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A minimization problem for the Nonlinear Schrödinger-Poisson type Equation

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Dedicated to Prof. Waldyr Muniz Oliva in occasion of his 80th birthday

Abstract. In this paper we consider the stationary solutions of the Schrödinger-Poisson equation:

$$i\psi_t + \Delta\psi - (|x|^{-1} * |\psi|^2)\psi + |\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}^3.$$

We are interested in the existence of standing waves, that is solutions of type $\psi(x,t) = u(x)e^{-i\omega t}$, where $\omega \in \mathbb{R}$, with fixed $L^2 - norm$. Then we are reduced to a constrained minimization problem. The main difficulty is the compactness of the minimizing sequences since the related functional is invariant by translations. By using some abstract results, we give a positive answer, showing that the minimum of the functional is achieved on small $L^2 - spheres$ in the case $2 < p < 3$ and large $L^2 - spheres$ in the case $3 < p < 10/3$. The results exposed here can be found with more details in [6] and [7].

1. Introduction

We consider the following Schrödinger-Poisson type equation

$$i\psi_t + \Delta\psi - (|x|^{-1} * |\psi|^2)\psi + |\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\psi(x,t) : \mathbb{R}^3 \times [0,T) \to \mathbb{C}$ is the wave function, $*$ denotes the convolution and $2 < p < 10/3$. Equation (1.1), known in the case $p = 8/3$ as

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Schrödinger-Poisson-Slater equation, has been used to analyze a wide variety of physical phenomena in Quantum-Chemistry and Solid State Physics. We refer to [11] and [13] for a detailed study of equations which model physical phenomena with nonlocal terms.

We are interested to the existence of particular class of solutions of the Schrödinger-Poisson equation: the *solitary waves*. By a solitary wave we mean a solution of (1.1) whose energy travels as a localized packet; if a solitary wave exhibits orbital stability it is called *soliton*. Actually we restrict to the *standing waves*, that is solutions of type

$$\psi(x,t) = e^{-i\omega t}u(x)\,, \quad \omega \in \mathbb{R}, \quad u(x) \in \mathbb{C}\,. \tag{1.2}$$

So we are reduced to study the following semilinear elliptic equation with a non local nonlinearity

$$-\Delta u + \phi_u u - |u|^{p-2}u = \omega u \quad \text{ in } \mathbb{R}^3, \tag{1.3}$$

where we have set

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy\,.$$

In the literature the Schrödinger-Poisson equation has been extensively studied. However many authors consider the case in which the frequency ω is a parameter (that is, a priori given) and not an unknown; then the energy functional they study is

$$F(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx$$

whose critical points are exactly the solutions of (1.3) with that given ω. See e.g. [1, 9, 10, 15, 16, 17, 18] and the references therein.

We recall that the energy and the charge associated to the wave function $\psi(x,t)$ evolving according to (1.1) are constants of motion and are given by

$$\begin{aligned} E(\psi(x,t)) : &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \psi|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} (|x|^{-1} * |\psi|^2)|\psi|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |\psi|^p dx \\ &= E(\psi(x,0)) \end{aligned}$$

and

$$Q(\psi(x,t)) := \frac{1}{2}\int_{\mathbb{R}^3} |\psi|^2 dx = Q(\psi(x,0)).$$

So it is physically relevant to study the critical points of E restricted on the manifold $Q = \mathit{constant}$. By using the ansatz (1.2), the natural way to attack this problem is to look for the constrained critical points of the functional

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u|u|^2dx-\frac{1}{p}\int_{\mathbb{R}^3}|u|^pdx \tag{1.4}$$

on the L^2-spheres in $H^1(\mathbb{R}^3;\mathbb{C})$

$$B_\rho=\{u\in H^1(\mathbb{R}^3;\mathbb{C}):\|u\|_2=\rho\}\quad \rho>0.$$

In this case ω is not a priori given but it is an unknown of the problem: so, now by a solution of (1.3) we mean a couple $(\omega_\rho,u_\rho)\in\mathbb{R}\times H^1(\mathbb{R}^3;\mathbb{C})$, where ω_ρ is the Lagrange multiplier associated to the critical point u_ρ on B_ρ. Once u_ρ is found, ω_ρ is given explicitly by

$$\omega_\rho=\frac{1}{\rho^2}\left(\|\nabla u_\rho\|_2^2+\int_{\mathbb{R}^3}\phi_{u_\rho}|u_\rho|^2dx-\int_{\mathbb{R}^3}|u_\rho|^pdx\right).$$

Note that this approach is more natural since the wave function ψ is an unknown, so u and ω has to be considered both as unknowns of the problem.

However, due to stability properties, we are interested in finding the critical points of I on B_ρ which are minima for the energy I. Therefore we study the minimization problem

$$I_{\rho^2}:=\inf_{B_\rho}I(u) \tag{1.5}$$

which makes sense for $2<p<10/3$ (see Proposition 2.1). Note that problem (1.5) is invariant by the action of noncompact group of translations in $\mathbb{R}^3$.

In a recent paper by Benci and Fortunato [4] the relevance of the energy/charge ratio for the existence of standing waves in field theories has been discussed under a general framework. In our context, the analogous is the function $s\mapsto\frac{I_{s^2}}{s^2}$ that will appear in Section 4.

There are only few papers concerning the minimization problem of the Schrödinger-Poisson functional I on the constraint B_ρ. There is just a result by Sanchez and Soler [19] in the case $p=8/3$ and by Catto and Lions in the case of nonhomogeneous nonlinearity of type $\|u\|_{10/3}^{10/3}-\|u\|_{8/3}^{8/3}$, see [8]. For $p=8/3$, the so called Schrödinger-Poisson-Slater equation, the existence of minimizers is proved in [19] only for ρ small, that is for small values of the charge. The difficulty, in considering all $\rho>0$, concerns the possibility of dichotomy for an arbitrary minimizing sequence.

We quote also [4] and [14] where the analogous problem in a bounded domain has been considered. In [4] the authors prove, by means of the Ljusternik-Schnirelmann theory, the existence of infinitely many solutions with Dirichelet boundary conditions on u and ϕ. In [14] a nonhomogeneous Neumann boundary condition on ϕ is considered.

The results we are going to prove here are the following.

Theorem 1.1. *Let $p \in (3, 10/3)$. Then there exists $\rho_2 > 0$ (depending on p) such that all the minimizing sequences for* (1.5) *are precompact in $H^1(\mathbb{R}^3; \mathbb{C})$, up to translations, provided that*

$$\rho_2 < \rho < +\infty.$$

In particular, there exists a couple $(\omega_\rho, u_\rho) \in \mathbb{R} \times H^1(\mathbb{R}^3; \mathbb{R})$ solution of (1.3).

We note explicitly that the solution u_ρ is real valued. The importance of the existence of the minimum of the functional I is related to its stability properties.

Theorem 1.2. *Let $p \in (3, 10/3)$. Then the set*

$$S_\rho = \{e^{i\theta}u(x) : \theta \in [0, 2\pi), \|u\|_2 = \rho,\ I(u) = I_{\rho^2}\} \quad \textit{for} \quad \rho > \rho_2$$

(with ρ_2 provided by Theorem 1.1) is orbitally stable.

The definition of orbital stability will be recalled in Subsection 3.1.

With a slightly different approach, we are able also to treat the case $2 < p < 3$.

Theorem 1.3. *Let $p \in (2, 3)$. Then there exists $\rho_1 > 0$ (depending on p) such that all the minimizing sequences for* (1.5) *are precompact in $H^1(\mathbb{R}^3; \mathbb{C})$, up to translations, provided that*

$$0 < \rho < \rho_1.$$

In particular, there exists a couple $(\omega_\rho, u_\rho) \in \mathbb{R} \times H^1(\mathbb{R}^3; \mathbb{R})$ solution of (1.3).

