_H^2 = 0, d_H d_V = -d_V d_H, d_V^2 = 0. \tag{3.13}$$

In local coordinates

$$d_H f = [\partial f/\partial x^i + u_{x^i}^\alpha \partial f/\partial u^\alpha + u_{x^i x^j}^\alpha \partial f/\partial u_{x^j}^\alpha + ...]dx^{x^i}, \tag{3.14}$$

$$d_V f = \partial f/\partial u^\alpha \theta^\alpha + \partial f/\partial u_{x^i}^\alpha \theta_{x^i}^\alpha + ... \tag{3.15}$$

I is locally given by:

$$I : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E)), \tag{3.16}$$

$$I(\omega) = \frac{1}{s}\theta^\alpha \wedge [(\partial/\partial u^\alpha \lrcorner \omega) - D_{x^i}((\partial/\partial u_{x^i}^\alpha \lrcorner \omega) + D_{x^i x^j}((\partial/\partial u_{x^i x^j}^\alpha \lrcorner \omega) - ...] \tag{3.17}$$

Definition 3.2. *The sequences of spaces*

$$
\begin{array}{c}
\uparrow d_V \\
0 \to \Omega^{0,3} \\
\uparrow d_V \\
0 \to \Omega^{0,2} \xrightarrow{d_H} \Omega^{1,2} \xrightarrow{d_H} \Omega^{2,2} \\
\uparrow d_V \qquad \uparrow d_V \qquad \uparrow d_V \\
0 \to \Omega^{0,1} \xrightarrow{d_H} \Omega^{1,1} \xrightarrow{d_H} \Omega^{2,1} \\
\uparrow d_V \qquad \uparrow d_V \qquad \uparrow d_V \\
0 \to \mathbb{R} \to \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \Omega^{2,0}
\end{array}
$$

$$
\begin{array}{ccc}
& \uparrow d_V & \uparrow \delta_V \\
... & \to \Omega^{n,3} & \xrightarrow{I} F^3 & \to 0 \\
\uparrow d_V ... & \uparrow d_V & \uparrow \delta_V \\
... \xrightarrow{d_H} \Omega^{n-1,2} & \xrightarrow{d_H} \Omega^{n,2} & \xrightarrow{I} F^2 & \to 0 \\
\uparrow d_V ... & \uparrow d_V & \uparrow \delta_V \\
... \xrightarrow{d_H} \Omega^{n-1,1} & \xrightarrow{d_H} \Omega^{n,1} & \xrightarrow{I} F^1 & \to 0 \\
\uparrow d_V ... & \uparrow d_V & \uparrow \delta_V \\
... \xrightarrow{d_H} \Omega^{n-1,0} & \xrightarrow{d_H} \Omega^{n,0}
\end{array}
$$

$$\tag{3.18}$$

is the Variational Bicomplex.

Therefore, we have:

$$0 \to R \to \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \Omega^{2,0} ... \xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{E} F^1 \xrightarrow{\delta_H} F^2 \xrightarrow{\delta_H} F^3 ... \quad (3.19)$$

3.3. Two other examples.

Example 2. *Let $T = T(x^i, u, u_{x^i}, ..., u_{x^{i_1} x^{i_2}})$ be a second order operator $1 \leq i_1, i_2 \leq n$. We assume that T is a smooth function. Let $L = L(x^i, u, u_{x^i})$ be a first-order operator, with L being a smooth function. $E[L] = \partial L / \partial u - D_{x^i} \partial L / \partial u_{x^i}$, where $D_{x^i} = \partial / \partial x^i + u_{x^i} \partial / \partial u + u_{x^i, x^{i_1}} \partial / \partial u_{x^{i_1}} + ...$*

Let v be a lift to the momentum space of an infinitesimal variation $F_*(\partial / \partial t)$ of $f = \pi o g$, where g is a solution of $(C(\Psi), \pi^* \omega)$. The Lie-derivative of $\psi = \pi^* \varphi + (\pi^\beta o i')^* [i^*(\chi)] \wedge \pi^* \omega_\beta$ along v is:

$$v \lrcorner d\psi + d(v \lrcorner \psi) = E[L](u) v^1 \pi^*(dx^1 \wedge ... \wedge dx^n))$$

$$+ d(\partial L / \partial u_{x^i} v^1 \pi^* ((-1)^{i-1} dx^1 \wedge ... \wedge \widehat{dx^i} ... \wedge dx^n)). \quad (3.20)$$

