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Exponential trichotomies and continuity of invariant manifolds

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Abstract. In this work, we consider the invariant manifolds for the family of equations

$$\dot{x} = Ax + f(\varepsilon, x),$$

where A the is generator of a strongly continuous semigroup of linear operators in a Banach space X and $f(\varepsilon, \cdot) : X \to X$ is continuous. The existence of stable (unstable) and center-stable (center-unstable) manifolds for a large class of these equations has been proved in [2]. We prove here that, if A admits a exponential trichotomy and f satisfies some suitable regularity hypotheses, then those manifolds are continuous with respect to the parameter ε.

1. Introduction

There exists a large literature on the existence and properties of stable (unstable) and center-stable (center-unstable) invariant manifolds for the problem

$$\dot{x} = Ax + f(x), \tag{1.1}$$

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under various assumptions.

We may cite for example, [1], [2], [5], [6], [7], [8], [10], [11], [13], [14], [15], [16] and [17]. The continuity of these sets with respect to parameters has also been investigated, assuming (uniform) exponential dicothomy for the operator A in [1], [10], [8] and [16], for instance. Existence and continuity, assuming nonuniform exponential dichotomy has been proved in [3] and [4]. For the case of exponential trichotomy, the continuity of these sets was proved in [5] and [14]. However, [14] treats only discrete systems and [5], as well as [14], use normal hyperbolicity.

As far as we know, the continuous case, without assuming normal hyperbolicity, has been considered for the first time in the first author's PhD's thesis (in Portuguese, see [18]). It is our aim here to present, among others, the result obtained there for this problem.

To be precise, we state below the hypotheses used throughout, and fix some notation, which are basically the same of [2].

Let X be a Banach space with norm $|\cdot|$, $A : D(A) \subset X \to X$ the generator of a strongly continuous semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on X and suppose $f : X \to X$ is a continuous function satisfying

(H1) $f(0) = 0$,

(H2) $\|f(\varphi) - f(\psi))\| \leq \eta(r)\|\varphi - \psi\|$, $\|\varphi\|, \|\psi\| < r$,

where η is non decreasing continuous function to real values on $[0, \infty)$ with $\eta(0) = 0$.

We also assume the following hypotheses for the semigroup.

(H3) (BU - Backwards Uniqueness). For each $t \geq 0$, $T(t)$ is injective;

(H4) X admits the following decomposition:

(H4a) $X = \pi_- X \oplus \pi_0 X \oplus \pi_+ X$, where π_-, π_0, π_+ are continuous linear projections on X.

The condition (H4a) implies

$$\pi_-\pi_0 = \pi_0\pi_- = \pi_-\pi_+ = \pi_+\pi_- = \pi_+\pi_0 = \pi_0\pi_+ = 0,$$

and

$$\pi_-\pi_- = \pi_-,\ \pi_0\pi_0 = \pi_0,\ \pi_+\pi_+,\ \pi_- + \pi_0 + \pi_+ = I.$$

If $\varphi \in X$, we write $\varphi_- = \pi_-\varphi$, $\varphi_0 = \pi_0\varphi$, $\varphi_+ = \pi_+\varphi$.

Throughout we shall use the equivalent norm $\|\cdot\|$ on X, where, for each $\varphi \in X$

$$\|\varphi\| = |\varphi_-| + |\varphi_0| + |\varphi_+|.$$

(H4b) For each $t \geq 0$, $T(t)$ commutes with the operators π_-, π_0, π_+ so that each of the subspaces $\pi_- X$, $\pi_0 X$, $\pi_+ X$ is invariant under $T(t)$. Furthermore, $T(t)$ may be extended to a continuous group of linear operators on $\pi_0 X \oplus \pi_+ X$;

(H4c) There exist constants

$$a_-,\ a_0,\ a_+,\ \min\{a_-, a_+\} > a_0 \geq 0 \text{ and } K > 1;$$

(without loss of generality we assume $a_0 > 0$), such that

(H4c.i) $\|T(t)\varphi_-\| \leq K e^{-a_- t}\|\varphi_-\|$, $\forall\ \varphi \in X$, $t \geq 0$;

(H4c.ii) $\|T(t)\varphi_0\| \leq K e^{a_0|t|}\|\varphi_0\|$, $\forall\ \varphi \in X$, $t \in \mathbb{R}$;

(H4c.iii) $\|T(t)\varphi_+\| \leq K e^{a_+ t}\|\varphi_+\|$, $\forall\ \varphi \in X$, $t \leq 0$.