Moreover we have

Theorem 1.4. *Let $p \in (2, 3)$. Then the set*

$$S_\rho = \{e^{i\theta}u(x) : \theta \in [0, 2\pi), \|u\|_2 = \rho,\ I(u) = I_{\rho^2}\} \quad \textit{for} \quad \rho < \rho_1$$

(with ρ_2 provided by Theorem 1.3) is orbitally stable.

1.1. **Notations.** As a matter of notations, in the paper it is understood that all the functions, unless otherwise stated, are complex-valued, but for simplicity we will write $L^s(\mathbb{R}^N), H^1(\mathbb{R}^N)$...., where $N \geq 3$ and for any $1 \leq s < +\infty$, $L^s(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_s^s := \int_{\mathbb{R}^N} |u|^s dx,$$

and $H^1(\mathbb{R}^N)$ the usual Sobolev space endowed with the norm

$$\|u\|_{H^1}^2 := \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 dx.$$

For our application, let us define the space $D^{1,2}(\mathbb{R}^N)$. It is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,2}}^2 := \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Moreover the letter c will be used to denote a suitable positive constant, whose value may change also in the same line, and the symbol $o(1)$ to denote a quantity which goes to zero. We also use $O(1)$ to denote a bounded sequence.

The paper is organized as follows. In the next section we give some general remarks and comments about the problems we are going to study. In Section 3 is considered the case $p \in (3, 10/3)$. Section 4 is devoted to the case $p \in (2, 3)$ which is more involved.

2. Preliminaries

First of all, the study of the minimization problem (1.5) is justified by the following

Proposition 2.1. *For every $\rho > 0$ and $p \in (2, 10/3)$ the functional I is bounded from below and coercive on B_ρ.*

Proof. We apply the following Sobolev inequality

$$\|u\|_q \leq b_q \|u\|_2^{1-\frac{N}{2}+\frac{N}{q}} \|\nabla u\|_2^{\frac{N}{2}-\frac{N}{q}}$$

that holds for $2 \leq q \leq 2^*$ when $N \geq 3$. Therefore if $\|u\|_2 = \rho$ it follows

$$\|u\|_p^p \leq b_p \rho^{\frac{6-p}{2}} \|\nabla u\|_2^{\frac{3p}{2}-3}$$

and

$$I(u) \geq \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p}|u|^p \right) dx \geq \frac{1}{2}\|\nabla u\|_2^2 - b_p \rho^{\frac{6-p}{2}} \|\nabla u\|_2^{\frac{3p}{2}-3}. \quad (2.1)$$

Since $p < 10/3$, it results $\frac{3p}{2} - 3 < 2$ and

$$I(u) \geq \frac{1}{2}\|\nabla u\|_2^2 + O(\|\nabla u\|_2^2).$$

which concludes the proof. □

As a consequence of this proposition, whenever ρ is fixed and $\{u_n\}$ is a minimizing sequence for I_{ρ^2}, we implicitly assume that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, so weakly convergent up to subsequences.

Note that, evidently, ϕ_u which appears in (1.3) satisfies $-\Delta\phi_u = 4\pi|u|^2$ and is usually interpreted as the scalar potential of the electrostatic field generated by the charge density $|u|^2$. Furthermore, it is useful to observe that, if we set

$$u_\lambda(\cdot) = \lambda^\alpha u(\lambda^\beta(\cdot)) \quad \alpha, \beta \in \mathbb{R}, \lambda > 0,$$

then

$$\begin{aligned}\phi_{u_\lambda}(x) &= \int_{\mathbb{R}^3} \frac{\lambda^{2\alpha+\beta}|u(\lambda^\beta y)|^2}{|\lambda^\beta x - \lambda^\beta y|} dy = \lambda^{2(\alpha-\beta)} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|\lambda^\beta x - y|} dy \\ &= \lambda^{2(\alpha-\beta)}\phi_u(\lambda^\beta x).\end{aligned}$$

To prove the theorems stated, we will make use of some abstract results. They concern the compactness condition in order to conclude that the minimizing sequences are (strongly) convergent. The main contribution to constrained minimization problems has been given by the celebrated *concentration-compactness* principle of Lions, see [12]. It is clear that the *relative compactness* of the minimizing sequences would give the existence of a minimizer for (1.5). However, for translation invariant functionals the minimizing sequence $\{u_n\}$ could run off to spatial infinity and/or spread uniformly in space. So even up to translations two possible bad scenarios are possible:

- (vanishing) $u_n \rightharpoonup 0$;
- (dichotomy) $u_n \rightharpoonup \bar{u} \neq 0$ and $0 < \|\bar{u}\|_2 < \rho$.

The general strategy in the applications is to prove that any minimizing sequence weakly converges, up to translation, to a function $\bar{u}$ which is different from zero, excluding the vanishing case. Then one has to show that $\|\bar{u}\|_2 = \rho$, which proves that dichotomy does not occur. As a consequence of the Lions' principle, the minimizing sequence converges, up to subsequence, to a minimizer which gives a solution of the problem.

In [12], Lions proved that the invariance by translations of the problem implies in many cases (as for our problem (1.5)) an inequality that the infima I_{ρ^2} have to satisfy and read as follows (*weak subadditivity inequality*)

$$I_{\rho^2} \leq I_{\mu^2} + I_{\rho^2-\mu^2} \quad \text{for all} \;\; 0 < \mu < \rho. \tag{2.2}$$

However the necessary and sufficient condition in order that any minimizing sequence on B_ρ is relatively compact is a stronger version of (2.2), that is

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2-\mu^2} \quad \text{for all} \;\; 0 < \mu < \rho. \tag{2.3}$$

In the literature it is referred as the *strong subadditivity inequality.* Our main affort concerns with the verification of (2.3).

Actually, Theorem 1.1 and Theorem 1.3 are consequence of general results (Lemma 3.1 and Theorem 4.1) which are applicable also in other situations. In contrast, Theorem 1.2 and Theorem 1.4 are quite expected; indeed their proofs are standard and based on two general facts

- the convergence of all the minimizing sequences,
- the conservation of energy and the L^2-norm.

During the proof of Theorems 1.1-1.3 we will use general results concerning the minimization of functionals of type

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + T(u)$$

on B_ρ, for some C^1 functional T on $H^1(\mathbb{R}^3)$. Clearly, our functional (1.4) is in this form.

3. The case $3 < p < 10/3$

To prove that the minimum in this case is achieved, we make use of some results contained in [6]. Here is crucial the condition

$$J_{\rho^2} < J_{\mu^2} + J_{\rho^2-\mu^2} \quad \text{for any} \quad 0 < \mu < \rho\,. \tag{3.1}$$

The next two lemma (the first of which is quite general and the second one is for our functional I) stated without proofs, will be used to prove Theorem 1.1.

Lemma 3.1. *Let T a C^1 functional defined on $H^1(\mathbb{R}^N)$ and*

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + T(u).$$

Let $\{u_n\} \subset B_\rho$ be a minimizing sequence for J_{ρ^2} such that $u_n \rightharpoonup \bar{u} \neq 0$ and let us set $\mu = \|\bar{u}\|_2 \in (0, \rho]$.

Assume (3.1) *and also that*

$$T(u_n - \bar{u}) + T(\bar{u}) = T(u_n) + o(1); \tag{3.2}$$

$$T(\alpha_n(u_n-\bar{u}))-T(u_n-\bar{u})=o(1), \tag{3.3}$$

where $\alpha_n=\frac{\rho^2-\mu^2}{\|u_n-\bar{u}\|_2}$. *Then* $\bar{u}\in B_\rho$.

Moreover if, as $n,m\to+\infty$

$$\langle T'(u_n)-T'(u_m),u_n-u_m\rangle=o(1) \tag{3.4}$$

$$\langle T'(u_n),u_n\rangle=O(1) \tag{3.5}$$

then $\|u_n-\bar{u}\|_{H^1(\mathbb{R}^N)}\to 0$.