If we identify e_i with $\pi^* ((-1)^{i-1} dx^1 \wedge ... \wedge \widehat{dx^i} ... \wedge dx^n)$ at each point of the integral manifold of $(C(\Psi), \pi^* \omega)$, we can write:

$$d(\partial L / \partial u_{x^i} v^1 \pi^* ((-1)^{i-1} dx^1 \wedge ... \wedge \widehat{dx^i} ... \wedge dx^n))$$

$$= Div V[u] \pi^* (dx^1 \wedge ... \wedge dx^n),$$

where

$$V[u] = \partial L / \partial u_{x^i} v^i e_i. \quad (3.21)$$

We have

$$E[L](u) = 0 \quad \text{if} \quad L[u] = Div W[u] \quad (3.22)$$

and

$$H(T) = 0 \quad \text{if} \quad T[u] \quad \text{is Euler-Lagrange.} \quad (3.23)$$

Helmholtz equations are:

$$\partial T / \partial u_{x^i} = D_{x^i} \partial T / \partial u_{x^i x^i} + 1/2 D_{x^{i_1}} \partial T / \partial u_{x^i x^{i_1}}. \quad (3.24)$$

There exists a sequence of spaces

$$V(u) \xrightarrow{Div} F(u) \xrightarrow{E} F(u) \xrightarrow{H} V(u) \quad (3.25)$$

that is a cochain complex, the Euler-Lagrange complex, where $F[u]$ is the set of smooth functions $F(x^i, u, u_{x^i}, ..., u_{x^{i_1} x^{i_2}})$, $V[u]$ is the set of vector fields defined in $\mathbb{R}^n$ with $F[u]$ coeficients. This is a particular case of (3.19).

This complex is exact and thus the inverse problem is solved in this second example.

Example 3. *Let* $T_\alpha(x^i, u^\beta, u^\beta_{x^i}, ..., u^\beta_{x^{i_1}x^{i_2}})$ *be second-order operators* $1 \leq i_1, i_2 \leq n$ *and* $1 \leq \beta, \alpha \leq m$. *We assume that* T_α *are smooth functions. Let* $L = L(x^i, u^\beta, u^\beta_{x^i})$ *be a first-order operator, with* L *being a smooth function.* $E_\alpha[L] = \partial L/\partial u^\alpha - D_{x^i}\partial L/\partial u^\alpha_{x^i}$ *where* $D_{x^i} = \partial/\partial x^i + u^\alpha_{x^i}\partial/\partial u^\alpha + u^\alpha_{x^i,x^{i_1}}\partial/\partial u^\alpha_{x^{i_1}} + ...$

Helmholtz equations are:

$$\partial T_\alpha/\partial u^\beta_{x^i x^{i_1}} = \partial T_\beta/\partial u^\alpha_{x^i x^{i_1}}, \tag{3.26}$$

$$\partial T_\alpha/\partial u^\beta_{x^i} = \partial T_\beta/\partial u^\alpha_{x^i} + 2D_{x^{i_1}}\partial T_\alpha/\partial u^\alpha_{x^i x^{i_1}}, \tag{3.27}$$

$$\partial T_\alpha/\partial u^\beta = \partial T_\beta/\partial u^\alpha - D_{x^i}\partial T_\beta/\partial u^\alpha_{x^i} + D_{x^i x^{i_1}}\partial T_\beta/\partial u^\alpha_{x^i x^{i_1}}. \tag{3.28}$$

4. **G.L. problem with non-holonomic constraints**

4.1. **G.L. problem with non-holonomic constraints.** Let us recall from [26] the generalized Lagrange problem with non-holonomic constraints for $n > 1, m > 1$.