It is well known (see for example [9] and [13]) that, if f satisfies (H1) and (H2), then the Cauchy problem

$$\begin{aligned} \dot{x} &= Ax + f(x) \\ x(0) &= x_0. \end{aligned} \tag{1.2}$$

has a unique local 'mild solution', that is, a solution of the integral equation

$$w(t) = T(t)w(0) + \int_0^t T(t-s) f(w(s)) ds. \tag{1.3}$$

defined for small positive $0 < t < t_1$, with $x(t) \to x_0$ as $t \to 0+$. If $x_0 \in D(A)$ and f is continuously differentiable, then the solution is also a strict solution (i.e $x : (0, t_1) \to X$ is C^1, $x(t) \in D(A)$, for $0 < t < t_1$ and the differential equation (1.2) is satisfied). If A is bounded, the solution is defined in a *open* interval around 0 (see [9]).

Define, for $\lambda > 0$,

$$f_\lambda(\varphi) = \begin{cases} f(\varphi), \ if \ \|\varphi\| \leq \lambda, \\ f\left(\frac{\lambda\varphi}{\|\varphi\|}\right), \ if \ \|\varphi\| > \lambda. \end{cases}$$

The properties on f and η ensure the existence of a non decreasing continuous function $\nu(\lambda)$, $\lambda \geq 0$, $\nu(0) = 0$ such that, for every $\varphi,\ \psi \in X$

$$\begin{gathered} \|f_\lambda(\varphi)\| \leq \nu(\lambda)\lambda, \\ \|f_\lambda(\varphi) - f_\lambda(\psi)\| \leq \nu(\lambda)\|\varphi - \psi\|. \end{gathered}$$

Thus, to study (1.3) locally it is enough to investigate the global behavior of the equation

$$w(t) = T(t)w(0) + \int_0^t T(t-s) f_\lambda(w(s)) ds. \tag{1.4}$$

This paper is organized as follows. In Section 2, we recall some results from [2]. In Section 3, we to prove the continuity of the invariant manifolds, with respect to a parameter.

2. Existence of invariant manifolds

We state below some results proved in [2], for completeness. We give below a slightly modified proof of the first result, just to introduce the ideas that will be used in the sequel.

Lemma 2.1. *Let $\tau > 0$ and $w : [0, \tau] \to X$ be continuous solutions of (1.3). Then*

$$y(t) = w(t + \tau),\ \forall\ t \in [-\tau, 0] \tag{2.5}$$

satisfies

$$T(-t)y(t) = y(0) + \int_0^t T(-s)f(y(s))ds,\ \forall\ t \in [-\tau, 0]. \tag{2.6}$$

Conversely, if (BU) holds and y satisfies (2.6), then w satisfies (1.3) in $[0, \tau]$.

Proof Suppose that w satisfies (1.3) and let y be given by (2.5). Then, if $t \in [-\tau, 0]$, it follows that

$$\begin{aligned} T(-t)y(t) &= T(-t)w(t+\tau) \\ &= T(-t)\left[T(t+\tau)w(0) + \int_0^{t+\tau} T(t+\tau-s)f(w(s))ds\right] \\ &= T(\tau)w(0) + \int_0^{t+\tau} T(\tau - s)f(w(s))ds. \end{aligned}$$