Proof. See [6]. □

By a straightforward computation, condition (3.1) can be proved for our functional I when $3<p<10/3$, indeed we have

Lemma 3.2. *If* $3<p<10/3$, *then there exists* $\rho_2>0$ *such that* I_{μ^2} *defined in* (1.5) *satisfies:*

a) $I_{\mu^2}<0$ *for all* $\mu>\rho_2$,
b) $I_{\rho^2}<I_{\mu^2}+I_{\rho^2-\mu^2}$ *for all* $\rho>\rho_2$ *and* $0<\mu<\rho$.

The verification of these two conditions is based on suitable rescaling properties of the functional defined in (1.4); it is technical and straightforward, hence omitted here; the interested reader is refereed to [6]. Let us see the consequences of this last lemma.

The condition $I_{\mu^2}<0$ is important to show that the weak limit of the minimizing sequences is not trivial (as required to apply the general Lemma 3.1). Indeed, fix $\mu\in(\rho_2,+\infty)$. Let $\{u_n\}$ be a minimizing sequence in B_μ. Notice that for any sequence $\{y_n\}\subset\mathbb{R}^3$ we have that $u_n(.+y_n)$ is still a minimizing sequence for I_{μ^2}. Now, if

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^3}\int_{B(y,1)}|u_n|^2dx\right)=0$$

then, by the Lions' Lemma (see [12]), $u_n\to 0$ in $L^q(\mathbb{R}^3)$ for any $q\in(2,2^*)$, where $B(a,r)=\{x\in\mathbb{R}^3:|x-a|\leq r\}$. Since $I_{\mu^2}<0$, this would address to a contradiction. Then it has to be

$$\sup_{y\in\mathbb{R}^3}\int_{B(y,1)}|u_n|^2dx\geq\delta>0$$

and we can choose $\{y_n\}\subset\mathbb{R}^3$ such that

$$\int_{B(0,1)}|u_n(.+y_n)|^2dx\geq\delta>0.$$

Consequently, due to the compactness of the embedding $H^1(B(0,1)) \subset L^2(B(0,1))$, we deduce that the weak limit of the sequence $u_n(.+y_n)$ is not the trivial function, so $u_n \rightharpoonup \bar{u} \neq 0$.

By setting now

$$T(u) := \frac{1}{4}B(u) + \frac{1}{p}C(u),$$

where

$$B(u) = \int_{\mathbb{R}^3} \phi_u |u|^2 dx, \quad C(u) = -\int_{\mathbb{R}^3} |u|^p dx,$$

our Schrödinger-Poisson functional can be written as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + T(u)$$

and satisfies the hypothesis of the abstract Lemma 3.1. Indeed, we have just seen that the minimizing sequences have a non trivial weak limit. Moreover condition (3.2) is satisfied by B and C as shown in Lemma 2.2 of [20]. Furthermore by the convolution and Sobolev inequalities we get

$$B(u_n) = \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^2 dx \leq c\|u_n\|_{12/5}^4 \leq c\|u_n\|_2^3 \|\nabla u_n\|_2 \tag{3.6}$$

and than the relation (3.3) follows from the homogeneity of B and C:

$$\begin{aligned} B(\alpha_n(u_n - \bar{u})) - B(u_n - u) &= (\alpha_n^4 - 1)B(u_n - \bar{u}) = o(1) \\ C(\alpha_n(u_n - \bar{u})) - C(u_n - u) &= (\alpha_n^p - 1)C(u_n - \bar{u}) = o(1) \end{aligned}$$

since $\alpha_n \to 1$.

Notice that thanks to the classical interpolation inequality we have

$$\|u_n - u_m\|_p \leq \|u_n - u_m\|_2^\alpha \|\nabla u_n - \nabla u_m\|_2^{1-\alpha} \quad \text{where } \frac{\alpha}{2} + \frac{(1-\alpha)}{2^*} = \frac{1}{p}$$

and then on the minimizing sequence we get $\|u_n - u_m\|_p = o(1)$.

We obtain, for $q = p/(p-1)$

$$\int_{\mathbb{R}^3} |u_n|^{p-1}|u_n - u| dx \leq \left(\int_{\mathbb{R}^3} |u_n|^q \, dx\right)^{1/q} \left(\int_{\mathbb{R}^3} |u_n - u|^p \, dx\right)^{1/p} = o(1)$$

and so

$$\left|\int_{\mathbb{R}^3} (|u_n|^{p-1} - |u_m|^{p-1})(u_n - u_m) \, dx\right| \leq c \, \|u_n - u_m\|_p = o(1).$$

This proves (3.4) for C. The verification of (3.4) for B follows from

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n-u_m)dx &\leq \|\phi_{u_n}\|_6\|u_n\|_2\|u_n-u_m\|_3\\ &\leq c\|u_n\|^2_{H^1}\|u_n\|_2\|u_n-u_m\|_3=o(1).\end{aligned}$$

Finally, condition (3.5) is trivial since, if $u_n \rightharpoonup \bar{u}$, then

$$\langle T'(u_n),u_n\rangle=\int_{\mathbb{R}^3}\phi_n|u_n|^2dx-\int_{\mathbb{R}^3}|u_n|^pdx$$

is bounded by (3.6) and the continuous inclusion of $H^1(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$. Then, applying Lemma 3.1 we deduce that the weak limit $\bar{u}$ of a minimizing sequence $\{u_n\}$ is in B_ρ. In accordance with the statement of Theorem 1.1, $\bar{u}$ is renamed u_ρ.

Remark 3.1. We remark here explicitly that the verification of (3.2)-(3.5) does not depend on the range in which p varies.

To conclude the proof of Theorem 1.1 we need to show that u_ρ is real valued. Notice that, in general, if z is a complex function written as $z(x,t)=|z(x,t)|e^{iS(x,t)}$ then

$$I(z(x,t))=I(|z(x,t)|)+\int_{\mathbb{R}^3}|z(x,t)|^2|\nabla S(x,t)|^2dx,$$

so we easily deduce that the minimizer u_ρ has to be real valued.

3.1. **The orbital stability.** We first recall the definition of orbital stability. Let us define

$$S_\rho=\{e^{i\theta}u(x):\theta\in[0,2\pi),\|u\|_2=\rho,\ I(u)=I_{\rho^2}\}.$$

We say that S_ρ is *orbitally stable* if for every $\varepsilon>0$ there exists $\delta>0$ such that for any $\psi_0\in H^1(\mathbb{R}^3)$ with $\inf_{v\in S_\rho}\|v-\psi_0\|_{H^1(\mathbb{R}^3)}<\delta$ we have

$$\forall\, t>0 \quad \inf_{v\in S_\rho}\|\psi(t,.)-v\|_{H^1(\mathbb{R}^3)}<\varepsilon,$$

where $\psi(t,.)$ is the solution of (1.1) with initial datum ψ_0. We notice explicitly that S_ρ is invariant by translations, i.e. if $v\in S_\rho$ then also $v(.-y)\in S_\rho$ for any $y\in\mathbb{R}^3$.
Since the energy and the charge associated to $\psi(x,t)$ evolving according to (1.1) are

$$\begin{aligned}E(\psi(x,t)) &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\psi|^2dx+\frac{1}{4}\int_{\mathbb{R}^3}(|x|^{-1}*|\psi|^2)|\psi|^2dx-\frac{1}{p}\int_{\mathbb{R}^3}|\psi|^pdx\\ &= E(\psi(x,0))\end{aligned}$$

and

$$Q(\psi(x,t)) = \frac{1}{2}\int_{\mathbb{R}^3} |\psi|^2 dx = Q(\psi(x,0)),$$

our action functional I is exactly the energy and Q is the L^2-norm.