Let us assume $g^\rho(x^i, u^\alpha, u^\alpha_{x^j}) = 0$, with $rank[\partial g^\rho/\partial u^\alpha_{x^j}] = mn - l$, $g^\rho(x^i, u^\alpha, u^\alpha_{x^j})$ are smooth functions, with $1 \leq i, j \leq n$, $1 \leq \alpha \leq m$, $1 \leq \rho \leq mn - l$ and $l \geq 0$. $(I^*, L^*, \varphi, I^*, L^*)$ is a well-posed valued differential system, where: $I^* = \text{span } \{\theta^\alpha\}$, and $L^* = \text{span } \{\theta^\alpha, dx^i | 1 \leq i \leq n\}$

$$\theta^\alpha = \{du^\alpha - u^\alpha_{x^i}dx^i | 1 \leq i \leq n\} \tag{4.1}$$

and

$$\theta^\alpha_j = \theta^\alpha \wedge \omega_j | 1 \leq i \leq n\}. \tag{4.2}$$

In this setting we have:

$$d\theta^\mu_i = -du^\mu_{x^i} \wedge \omega, \tag{4.3}$$

$$d\theta^\sigma_{i_\sigma} = -du^\sigma_{x^{i_\sigma}} \wedge \omega \quad \text{with} \quad i_\sigma \in L_\sigma \subset \{1,...,n\}, \tag{4.4}$$

$$d\theta^\sigma_{j_\sigma} = +A^{i\sigma}_{j_\sigma\mu}du^\mu_{x^i} \wedge \omega + A^{i_{\sigma'}\sigma}_{j_\sigma\sigma'}du^{\sigma'}_{x^{i_{\sigma'}}} \wedge \omega$$
$$+B^\sigma_{j_\sigma\alpha}\theta^\alpha \wedge \omega \mod I \wedge I \text{ with } j_\sigma \notin L_\sigma. \tag{4.5}$$

$$L^\mu_{x^i} = (\partial/\partial u^\mu_{x^i} - A^{i\sigma}_{j_\sigma\mu}\partial/\partial u^\sigma_{x^{j_\sigma}})L, \tag{4.6}$$

$$L^{\sigma'}_{x^{i_{\sigma'}}} = (\partial/\partial u^{\sigma'}_{x^{i_{\sigma'}}} - A^{i_{\sigma'}\sigma}_{j_\sigma\sigma'}\partial/\partial u^\sigma_{x^{j_\sigma}})L, \tag{4.7}$$

$$L^{\nu\mu}_{x^{i'}x^i} = (\partial/\partial u^\mu_{x^i} - A^{i\sigma}_{j_\sigma\mu}\partial/\partial u^\sigma_{x^{j_\sigma}})L^\nu_{x^{i'}}, \tag{4.8}$$

$$L^{\mu\sigma'}_{x^i x^i_{\sigma'}} = (\partial/\partial u^{\sigma'}_{x^i_{\sigma'}} - A^{i_{\sigma'}\sigma}_{j_\sigma\sigma'}\partial/\partial u^\sigma_{x^{j_\sigma}})L^\mu_{x^i}, \tag{4.9}$$

$$L^{\sigma'\mu}_{x^i_{\sigma'}x^i} = (\partial/\partial u^\mu_{x^i} - A^{i\sigma}_{j_\sigma\mu}\partial/\partial u^\sigma_{x^{j_\sigma}})L^{\sigma'}_{x^i_{\sigma'}}, \tag{4.10}$$

$$L^{\sigma'\sigma^{''}}_{x^i_{\sigma'}x^i_{\sigma''}} = (\partial/\partial u^{\sigma^{''}}_{x^i_{\sigma''}} - A^{i\sigma}_{j_\sigma\sigma^{''}}\partial/\partial u^\sigma_{x^{j_\sigma}})L^{\sigma'}_{x^i_{\sigma'}}, \tag{4.11}$$

with $1 \leq \sigma, \sigma'\sigma'' \leq m - m_l$ and $m - m_l + 1 \leq \mu, \nu \leq m$. (4.12)
We have $\sum_{\sigma=1}^{m_l} n_\sigma = l$, where $n_\sigma = n - \#L_\sigma$.