But, from (1.3) we have

$$T(\tau)w(0) = w(\tau) - \int_0^{\tau} T(\tau - s)f(w(s))ds.$$

Therefore

$$\begin{aligned} T(-t)y(t) &= w(\tau) - \int_0^{\tau} T(\tau-s)f(w(s))ds + \int_0^{t+\tau} T(\tau-s)f(w(s))ds \\ &= y(0) + \int_{\tau}^{t+\tau} T(\tau-s)f(w(s))ds \\ &= y(0) + \int_{\tau}^{t+\tau} T(\tau-s)f(w(s))ds \\ &= y(0) + \int_0^{t} T(-r)f(w(r+\tau))dr \\ &= y(0) + \int_0^{t} T(-r)f(y(r))dr, \end{aligned}$$

that is, y satisfies (2.6). Conversely, suppose that y satisfies (2.6). Then

$$y(0) = T(-t)y(t) - \int_0^t T(-s)f(y(s))ds, \ t \in [-\tau, 0],$$

which, with $t = -\tau$, becomes

$$y(0) = T(\tau)y(-\tau) - \int_0^{-\tau} T(-s)f(y(s))ds.$$

Since $y(0) = w(\tau)$ and $y(-\tau) = w(0)$, it follows that

$$w(\tau) = T(\tau)w(0) - \int_0^{-\tau} T(-s)f(w(s+\tau))ds. \tag{2.7}$$

On the other hand, from (2.6), we have

$$T(-t)w(t+\tau) = w(\tau) + \int_0^t T(-s)f(w(s+\tau))ds.$$

Letting $-t = \tau - \theta$, we obtain

$$T(\tau-\theta)w(\theta) = w(\tau) + \int_0^{\theta-\tau} T(-s)f(w(s+\tau))ds.$$

Using (2.7), we obtain

$$\begin{aligned} T(\tau-\theta)w(\theta) &= T(\tau)w(0) - \int_0^{-\tau} T(-s)f(w(s+\tau))ds \\ &+ \int_0^{\theta-\tau} T(-s)f(w(s+\tau))ds \\ &= T(\tau)w(0) - \int_{\theta-\tau}^{-\tau} T(-s)f(w(s+\tau))ds. \end{aligned}$$

Thus

$$T(\tau-\theta)[w(\theta)-T(\theta)w(0)+\int_{\theta-\tau}^{-\tau}T(\theta-\tau-s)f(w(s+\tau))ds]=0.$$

Changing variables to $r=s+\tau$, we have

$$T(\tau-\theta)[w(\theta)-T(\theta)w(0)-\int_{0}^{\theta}T(\theta-r)f(w(r))dr]=0.$$

Thus, by (BU), it follows that

$$w(\theta)-T(\theta)w(0)-\int_{0}^{\theta}T(\theta-r)f(w(r))dr=0.$$

Therefore, w satisfies (1.3). □

Definition 2.2. *A solution of (1.3) in an interval $[-\tau,0]$, $\tau>0$ is a function $y:[-\tau,0]\to X$ such that w, given by (2.5), is a solution of (1.3) in $[0,\tau]$.*

In the definition below, $B(\varphi,\varepsilon)$ denotes the ball of radius ε and center at φ.

Definition 2.3. *A subset $K\subset X$ is said to be locally positively invariant under the flow of (1.3) if there exists $\varepsilon>0$ such that, for any $\varphi\in K\cap B(\varphi,\varepsilon)$:*

(i) *for sufficiently small $t>0$ a solution $w(t)$, of (1.3), exists with $w(0)=\varphi$;*

(ii) *if for $\tau>0$, $w(t)$ exists and belongs to $B(\varphi,\varepsilon)$ for all $t\in[0,\tau]$ then $w(t)\in K$, for all $t\in[0,\tau]$.*

Negatively invariant subsets are defined by substituting $<$ for $>$ and $t\in[0,\tau]$ for $t\in[\tau,0]$ in (i) and (ii).

Definition 2.4. *Suppose that a Banach space Y is decomposed as $Y=\pi_1Y\oplus\pi_2Y$ for continuous linear projection operators π_1 and π_2. Then, a subset $S\subset Y$ of Y containing y_0 is said to be tangent to π_2Y at y_0 if*

$$\frac{\|\pi_1(y-y_0)\|}{\|\pi_2(y-y_0)\|}\to 0,\ \text{as } y\to y_0 \text{ in } S.$$

The following results has been proved in [2].