In order to prove Theorem 1.2 we argue by contradiction assuming that there exists a ρ such that S_ρ is not orbitally stable. This means that there exists $\varepsilon > 0$ and a sequence of initial data $\{\psi_{n,0}\} \subset H^1(\mathbb{R}^3)$ and $\{t_n\} \subset \mathbb{R}$ such that the maximal solution ψ_n, which is global in time and $\psi_n(0,.) = \psi_{n,0}$, satisfies

$$\lim_{n\to+\infty} \inf_{v\in S_\rho} \|\psi_{n,0} - v\|_{H^1(\mathbb{R}^3)} = 0 \quad \text{and} \quad \inf_{v\in S_\rho} \|\psi_n(t_n,.) - v\|_{H^1(\mathbb{R}^3)} \geq \varepsilon$$

Then there exists $u_\rho \in H^1(\mathbb{R}^3)$ minimizer of I_ρ and $\theta \in \mathbb{R}$ such that $v = e^{i\theta}u_\rho$ and

$$\|\psi_{n,0}\|_2 \to \|v\|_2 = \rho \quad \text{and} \quad I(\psi_{n,0}) \to I(v) = I_{\rho^2}$$

Actually we can assume that $\psi_{n,0} \in B_\rho$ (there exist $\alpha_n = \rho/\|\psi_{n,0}\|_2 \to 1$ so that $\alpha_n\psi_{n,0} \in B_\rho$ and $I(\alpha_n\psi_{n,0}) \to I_{\rho^2}$, i.e. we can replace $\psi_{n,0}$ with $\alpha_n\psi_{n,0}$). So $\{\psi_{n,0}\}$ is a minimizing sequence for I_{ρ^2}, and since

$$I(\psi_n(.,t_n)) = I(\psi_{n,0})$$

also $\{\psi_n(.,t_n)\}$ is a minimizing sequence for I_{ρ^2}. Since we have proved that every minimizing sequence has a subsequence converging (up to translation) in H^1-norm to a minimum on the sphere B_ρ, we readily have a contradiction, proving Theorem 1.2.

4. The case $2 < p < 3$

The proof of the existence of a minimizer for I in this case is more involved. Indeed the main problem here is the subadditivity condition which is not easy to verify when $2 < p < 3$ and indeed the possibility of dichotomy for an arbitrary minimizing sequence cannot be excluded. In this case the computations of the proof of Lemma 3.2 to prove the strong subadditivity inequality (2.3), fail due to the limitations on p. In fact we will recover (2.3) indirectly.

The results of this section are contained in [7] to which the reader is referred for details and to deal with a more general case.

Turning back to (2.3), a classical approach to prove it, is to ensure that

(MD) the function $s \mapsto \frac{I_{s^2}}{s^2}$ is monotone decreasing.

Indeed, in case (MD) holds, for $\mu \in (0,\rho)$ we get

$$\frac{\mu^2}{\rho^2} I_{\rho^2} < I_{\mu^2} \quad \text{and} \quad \frac{\rho^2-\mu^2}{\rho^2} I_{\rho^2} < I_{\rho^2-\mu^2}$$

and hence

$$I_{\rho^2} = \frac{\mu^2}{\rho^2} I_{\rho^2} + \frac{\rho^2-\mu^2}{\rho^2} I_{\rho^2} < I_{\mu^2} + I_{\rho^2-\mu^2} \quad \forall\, \mu \in (0,\rho),$$

i.e. (2.3). Our aim, is then to give sufficient conditions that guarantee (MD).

Let us start with the following abstract situation referred to the C^1 functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + T(u).$$

Definition 4.1. *Let $u \in H^1(\mathbb{R}^N), u \neq 0$. A continuous path $g_u : \theta \in \mathbb{R}^+ \mapsto g_u(\theta) \in H^1(\mathbb{R}^N)$ such that $g_u(1) = u$ is said to be a* scaling path of u *if*

$$\Theta_{g_u}(\theta) := \|g_u(\theta)\|_2^2 \|u\|_2^{-2} \quad \textit{is differentiable and } H'_{g_u}(1) \neq 0$$

where the prime denotes the derivative. We denote with $\mathcal{G}_u$ the set of the scaling paths of u.

The set $\mathcal{G}_u$ is nonempty and indeed it contains a lot of elements: for example, $g_u(\theta) = \theta u(x) \in \mathcal{G}_u$, since $\Theta_{g_u}(\theta) = \theta^2$. Also $g_u(\theta) = u(x/\theta)$ is an element of $\mathcal{G}_u$ since $\Theta_{g_u}(\theta) = \theta^N$. As we will see in our application, it is relevant to consider the family of scaling paths of u parametrized with $\beta \in \mathbb{R}$ given by

$$\mathcal{G}_u^\beta := \{g_u(\theta) = \theta^{1-\frac{N}{2}\beta} u(x/\theta^\beta)\} \subset \mathcal{G}_u. \tag{4.1}$$

Notice that all the paths of this family have as associated function $\Theta(\theta) = \theta^2$.

Moreover, fixed $u \neq 0$, we define the following real valued function which is crucial for our purpose:

$$h_{g_u}(\theta) := J(g_u(\theta)) - \Theta_{g_u}(\theta) J(u)\,,\ \theta \geq 0. \tag{4.2}$$

Definition 4.2. *Let $u \neq 0$ be fixed and $g_u \in \mathcal{G}_u$. We say that the scaling path g_u is admissible for the functional J if h_{g_u} is a differentiable function.*

In our application the function h_{g_u} will be obviously differentiable; this is due to the special form of the scaling path we choose; indeed we will work with the subfamily $\mathcal{G}_u^\beta$.

Our main abstract theorem is now the following.

Theorem 4.1 (Avoiding Dichotomy)**.** *Assume that for every $\rho > 0$, all the minimizing sequences $\{u_n\}$ for J_{ρ^2} have a weak limit, up to translations, different from zero. Assume that T satisfies assumptions* (3.2)*,* (3.3)*,* (3.4) *and* (3.5) *of Lemma 3.1.*

Assume finally (2.2) *and the following conditions*

$$-\infty < J_{s^2} < 0 \quad \text{for all } s > 0, \tag{4.3}$$

$$s \mapsto J_{s^2} \quad \text{is continuous}, \tag{4.4}$$

$$\lim_{s\to 0} \frac{J_{s^2}}{s^2} = 0. \tag{4.5}$$

Then for every $\rho > 0$ the set

$$M(\rho) = \bigcup_{\mu\in(0,\rho]} \{u \in B_\mu : J(u) = J_{\mu^2}\}$$

is non empty.

If in addition

$$\forall\, u \in M(\rho) \quad \exists\, g_u \in \mathcal{G}_u \text{ admissible, such that } \quad \frac{d}{d\theta} h_{g_u}(\theta)_{|\theta=1} \neq 0, \tag{4.6}$$

then **(MD)** *holds. Moreover, if $\{u_n\}$ is a minimizing sequence weakly convergent to a certain $\bar{u}$ (necessarily $\neq 0$) then $\|u_n - \bar{u}\|_{H^1} \to 0$ and $J(\bar{u}) = J_{\rho^2}$.*

Remark 4.1. We have seen in the previous section that (4.3) ensures that the weak limit of the minimizing sequences is not zero (this is independent of the range in which p varies). Notice that to recover (4.3), it is sufficient the weak subadditivity condition (2.2) in $[0, +\infty)$ and the fact that $J_{s^2} < 0$ only for s in a certain interval $(0, \bar{\rho}]$. Indeed, let $\rho \in (\bar{\rho}, \sqrt{2}\bar{\rho}]$: then for every $s \in (\bar{\rho}, \rho]$ we get

$$J_{s^2} \leq J_{\bar{\rho}^2} + J_{s^2-\bar{\rho}^2} < 0$$

since $s^2 - \bar{\rho}^2 < \bar{\rho}^2$. This shows that $J_{s^2} < 0$ for s in the larger interval $(0, \rho]$. Iterating this procedure it follows that $J_{s^2} < 0$ for every $s > 0$.