$$\Psi \equiv (L^\mu_{x^i} - \lambda^\mu_{x^i} - \lambda^\sigma_{j_\sigma}A^{i\sigma}_{j_\sigma\mu})\pi^*(du^\mu_{x^i} \wedge \omega)$$
$$+(L^{\sigma'}_{x^i_{\sigma'}} - \lambda^{\sigma'}_{x^i_{\sigma'}} - \lambda^\sigma_{j_\sigma}A^{i_{\sigma'}\sigma}_{j_\sigma\sigma'})\pi^*(du^\sigma_{x^{i_\sigma}} \wedge \omega)$$
$$+d\lambda^\mu_i \wedge \pi^*(\theta^\mu_i) + d\lambda^\sigma_i \wedge \pi^*(\theta^\sigma_i)$$
$$+(L_{u^\alpha} - \lambda^\sigma_{j_\sigma}B^\sigma_{j_\sigma\alpha} + L_{u^{\sigma'}_{x^{j_{\sigma'}}}}B^\sigma_{j_\sigma\alpha})\pi^*(\theta^\alpha \wedge \omega) \bmod I \wedge I, \tag{4.13}$$

The Cartan system is:

$$\pi^*\theta^\alpha_i \quad (1 \leq \alpha \leq m \quad \text{and} \quad 1 \leq i \leq n), \tag{4.14}$$

$$(L^\mu_{x^i} - \lambda^\mu_{x^i} - \lambda^\sigma_{j_\sigma}A^{i\sigma}_{j_\sigma\mu})\pi^*\omega \quad (m - m_l + 1 \leq \mu \leq m), \tag{4.15}$$

$$(L^{\sigma'}_{x^i_{\sigma'}} - \lambda^{\sigma'}_{x^i_{\sigma'}} - \lambda^\sigma_{j_\sigma}A^{i_{\sigma'}\sigma}_{j_\sigma\sigma'})\pi^*\omega, \tag{4.16}$$

$$(1 \leq \sigma, \sigma', \sigma^{''} \leq m_\sigma \text{ with } i_{\sigma'} \in L_\sigma \text{ and } j_\sigma \notin L_\sigma)$$

$$(-d\lambda^\mu_{x^i} \wedge \pi^*\omega_i) + (L_{u^\mu} - \lambda^\sigma_{j_\sigma}B^\sigma_{j_\sigma\mu} + L_{u^{\sigma'}_{x^{j_{\sigma'}}}}B^\sigma_{j_\sigma\mu}) \wedge \pi^*\omega, \tag{4.17}$$

$$(-d\lambda^{\sigma^{''}}_{x^{j_{\sigma''}}} \wedge \pi^*\omega_{j_{\sigma''}} - d\lambda^{\sigma''}_{x^{j_{\sigma''}}} \wedge \pi^*\omega_{i_{\sigma''}})$$
$$+(L_{u^{\sigma''}} - \lambda^\sigma_{j_\sigma}B^\sigma_{j_\sigma\sigma''} + L_{u^{\sigma'}_{x^{j_{\sigma'}}}}B^\sigma_{j_\sigma\sigma''}) \wedge \pi^*\omega. \tag{4.18}$$

Let us assume $g^\rho/\partial u^\mu_{x^i} = 0$ for all $m_l + 1 \leq \mu \leq m$ and $g^\rho/\partial u^\sigma_{x^i_\sigma} = 0$ for all $i_\sigma \in L_\sigma$ and $1 \leq \sigma \leq m_l$. Then the Euler-Lagrange equations are:

$$E^\mu(L) = \partial L/\partial u^\mu - D_{x^i}\partial L/\partial u^\mu_{x^i} + \partial L/\partial u^{\sigma'}_{x^i_{\sigma'}}B^\sigma_{j_\sigma\mu} + \lambda^\sigma_{j_\sigma}B^\sigma_{j_\sigma\mu}, \tag{4.19}$$

$$E^\sigma(L) = \partial L/\partial u^\sigma - D_{x^{i_\sigma}} \partial L/\partial u^\sigma_{x^{i_\sigma}} + \partial L/\partial u^{\sigma'}_{x^{i_{\sigma'}}} B^{\sigma'}_{j'_\sigma \sigma} + \lambda^{\sigma'}_{j'_\sigma} B^{\sigma'}_{j'_\sigma \sigma} - \lambda^\sigma_{j_\sigma x^{i_\sigma}}.$$
(4.20)