Theorem 2.5. *Assume the hypotheses (H1) , (H2), (H3) and (H4) hold. Let $\varepsilon>0$ be such that $\min(a_-,a_+)>\varepsilon$. Then, for $\delta>0$ sufficiently small, there exist locally invariant sets*

$$S=\{\varphi\in B(0,\delta)\ :\ \|\varphi_-\|<\frac{\delta}{2K},\ \varphi_0+\varphi_+=p_\lambda(\varphi_-)\},$$

and

$$U = \{\varphi \in B(0,\delta) \ : \ \|\varphi_+\| < \frac{\delta}{2K},\ \varphi_- + \varphi_0 = q_\lambda(\varphi_+)\},$$

termed the ***stable and unstable manifold*** *respectively, where* p_λ, q_λ *are Lipschitz function defined for* $\|\varphi_-\| < \frac{\delta}{2K}$, $\|\varphi_+\| < \frac{\delta}{2K}$,*respectively. If* $\varphi \in S$ *then a unique solution* $w(t)$ *of (1.3) with* $w(0) = \varphi$ *exists for* $t \geq 0$ *and*

$$\|w(t)\| \leq 2Ke^{-(a_- - \varepsilon)t}\|w_-(0)\|,\ t \geq 0.$$

If the hypothesis (BU) holds and $\varphi \in U$ *then an unique solution* $w(t)$ *of (1.3) with* $w(0) = \varphi$ *exists for* $t \leq 0$ *and*

$$\|w(t)\| \leq 2Ke^{(a_+ - \varepsilon)t}\|w_+(0)\|,\ t \leq 0.$$

Furthermore, S *is tangent at zero to* $\pi_- X$*,* U *is tangent at zero to* $\pi_+ X$ *and* $(p_\lambda, w_+^{p_\lambda})$ *is the unique solution of the system*

$$\begin{aligned} w_-(t) &= T(t)\varphi_- + \textstyle\int_0^t T(t-s)\pi_- f_\lambda(w_-(s) + p_\lambda(w_-(s)))ds, \\ p_\lambda(\varphi_-) &= \textstyle\int_\infty^0 T(-s)(\pi_0 + \pi_+) f_\lambda(w_-(s) + p_\lambda(w_-(s)))ds,\ t \geq 0 \end{aligned} \tag{2.8}$$

and $(q_\lambda, w_+^{q_\lambda})$ *is the unique solution of the system*

$$q_\lambda(\varphi_+) = \textstyle\int_{-\infty}^0 T(-s)(\pi_- + \pi_0) f_\lambda(w_+(s) + q_\lambda(w_+(s)))ds,$$

$$w_+(t) = T(t)\varphi_+ + \textstyle\int_0^t T(t-s)\pi_+ f_\lambda(w_+(s) + q_\lambda(w_+(s)))ds,\ t \leq 0,$$

The functions p_λ, q_λ *have Lipschitz constants smaller or equal to 1,* $p_\lambda(0) = 0$, $q_\lambda(0) = 0$ *and* K *is the constant given in the hypothesis (2.c).*

In what follows, we denote by $B(0,\delta)$, $B_{(\pi_-\oplus\pi_0)}(0,\delta)$, $B_{(\pi_0\oplus\pi_+)}(0,\delta)$, the ball of radius δ and center in the origin of X, $\pi_- X \oplus \pi_0 X$, $\pi_0 X \oplus \pi_+ X$, respectively.

Theorem 2.6. *Assume the hypotheses (H1) , (H2), (H3) and (H4) hold. Then, there exists* $\delta > 0$ *and sets*

$$W^{*s} = \{\varphi \in X \ : \ \|\varphi_- + \varphi_0\| < \delta,\ \varphi_+ = p^*(\varphi_- + \varphi_0)\},$$
$$W^{*u} = \{\varphi \in X \ : \ \|\varphi_0 + \varphi_+\| < \delta,\ \varphi_- = q^*(\varphi_0 + \varphi_+)\},$$