Before to prove this theorem, we think it is interesting to address the dichotomy case, i.e. when the minimizing sequences for I_{ρ^2} weakly converge to a non zero function $\bar{u}$ which is not on the right constraint but satisfies $\|\bar{u}\|_2 = \mu_0 < \rho$. The result is not surprising in view of the Lions' principle.

Proposition 4.1 (Dichotomy)**.** *Let $T \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ satisfying* (3.2) *and* (3.3)*. Let $\rho > 0$ and $\{u_n\} \subset B_\rho$ be a minimizing sequence for J_{ρ^2} such that $u_n \rightharpoonup \bar{u} \neq 0$ and assume that $\mu_0 = \|\bar{u}\|_2 \in (0, \rho)$. Assume also that* (2.2) *holds. Then*

$$J_{\rho^2} = J_{\mu_0^2} + J_{\rho^2-\mu_0^2} \tag{4.7}$$

and $J(\bar{u}) = J_{\mu_0^2}$.

This proposition shows that in the dichotomy case, in (2.2) the equality holds and the weak limit $\bar{u}$ is a minimizer on the manifold given by the constraint $\|u\|_2 = \mu_0$. Although B_{μ_0} is not the original constraint, we can take advantage of the fact that $\bar{u}$ is a minimizer on $\|u\|_2 = \mu_0$.

Proof of Proposition 4.1. Since $u_n - \bar{u} \rightharpoonup 0$, we get

$$\|u_n - \bar{u}\|_2^2 + \|\bar{u}\|_2^2 = \|u_n\|_2^2 + o(1)$$

therefore

$$\alpha_n = \frac{\rho^2 - \mu_0^2}{\|u_n - \bar{u}\|_2} \to 1. \tag{4.8}$$

On the other hand, $\{u_n\}$ is a minimizing sequence for I_{ρ^2}, so

$$\frac{1}{2}\|u_n\|_{D^{1,2}}^2 + T(u_n) = I_{\rho^2} + o(1)$$

and by (3.2), we deduce also

$$\frac{1}{2}\|u_n - \bar{u}\|_{D^{1,2}}^2 + \frac{1}{2}\|\bar{u}\|_{D^{1,2}}^2 + T(u_n - \bar{u}) + T(\bar{u}) = J_{\rho^2} + o(1).$$

Hence using (4.8) and (3.3) we infer

$$\frac{1}{2}\|\alpha_n(u_n - \bar{u})\|_{D^{1,2}}^2 + \frac{1}{2}\|\bar{u}\|_{D^{1,2}}^2 + T(\alpha_n(u_n - \bar{u})) + T(\bar{u}) = J_{\rho^2} + o(1)$$

that is,

$$J(\alpha_n(u_n - \bar{u})) + J(\bar{u}) = J_{\rho^2} + o(1). \tag{4.9}$$

Then, since $\|\alpha_n(u_n - \bar{u})\|_2 = \rho^2 - \mu_0^2$ and (2.2) we get

$$J_{\rho^2-\mu_0^2} + J(\bar{u}) \leq J(\alpha_n(u_n - \bar{u})) + J(\bar{u}) = J_{\rho^2} + o(1) \leq J_{\rho^2-\mu_0^2} + J_{\mu_0^2} + o(1)$$

which implies $J(\bar{u}) = J_{\mu_0^2}$ and consequently (4.7). □

A crucial remark now for our purpose is in order. The strong subadditivity inequality (2.3) holds if the following condition is satisfied

(I) *the function* $s \mapsto \frac{J_{s^2}}{s^2}$ *in the interval* $[0,\rho]$ *achieves its unique minimum in* $s = \rho$.

Indeed for $\mu \in (0,\rho)$ we get $\frac{\mu^2}{\rho^2} J_{\rho^2} < J_{\mu^2}$ and $\frac{\rho^2-\mu^2}{\rho^2} J_{\rho^2} < J_{\rho^2-\mu^2}$. Therefore

$$J_{\rho^2} = \frac{\mu^2}{\rho^2} J_{\rho^2} + \frac{\rho^2 - \mu^2}{\rho^2} J_{\rho^2} < J_{\mu^2} + J_{\rho^2-\mu^2} \quad \forall\, \mu \in (0,\rho).$$

We now show a lemma that asserts that the behavior of the function $s \mapsto J_{s^2}$ near zero is sufficient to deduce "almost" (2.3), the strong subadditivity inequality. Moreover this Lemma will be useful also to show Theorem 4.1.

Lemma 4.1. *Let us assume that condition* (4.3) *is satisfied in a certain interval* $[0, \rho]$ *and that* (4.4) *and* (4.5) *hold. Then for every* $\rho > 0$ *there exists* $\rho_0 \in (0, \rho]$ *such that for every* $\mu \in (0, \rho_0)$

$$J_{\rho_0^2} < J_{\mu^2} + J_{\rho_0^2-\mu^2}.$$

Proof. Let us fix $\rho > 0$ and define

$$\rho_0 := \min\left\{ s \in [0, \rho] \text{ s.t } \frac{J_{s^2}}{s^2} = \frac{I_{\rho^2}}{\rho^2} \right\}$$

which is strictly positive in virtue of (4.4) and (4.5).

We claim that the function $s \mapsto \frac{J_{s^2}}{s^2}$ in the interval $[0, \rho_0]$ achieves the minimum only in $s = \rho_0$. By the claim follows, as noticed before, that $J_{\rho_0^2} < J_{\mu^2} + J_{\rho_0^2-\mu^2}$ for every $\mu \in (0, \rho_0)$. In order to prove the claim we notice that if there exists $\rho_* < \rho_0$ such that $\frac{J_{\rho_*^2}}{\rho_*^2} < \frac{J_{\rho_0^2}}{\rho_0^2}$ it will exists by continuity a $\bar{\rho} < \rho_0$ such that $\frac{J_{\bar{\rho}^2}}{\bar{\rho}^2} = \frac{J_{\rho_0^2}}{\rho_0^2}$ which contradicts the definition of ρ_0. $\square$

With this result in hands we can give now the

Proof of Theorem 4.1. To prove that $M(\rho) \neq \emptyset$ let us fix $\rho > 0$. By Lemma 4.1 there exists $\rho_0 \in (0, \rho]$ such that for every $\mu \in (0, \rho_0)$

$$J_{\rho_0^2} < J_{\rho_0^2-\mu_0^2} + J_{\mu_0^2}.$$

Then by Lemma 3.1 we get $\{u \in B_{\mu_0} : J(u) = J_{\mu_0^2}\} \neq \emptyset$.

To get (MD) it is sufficient to prove condition (I) on every interval $[0, \rho]$. So let us fix $\rho > 0$ and call $\alpha := \min_{[0,\rho]} \frac{I_{s^2}}{s^2} < 0$, by (4.3). Let

$$\rho_0 := \min\left\{ s \in [0, \rho] \text{ s.t } \frac{J_{s^2}}{s^2} = \alpha \right\}.$$

We have to prove that $\rho_0 = \rho$.