Proposition 4.1. *Let* (I^*, L^*) *be a locally embeddable differential system defined on* $X = J^1(\mathbb{R}^n, \mathbb{R}^m)|_{g^\rho(x^i, u^\alpha, u^\alpha_{x^j})=0}$, $rank[\partial g^\rho/\partial u^\alpha_{x^j}] = mn - l$, $g^\rho(x^i, u^\alpha, u^\alpha_{x^j})$ *are smooth functions* $(1 \le i, j \le n, 1 \le \alpha \le m, 1 \le \rho \le mn - l, l \ge 0)$, *and* $g^\rho/\partial u^\mu_{x^i} = 0$ *for all* $m_l + 1 \le \mu \le m$ *and* $g^\rho/\partial u^\sigma_{x^i_\sigma} = 0$ *and for all* $i_\sigma \in L_\sigma$ *and* $1 \le \sigma \le m_l$, *where* $I^* = span\ \{\theta^\alpha\}$, $L^* = span$ $\{\theta^\alpha, dx^i | 1 \le i \le n\}$,

$$\theta^\alpha = du^\alpha - u^\alpha_{x^j} dx^j \quad 1 \le j \le n. \tag{4.21}$$

Let

$$Q_\mu(x^i, u^\mu, u^\mu_{x^i}, u^\mu_{x^i x^j}, u^\sigma, u^\sigma_{x^{i_\sigma}}, u^\sigma_{x^{i_\sigma} x^{i'_\sigma}}, \lambda^\sigma_{j_\sigma}, \lambda^\sigma_{j_\sigma x^{i_\sigma}})$$

and

$$Q_\sigma(x^i, u^\mu, u^\mu_{x^i}, u^\mu_{x^i x^j}, u^\sigma, u^\sigma_{x^{i_\sigma}}, u^\sigma_{x^{i_\sigma} x^{i'_\sigma}}, \lambda^\sigma_{j_\sigma}, \lambda^\sigma_{j_\sigma x^{i_\sigma}}),$$

with $m_l + 1 \le \mu \le m$, $1 \le \sigma \le m_l$ *and* $1 \le i \le n$ *and* $(i_\sigma, i_{\sigma'}) \in (L_\sigma)^2$, *with*

$$Q_\alpha(x^i, u^\sigma, u^\mu_{x^i}, t u^\mu_{x^i x^j}, u^\sigma, u^\sigma_{x^{i_\sigma}}, t u^\sigma_{x^{i_\sigma} x^{i'_\sigma}}, \lambda^\sigma_{j_\sigma}, \lambda^\sigma_{j_\sigma x^{i_\sigma}}),$$

$1 \le \alpha \le m$ *being polynomials in* t *of degree less or equal to 1, and*

$$P_\sigma = Q_\sigma + \lambda^{\sigma'}_{j'_\sigma} B^{\sigma'}_{j_{\sigma'} \sigma} - \lambda^\sigma_{j_\sigma x^{i_\sigma}}, \tag{4.22}$$

$$P_\mu = Q_\mu + \lambda^\sigma_{j_\sigma} B^\sigma_{j_\sigma \mu}. \tag{4.23}$$

Furthermore, if we assume that P_α *satisfy Helmholtz conditions and do not depend on* $\lambda^\sigma_{j_\sigma}$ *and* $\lambda^\sigma_{j_\sigma x^{i_\sigma}}$ *coordinates, then* Q_α *are locally Euler-Lagrange operators for a Lagrangian* $L(x^i, u^\mu, u^\mu_{x^i}, u^\sigma, u^\sigma_{x^i})$.