termed the ***center-stable*** *and* ***center-unstable*** *manifolds of (1.1), respectively, where* p^* *and* q^* *are Lipschitz functions defined on* $B_{(\pi_-\oplus\pi_0)}(0,\delta) \subset \pi_- X \oplus \pi_0 X$, $B_{(\pi_0\oplus\pi_+)}(0,\delta) \subset \pi_0 X \oplus \pi_+ X$*, respectively; The set* W^{*s} *is locally positively invariant under the flow (1.3), while if (BU) holds* W^{*u} *is locally negatively invariant. Any solution of (1.3) which* *exists and remains in $B(0,\delta)$ fort ≥ 0 lies on W^{*s}, and any solution of (1.3) which exists and remains in $B(0,\delta)$ for $t \leq 0$ lies on W^{*u}. Furthermore, the tangent space to W^{*s} at zero is $\pi_- X \oplus \pi_0 X$ and the tangent space to W^{*u} at zero is $\pi_0 X \oplus \pi_+ X$.*

Proof This theorem has also been proved in [2], where W^{*s} for (1.4) is shown to exist, for λ small enough to imply

$$\max\left\{\frac{2K^2\nu(\lambda)}{a_+ - a_0 - 4K\nu(\lambda)}, \frac{2(K\nu(\lambda))^2}{a_0(a_+ - a_0 - 4K\nu(\lambda))} + \frac{K\nu(\lambda)}{a_+}\right\} < 1,$$

and p^*_λ, for $t \geq 0$, is the unique solution of the system below

$$w_-(t) = T(t)\varphi_- + \int_0^t T(t-s)\pi_- f_\lambda(w_-(s) + w_0(s) + p^*_\lambda(w_-(s) + w_0(s)))ds,$$

$$w_0(t) = T(t)\varphi_0 + \int_0^t T(t-s)\pi_0 f_\lambda(w_-(s) + w_0(s) + p^*_\lambda(w_-(s) + w_0(s)))ds,$$

$$p^*_\lambda(\varphi_- + \varphi_0) = \int_\infty^0 T(-s)\pi_+ f_\lambda(w_-(s) + w_0(s) + p^*_\lambda(w_-(s) + w_0(s)))ds,$$

where $w_-(t) = \pi_- w(t)$, $w_0(t) = \pi_0 w(t)$ e $w(t)$ satisfies (1.4). Furthermore, p^*_λ has Lipschitz constant smaller or equal to 1 and $p^*_\lambda(0) = 0$.

Similarly, W^{*u} can be shown to exist as a graph of q^*_λ, the unique solution of the system

$$q^*_\lambda(\varphi_0 + \varphi_+) = \int_{-\infty}^0 T(-s)\pi_- f_\lambda(q^*_\lambda(w_0(s) + w_+(s)) + w_0(s) + w_+(s))ds,$$

$$w_0(t) = T(t)\varphi_0 + \int_0^t T(t-s)\pi_0 f_\lambda(q^*_\lambda(w_0(s) + w_+(s)) + w_0(s) + w_+(s))ds,$$

$$w_+(t) = T(t)\varphi_+ + \int_0^t T(t-s)\pi_+ f_\lambda(q^*_\lambda(w_0(s) + w_+(s)) + w_0(s) + w_+(s))ds,$$

for $t \leq 0$, where $w_0(t) = \pi_0 w(t)$, $w_+(t) = \pi_+ w(t)$ and $w(t)$ satisfies (1.4).

3. Continuity of the invariant manifolds

We prove in this section that the manifolds of the previous section are continuous with respect to the parameter ε.

3.1. Continuity of the stable (unstable) manifolds.

We now state one of the main results of this section.

Theorem 3.1. *(Continuity of the stable (unstable) manifolds). Suppose that the functions f_λ, in (1.4), also depend on a parameter $\varepsilon \in \Lambda$, where Λ is an open set in a Banach space, and $f_\lambda = f_\lambda^\varepsilon$ satisfies the estimates*

$$\|f_\lambda^\varepsilon(u) - f_\lambda^{\varepsilon_0}(u)\| \le C_1(\varepsilon)\|u\|,\ C_1(\varepsilon) \to 0,\ \varepsilon \to \varepsilon_0; \tag{3.9}$$

$$\|f_\lambda^\varepsilon(u) - f_\lambda^\varepsilon(v)\| \le \nu(\lambda)\|u - v\|,\ for\ each\ \varepsilon \in \Lambda, \tag{3.10}$$

where $\nu(\cdot)$ is a non decreasing continuous function with $\nu(0) = 0$. Then the stable (unstable) manifold S^ε, (U^ε), is continuous with respect to the parameter ε at ε_0.