Thanks to (4.4) and (4.5), $\rho_0 > 0$ and

$$\forall\, s \in [0, \rho_0) : \frac{J_{\rho_0^2}}{\rho_0^2} < \frac{J_{s^2}}{s^2} \tag{4.10}$$

namely, the function $[0,\rho_0]\ni s\mapsto \frac{J_{s^2}}{s^2}\in\mathbb{R}_-$ achieves the minimum only in $s=\rho_0$, by definition of ρ_0. Since condition (I) is satisfied in $[0,\rho_0]$ we have the strong subadditivity inequality

$$J_{\rho_0^2}<J_{\mu^2}+J_{\rho_0^2-\mu^2}\quad\forall\,\mu\in(0,\rho_0).$$

Therefore we can apply Lemma 3.1 to the minimization problem

$$J_{\rho_0^2}=\inf_{B_{\rho_0}}J(u)$$

and we deduce the existence of $\bar u\in B_{\rho_0}$ such that $J(\bar u)=J_{\rho_0^2}$. In particular $\bar u\in M(\rho)$. Now we argue by contradiction by assuming that $\rho_0<\rho$. Then fixed $g_{\bar u}\in\mathcal{G}_{\bar u}$ with its associated Θ, by (4.10) and the definition of ρ_0:

$$\frac{J_{\rho_0^2}}{\rho_0^2}\leq\frac{J_{\Theta(\theta)\rho_0^2}}{\Theta(\theta)\rho_0^2}\quad\text{for all}\quad\theta\in(1-\varepsilon,1+\varepsilon).$$

Therefore we have

$$\frac{J(g_{\bar u}(\theta))}{\Theta(\theta)\rho_0^2}\geq\frac{J_{\Theta(\theta)\rho_0^2}}{\Theta(\theta)\rho_0^2}\geq\frac{J_{\rho_0^2}}{\rho_0^2}=\frac{J(\bar u)}{\rho_0^2}\quad\text{for every }\theta\in(1-\varepsilon,1+\varepsilon).$$

This means that the map $h_{g_{\bar u}}(\theta)=J(g_{\bar u}(\theta))-\Theta(\theta)J(\bar u)$, defined in a neighborhood of $\theta=1$, is non negative and has a global minimum in $\theta=1$ with $h_{g_{\bar u}}(1)=0$. Then we get

$$h'_{g_{\bar u}}(1)=0.$$

Since $g_{\bar u}$ is arbitrary this relation has to be true for every map $g_{\bar u}$, so we have found a $\bar u\in M(\rho)$ such that for every $g_{\bar u}\in\mathcal{G}_{\bar u}$ it results $h'_{g_{\bar u}}(1)=0$; this clearly contradicts (4.6) and so $\rho_0=\rho$. This implies condition (I) on every interval of type $[0,\rho]$ and so (MD), that is, $s\mapsto J_{s^2}/s^2$ is monotone decreasing in $[0,+\infty)$.

To prove the final part, let $\{u_n\}$ be a minimizing sequence for J_{ρ^2} weakly convergent to a certain $\bar u$. We already know that $\bar u\neq 0$. Since we have just shown that in $(0,\rho)$ the strong subadditivity condition is satisfied we can apply Lemma 3.1 and conclude the proof. □

To prove Theorem 1.3 we show that all the hypothesis of Theorem 4.1 are satisfied.

As before, for simplicity we define

$$A(u):=\int_{\mathbb{R}^3}|\nabla u|^2dx,\quad B(u):=\int_{\mathbb{R}^3}\phi_u|u|^2dx,\quad C(u):=-\int_{\mathbb{R}^3}|u|^pdx.$$

so that

$$I(u)=\frac{1}{2}A(u)+\frac{1}{4}B(u)+\frac{1}{p}C(u).$$

We divide the proof in various steps.

Step 1 Condition (2.2) holds and the functional $T = \frac{1}{4}B + \frac{1}{p}C$ satisfies (3.2), (3.3), (3.4) and (3.5).

These facts are proved in [19] (Proposition 2.3) and Section 3 (see Remark 3.1) respectively.

Step 2 If $2 < p < 3$, then condition (4.3) is satisfied.

We already know that $I_{s^2} > -\infty$ for all $s > 0$ so we just have to prove that $I_{s^2} < 0$ for every $s > 0$. Let $u \in H^1(\mathbb{R}^3)$ and choose the family of scaling paths given in (4.1)

$$g_u(\theta) = \theta^{1-\frac{3}{2}\beta} u(x/\theta^\beta)$$

such that $\Theta(\theta) = \theta^2$ and $\|g_u(\theta)\|_2 = \theta$. We easily find the following scaling laws:

$$A(g_u(\theta)) = \theta^{2-2\beta} A(u),$$
$$B(g_u(\theta)) = \theta^{4-\beta} B(u),$$
$$C(g_u(\theta)) = \theta^{(1-\frac{3}{2}\beta)p+3\beta} C(u).$$

For $\beta = -2$ we get

$$I(g_u(\theta)) = \frac{\theta^6}{2} A(u) + \frac{\theta^6}{4} B(u) + \frac{\theta^{4p-6}}{p} C(u) \to 0^- \text{ for } \theta \to 0,$$

since $4p - 6 < 6$ and $C(u) < 0$. This proves that there exists a small θ_0 such that

$$I_{s^2} < 0 \quad \forall\, s \in (0, \theta_0].$$

Then by Step 1 and Remark 4.1 we conclude that $I_{s^2} < 0$ for every $s > 0$.

Step 3 For every $\rho > 0$, all the minimizing sequences $\{v_n\}$ for I_{ρ^2} have a weak limit, up to translations, different from zero. Furthermore the weak limit is in $M(\rho)$.

The proof of this step is the same as in the case $3 < p < 10/3$ but we give it for completeness. Let $\{v_n\}$ be a minimizing sequence in B_ρ for I_{ρ^2}. For any sequence $\{y_n\} \subset \mathbb{R}^3$ we have that $v_n(. + y_n)$ is still a minimizing sequence for I_{ρ^2}. Again we will show that there exist a sequence $\{y_n\} \subset \mathbb{R}^3$ such that the weak limit of $v_n(. + y_n)$ is different from zero.

By the well-known Lions' lemma it follows that if

$$\lim_{n\to\infty} \left(\sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |v_n|^2 dx \right) = 0,$$

then $v_n \to 0$ in $L^q(\mathbb{R}^3)$ for any $q \in (2, 2^*)$ and so $C(v_n) \to 0$. On the other hand, by Step 2, $I_{\rho^2} < 0$ so we have necessarily that

$$\sup_{y \in \mathbb{R}^3} \int_{B(y,1)} |v_n|^2 dx \geq \delta > 0.$$

In this case we can choose $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B(0,1)} |v_n(. + y_n)|^2 dx \geq \delta > 0$$

and hence, due to the compact embedding $H^1(B(0,1)) \hookrightarrow L^2(B(0,1))$, we deduce that the sequence $v_n(. + y_n)$ weakly converges to a nonzero v.

From the previous step it follows that $v \in M(\rho) \neq \emptyset$: if $\|v\|_2 = \rho$ it is obvious, otherwise use Proposition 4.1.

Before going to Step 4, we prove a lemma about the behavior of the levels of minima of the functional associated to the nonlinear Schrödinger equation without the nonlocal term. Let us define

$$G(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx,$$

where $2 < p < 10/3$ and let

$$G_{\rho^2} = \inf_{B_\rho} \ G(u). \tag{4.11}$$

It is known that, for every $\rho > 0$

$$\exists\, u_\rho \in B_\rho \quad \text{such that} \quad G_{\rho^2} = G(u_\rho) < 0$$

(see [2]); moreover by (4.9)

$$\forall\, u \in B_\rho : G(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - b_p \rho^{\frac{6-p}{2}} \|\nabla u\|_2^{\frac{3(p-2)}{2}}.$$

As a consequence we get

$$0 > G(u_\rho) \geq (\frac{1}{2} - b_p \rho^{\frac{6-p}{p}} \|\nabla u_\rho\|_2^{\frac{3p-10}{2}})\|\nabla u_\rho\|_2^2 \tag{4.12}$$

which implies, since $p < 10/3$, that

$$\{\|\nabla u_\rho\|_2\}_{\rho > 0} \ \text{ is bounded } \ \text{for } \rho \to 0. \tag{4.13}$$

Lemma 4.2. *We have* $\lim_{\rho \to 0} \frac{G_{\rho^2}}{\rho^2} = 0$.