Proof: In this case the Helmholtz conditions are:

$$\partial P_\alpha/\partial u^\beta_{x^i x^{i_1}} = \partial P_\beta/\partial u^\alpha_{x^i x^{i_1}}, \tag{4.24}$$

$$\partial P_\alpha/\partial u^\beta_{x^i} = \partial P_\beta/\partial u^\alpha_{x^i} + 2 D_{x^{i_1}} \partial P_\alpha/\partial u^\alpha_{x^i x^{i_1}}, \tag{4.25}$$

$$\partial P_\alpha/\partial u^\beta = \partial P_\beta/\partial u^\alpha - D_{x^i} \partial P_\beta/\partial u^\alpha_x + D_{x^i x^{i_1}} \partial P_\beta/\partial u^\alpha_{x^i x^{i_1}}. \tag{4.26}$$

From Theorem 3.1 we know that a function $F(x^i, u^\mu, u^\mu_{x^i}, u^\sigma, u^\sigma_{x^{i_\sigma}})$ can be found such that $E_\alpha[F] = P_\alpha$.

In addition, if in the domain of P_α the sequence of spaces is exact

$$\Omega^{n,0} \xrightarrow{E} F^1 \xrightarrow{H} 0, \tag{4.27}$$

then we have a global solution for the inverse problem.

Example 4. *Let $X = J^1(\mathbb{R}^n, \mathbb{R}^m)|_{g^\rho(x^i, u^\alpha, u^\alpha_{x^i})=0}$, $rank[\partial g^\rho/\partial u^\alpha_{x^j}] = mn - l$, $g^\rho(x^i, u^\alpha, u^\alpha_{x^i})$ are smooth functions ($1 \leq i, j \leq n$, $1 \leq \alpha \leq m$, $1 \leq \rho \leq n - l, l \geq 0$). Furthermore, let us assume that $g^\rho/\partial u^\mu_{x^i} = 0$ for all $m_l + 1 \leq \mu \leq m$ and $g^\rho/\partial u^\sigma_{x^\sigma_i} = 0$ for all $i_\sigma \in L_\sigma$, $1 \leq \sigma \leq m_l$. $I^* = span\ \{\theta^\alpha\}$, $L^* = span\ \{\theta^\alpha, dx^i | 1 \leq i \leq n\}$.*

$$Q_\sigma(x^i, u^\mu, u^\mu_{x^i}, u^\mu_{x^i} x^j, u^\sigma, u^\sigma_{x^{i_\sigma}}, u^\sigma_{x^{i_\sigma} x^{i'_\sigma}}, \lambda^\sigma_{j_\sigma}, \lambda^\sigma_{j_\sigma x^{i_\sigma}}) =$$

$$2u^{\sigma'}_{x^{j_{\sigma'}}} B^\sigma_{j_{\sigma'}\alpha} + \sum_{i_\sigma} 2u^\sigma_{x^{i_\sigma} x^{i_\sigma}} - \lambda^\sigma_{j_\sigma} B^\sigma_{j_\sigma \sigma} - \lambda^\sigma_{j_\sigma x^{i_\sigma}}, \qquad (4.28)$$

$$Q_\mu(x^i, u^\mu, u^\mu_{x^i}, u^\mu_{x^i} x^i, u^\sigma, u^\sigma_{x^{i_\sigma}}, u^\sigma_{x^{i_\sigma} x^{i'_\sigma}}, \lambda^\sigma_{j_\sigma}, \lambda^\sigma_{j_\sigma x^{i_\sigma}}) =$$

$$2u^\sigma_{x^{j_\sigma}}(x^i, u) B^\mu_{j_\sigma \alpha} + \sum_i 2u^\mu_{x^i x^i} - \lambda^\sigma_{j_\sigma} B^\sigma_{j_\sigma \sigma}. \qquad (4.29)$$

$Q_\sigma - \lambda^\sigma_{j_\sigma} B^{\sigma'}_{j'_\sigma \sigma} - \lambda^{\sigma'}_{j'_\sigma x^{i_\sigma}}$ and $Q_\mu - \lambda^\sigma_{j_\sigma} B^\mu_{j_\sigma \mu}$ satisfy Helmhotz equations and are globally Euler-Lagrange operators for $L = \sum_{i_\sigma} (u^\sigma_{x^{i_\sigma}})^2 + \sum_{j^\sigma} (u^\sigma_{x^{j_\sigma}})^2 + \sum_i (u^\mu_{x^i})^2$.

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