Proof. Since the minimizer u_ρ for G_{ρ^2} satisfies

$$-\Delta u_\rho - |u_\rho|^{p-2}u_\rho = \omega_\rho u_\rho, \tag{4.14}$$

we get, taking into account (4.12),

$$\frac{\omega_\rho}{2} = \frac{\|\nabla u_\rho\|_2^2 - \int_{\mathbb{R}^3}|u_\rho|^p dx}{2\int_{\mathbb{R}^3}|u_\rho|^2 dx} \le \frac{\frac{1}{2}\|\nabla u_\rho\|_2^2 - \frac{1}{p}\int_{\mathbb{R}^3}|u_\rho|^p dx}{\int_{\mathbb{R}^3}|u_\rho|^2 dx} = \frac{G(u_\rho)}{\rho^2} < 0 \tag{4.15}$$

where ω_ρ is the Lagrange multiplier associated to the minimizer. Actually we prove that $\lim_{\rho\to 0}\omega_\rho = 0$, so by comparison in (4.15) we get the Lemma.

To show that $\lim_{\rho\to 0}\omega_\rho = 0$ we argue by contradiction by assuming that there exists a sequence $\rho_n \to 0$ such that $\omega_{\rho_n} < -c$ for some $c \in (0,1)$. Since the minimizers $u_n := u_{\rho_n}$ satisfy the equation (4.14), we get

$$\begin{aligned} c\|u_n\|_{H^1}^2 &\le \int_{\mathbb{R}^3}|\nabla u_n|^2 dx + c\int_{\mathbb{R}^3}|u_n|^2 dx \\ &\le \int_{\mathbb{R}^3}|\nabla u_n|^2 dx - \omega_{\rho_n}\int_{\mathbb{R}^3}|u_n|^2 dx = \int_{\mathbb{R}^3}|u_n|^p dx \le c\|u_n\|_{H^1}^p, \end{aligned}$$

which implies that there exists $c' > 0$ such that $\|\nabla u_n\|_2 > c' > 0$. But then, by using (4.12) and (4.13)

$$0 \ge G(u_n) \ge \frac{1}{2}c' - o(1)$$

with $o(1) \to 0$ for $n \to \infty$ and this yields to a contradiction, finishing the proof. □

Now we can proceed.

Step 4 The function $s \mapsto I_{s^2}$ satisfies (4.4) and (4.5).

We first prove that if $\rho_n \to \rho$ then $\lim_{n\to\infty} I_{\rho_n^2} = I_{\rho^2}$. For every $n \in \mathbb{N}$, let $w_n \in B_{\rho_n}$ such that $I(w_n) < I_{\rho_n^2} + \frac{1}{n} < \frac{1}{n}$. Therefore, by using the interpolation and the Sobolev inequality, we get

$$\frac{1}{2}\|\nabla w_n\|_2^2 - C\rho_n^{\frac{6-p}{2}}\|\nabla w_n\|_2^{\frac{3(p-2)}{2}} \le \frac{1}{2}\|\nabla w_n\|_2^2 - \frac{1}{p}\|w_n\|_p^p \le I(w_n) < \frac{1}{n}.$$

Since $\frac{3(p-2)}{2} < 2$ and $\{\rho_n\}$ is bounded, we deduce that

$$\{w_n\} \text{ is bounded in } H^1(\mathbb{R}^3).$$

In particular $\{A(w_n)\}$ and $\{C(w_n)\}$ are bounded sequences, and also $\{B(w_n)\}$ since in general,

$$\forall\, u \in H^1(\mathbb{R}^3) : B(u) = \int_{\mathbb{R}^3} \phi_u |u|^2 dx \leq c\|u\|^4_{H^1(\mathbb{R}^3)},$$

see e.g. [16]. So we easily find

$$\begin{aligned} I_{\rho^2} &\leq I(\frac{\rho}{\rho_n} w_n) = \frac{1}{2}\left(\frac{\rho}{\rho_n}\right)^2 A(w_n) + \frac{1}{4}\left(\frac{\rho}{\rho_n}\right)^4 B(w_n) + \frac{1}{p}\left(\frac{\rho}{\rho_n}\right)^p C(w_n) \\ &= I(w_n) + o(1) < I_{\rho_n^2} + o(1). \end{aligned}$$

On the other hand, given a minimizing sequence $\{v_n\} \subset B_\rho$ for I_{ρ^2}, we have

$$I_{\rho_n^2} \leq I(\frac{\rho_n}{\rho} v_n) = I(v_n) + o(1) = I_{\rho^2} + o(1)$$

which, joint to the previous computation, gives $\lim_{n\to\infty} I_{\rho_n^2} = I_{\rho^2}$.

In order to show that $\lim_{\rho\to 0} \frac{I_{\rho^2}}{\rho^2} = 0$, we notice that (see (4.11))

$$\frac{G_{\rho^2}}{\rho^2} \leq \frac{I_{\rho^2}}{\rho^2} < 0.$$

Since $G_{\rho^2}/\rho^2 \to 0$ (see Lemma 4.2) we easily conclude the proof of (4.5).

Step 5 For small ρ the functional I satisfies (4.6)

First recall by Step 3 that $M(\rho) \neq \emptyset$; moreover since $0 \notin M(\rho)$, $A(u), B(u)$ and $C(u)$ are all different from zero whenever $u \in M(\rho)$.

We claim now that

$$\forall\, u \in M(\rho) : -A(u) - \frac{1}{4}B(u) + \frac{6-3p}{2p}C(u) = 0 \tag{4.16}$$

Indeed, for $u \in M(\rho)$ (i.e $\|u\|_2 = \mu \in (0,\rho]$ and $I(u) = I_{\mu^2}$) we define $v(\theta,u) = \theta^{-\frac{3}{2}} u(\frac{x}{\theta})$ so that $\|v(\theta,u)\|_2 = \|u\|_2$. It follows that

$$A(v(\theta,u)) = \theta^{-2}A(u), \quad B(v(\theta,u)) = \theta^{-1}B(u), \quad C(v(\theta,u)) = \theta^{3-\frac{3}{2}p}C(u).$$

Since the map $\theta \mapsto I(v(\theta,u))$ is differentiable and u achieves the minimum on B_μ, we get

$$\frac{d}{d\theta} I(v(\theta,u))|_{\theta=1} = 0$$

which is exactly our claim (4.16).

Now, for $u \neq 0$ we compute explicitly $h_{g_u}(\theta)$ by choosing the family of scaling paths of u parametrized with $\beta \in \mathbb{R}$ given by

$$\mathcal{G}_u^\beta = \{g_u(\theta) = \theta^{1-\frac{3}{2}\beta} u(x/\theta^\beta)\} \subset \mathcal{G}_u.$$

All the paths of this family have as associated function $\Theta(\theta) = \theta^2$. We get (see (4.2))

$$h_{g_u}(\theta) = \frac{1}{2}(\theta^{2-2\beta} - \theta^2)A(u) + \frac{1}{4}(\theta^{4-\beta} - \theta^2)B(u) + \frac{1}{p}(\theta^{(1-\frac{3}{2}\beta)p+3\beta} - \theta^2)C(u),$$

which shows that the paths in $\mathcal{G}_u^\beta$ are admissible, i.e. h_{g_u} is differentiable for every $g_u \in \mathcal{G}_u^\beta$. We have also, for $g_u \in \mathcal{G}_u^\beta$:

$$h'_{g_u}(1) = -\beta A(u) + \frac{2-\beta}{4}B(u) + \frac{(1-\frac{3}{2}\beta)p + 3\beta - 2}{p}C(u).$$

We will show that the admissible scaling path satisfying $\frac{d}{d\theta}h_{g_u}(\theta))_{|\theta=1} \neq 0$ can be chosen in $\mathcal{G}_u^\beta$.

For future reference we compute

$$\begin{aligned} \frac{I(g_u(\theta))}{\theta^2\|u\|_2^2} &= \frac{h_{g_u}(\theta)}{\theta^2\|u\|_2^2} + \frac{I(u)}{\|u\|_2^2} \\ &= \frac{1}{\|u\|_2^2}\left(\frac{1}{2}\theta^{-2\beta}A(u) + \frac{1}{4}\theta^{2-\beta}B(u) + \frac{1}{p}\theta^{(1-\frac{3}{2}\beta)p+3\beta-2}C(u)\right). \end{aligned} \tag{4.17}$$

To prove (4.6) we argue now by contradiction. Assume that there exists a sequence $\{u_n\} \subset M(\rho)$ with $\rho \geq \|u_n\|_2 = \rho_n \to 0$ such that for all $\beta \in \mathbb{R}$ (that is: for all $g_{u_n} \in \mathcal{G}_{u_n}^\beta$)

$$h'_{g_{u_n}}(1) = -\beta A(u_n) + \frac{2-\beta}{4}B(u_n) + \frac{(1-\frac{3}{2}\beta)p + 3\beta - 2}{p}C(u_n) = 0$$

then, by using (4.16) we get

$$\frac{1}{2}B(u_n) + \frac{p-2}{p}C(u_n) = 0$$

and hence (again by (4.16))

$$\begin{aligned} &B(u_n) = 2A(u_n)\,, \quad C(u_n) = \frac{p}{2-p}A(u_n), \\ &I(u_n) = \frac{A(u_n)}{2} + \frac{B(u_n)}{4} + \frac{C(u_n)}{p} = \frac{3-p}{2-p}A(u_n). \end{aligned} \tag{4.18}$$

The contradiction is achieved by showing that relations (4.18) are impossible for $p \in (2,3)$ for small ρ. We know that

$$\begin{cases} I(u_n) = I_{\rho_n^2} \to 0 & \text{(by continuity)} \\ A(u_n), B(u_n), C(u_n) \to 0 & \text{(by (4.18))} \end{cases} \tag{4.19}$$

Because of the following Hardy-Littlewood-Sobolev inequality

$$B(u_n) = \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|} dxdy \leq c\|u_n\|_{12/5}^4$$

(that we will frequently use), it is convenient to consider some cases.

- *Case a)* $2 < p < 12/5$.

Then

$$B(u_n) \leq c\|u_n\|_{12/5}^4 \leq c\|u_n\|_p^{4\alpha}\|u_n\|_6^{4(1-\alpha)}, \quad \alpha = \frac{3p}{2(6-p)}.$$

We get, thanks to (4.18) and the Sobolev inequality $\|u_n\|_6^2 \leq SA(u_n)$ (here S is the best Sobolev constant),

$$B(u_n) \leq cB(u_n)^{\frac{4\alpha}{p}} B(u_n)^{\frac{4(1-\alpha)}{2}}.$$

This is in contradiction with (4.19) since $\frac{4\alpha}{p} + \frac{4(1-\alpha)}{2} > 1$, being $p < 3$.

- *Case b)* $p = 12/5$.

This case is simpler: thanks to (4.18) we get

$$\|u_n\|_{12/5}^{12/5} = cB(u_n) \leq c\|u_n\|_{12/5}^4$$

which contradicts (4.19).

- *Case c)* $12/5 < p < 8/3$.

Interpolating $L^{12/5}$ between L^2 and L^p we get

$$\|u_n\|_p^p = cB(u_n) \leq c\|u_n\|_{12/5}^4 \leq c\|u_n\|_2^{4\alpha}\|u_n\|_p^{4(1-\alpha)}, \ \alpha = \frac{5p-12}{6(p-2)}$$

i.e. $\|u_n\|_p^p \leq \rho_n^{4\alpha}\|u_n\|_p^{4(1-\alpha)}$. Since $p < 4(1-\alpha)$, i.e. $p < 8/3$, we get a contradiction with (4.19).

- *Case d)* $p = 8/3$

Again by interpolation we get

$$B(u_n) \leq c\|u_n\|_{12/5}^4 \leq c\rho_n^{4/3}\|u_n\|_{8/3}^{8/3},$$

and again, using that $B(u_n) = \|u_n\|_{8/3}^{8/3}$ we get a contradiction.

- *Case e)* $8/3 < p < 3$.

In this case for u_0 satisfying (4.18), with $\|u_0\|_2 = \rho_0$ we get (see (4.17))

$$\begin{aligned}\frac{I_{\theta^2\rho_0^2}}{\theta^2\rho_0^2} &\leq \frac{I(g_{u_0}(\theta))}{\theta^2\rho_0^2} \\ &= \frac{1}{\rho_0^2}\left(\frac{1}{2}\theta^{-2\beta}A(u_0) + \frac{1}{2}\theta^{2-\beta}A(u_0) + \frac{A(u_0)}{2-p}\theta^{(1-\frac{3}{2}\beta)p+3\beta-2}\right).\end{aligned}$$

Now let us choose $\beta = \frac{2(2-p)}{10-3p}$ so that

$$0 < -2\beta = (1-\frac{3}{2}\beta)p + 3\beta - 2 < 2 - \beta.$$

Hence we obtain

$$\begin{aligned}\frac{I_{\theta^2\rho_0^2}}{\theta^2\rho_0^2} \leq \frac{I(g_{u_0}(\theta))}{\theta^2\rho_0^2} &= \frac{A(u_0)}{\rho_0^2}\left[\frac{4-p}{2(2-p)}\theta^{\frac{4(p-2)}{10-3p}} + \frac{1}{2}\theta^{\frac{4(4-p)}{10-3p}}\right] \\ &= \frac{2-p}{3-p}\frac{I(u_0)}{\rho_0^2}\left[\frac{4-p}{2(2-p)}\theta^{\frac{4(p-2)}{10-3p}} + \frac{1}{2}\theta^{\frac{4(4-p)}{10-3p}}\right]\end{aligned}$$

and so renaming $\theta^2\rho_0^2 = s^2$ we get

$$\frac{I_{s^2}}{s^2} \leq -cs^{\frac{4(p-2)}{10-3p}} + o(s^{\frac{4(p-2)}{10-3p}}) \tag{4.20}$$

for sufficiently small s.

On the other hand for u_n satisfying (4.18) we have

$$\|u_n\|_p^p = cB(u_n) \leq c\|u_n\|_{12/5}^4 \leq c\|u_n\|_2^{4\alpha}\|u_n\|_p^{4(1-\alpha)} \quad \alpha = \frac{5p-12}{6(p-2)},$$

that is

$$\|u_n\|_p^p \leq c\rho_n^{4\alpha}\|u_n\|_p^{4(1-\alpha)}. \tag{4.21}$$

Since now $8/3 < p$ (that is $4(1-\alpha) < p$) we cannot argue as in *Case c)* to get the contradiction. But we deduce from (4.21) that $\|u_n\|_p^p \leq c\rho_n^{\frac{2(5p-12)}{3p-8}}$, and hence using (4.18),

$$\frac{I_{\rho_n^2}}{\rho_n^2} \geq -c\rho_n^{\frac{4(p-2)}{3p-8}}. \tag{4.22}$$

Combining (4.22) with (4.20) we find

$$-c\rho_n^{\frac{4(p-2)}{3p-8}} \leq \frac{I_{\rho_n^2}}{\rho_n^2} \leq -c\rho_n^{\frac{4(p-2)}{10-3p}} + o(\rho_n^{\frac{4(p-2)}{10-3p}})$$

This drives to a contradiction for $\rho_n \to 0$ since

$$\frac{4(p-2)}{3p-8} > \frac{4(p-2)}{10-3p}.$$

Summing up, we have verified all the hypothesis of Theorem 4.1 so the minimizing sequence $\{u_n\}$ is strongly convergent in $H^1(\mathbb{R}^3)$. Moreover the minimizer is real-valued and this finishes the proof of Theorem 1.3.

4.1. **The orbital stability.** The proof of the orbital stability is exactly as in the case $3 < p < 10/3$, since we have never used this restriction. The unique fact used in the proof of the orbital stability is just the convergence of every minimizing sequence and the conservation of energy and charge.

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