Proof We prove the continuity of S^ε; the proof for U^ε is analogous. By Theorem 2.5, S^ε is the graph of a Lipschitz function $p_\lambda = p_\lambda^\varepsilon$, where $(p_\lambda, w_-^{p_\lambda})$ is the unique solution of (2.8). From (2.8) and (3.10), we have

$$\begin{aligned}
&\|w_-(t,\varepsilon)\| \\
&\le Ke^{-a_-t}\|\varphi_-\| + \int_0^t Ke^{-a_-(t-s)}\nu(\lambda)\|w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon))\|ds \\
&\le Ke^{-a_-t}\|\varphi_-\| + \int_0^t Ke^{-a_-(t-s)}\nu(\lambda)[\|w_-(s,\varepsilon)\| + \|p_\lambda^\varepsilon(w_-(s,\varepsilon))\|]ds \\
&\le Ke^{-a_-t}\|\varphi_-\| + \int_0^t 2K\nu(\lambda)e^{-a_-(t-s)}\|w_-(s,\varepsilon)\|ds.
\end{aligned}$$

By Gronwall's Lemma, we obtain

$$\|w_-(t,\varepsilon)\| \le K\|\varphi_-\|e^{-(a_- - 2K\nu(\lambda))t},\ t \ge 0. \tag{3.11}$$

We will use the metric ρ given by

$$\rho(h_1, h_2) = \sup_{\substack{\varphi \in X \\ \varphi_- \neq 0}} \frac{\|h_1(\varphi_-) - h_2(\varphi_-)\|}{\|\varphi_-\|},$$

equipped with which, the set

$$G = \{h : \pi_-X \to \pi_0X \oplus \pi_+X,\ \|h(\varphi_-) - h(\psi_+)\| \le \|\varphi_- - \psi_+\|,$$

$\forall\, \varphi,\, \psi \in X,\ h(0) = 0\}$ becomes a complete metric space.

Let $\theta(t) = \|w_-(t,\varepsilon) - w_-(t,\varepsilon_0)\|$, $t \geq 0$. Then

$$\begin{aligned}\theta(t) \leq \int_0^t & \|T(t-s)\pi_-\{f_\lambda^\varepsilon[w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon))] \\ & \qquad - f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\}\|ds \\ & \leq \int_0^t Ke^{-(t-s)a_-}\|f_\lambda^\varepsilon[w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon))] \\ & \qquad - f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\|ds \\ & \leq \int_0^t Ke^{-(t-s)a_-}\|f_\lambda^\varepsilon[w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon))] \\ & \qquad - f_\lambda^{\varepsilon}[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\|ds \\ & + \int_0^t Ke^{-(t-s)a_-}\|f_\lambda^\varepsilon[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))] \\ & \qquad - f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\|ds.\end{aligned}$$

From (3.9) and (3.10), it follows that

$$\begin{aligned}\theta(t) \leq \int_0^t & K\nu(\lambda)e^{-(t-s)a_-}\Big[\|w_-(s,\varepsilon) - w_-(s,\varepsilon_0)\| + \\ & \qquad \|p_\lambda^\varepsilon(w_-(s,\varepsilon)) - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|\Big]ds \\ & + \int_0^t Ke^{-(t-s)a_-}C_1(\varepsilon)\|w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|ds \\ = \int_0^t & K\nu(\lambda)e^{-(t-s)a_-}\Big[\theta(s) + \|p_\lambda^\varepsilon(w_-(s,\varepsilon)) - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|\Big]ds \\ & + \int_0^t Ke^{-(t-s)a_-}C_1(\varepsilon)\|w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|ds.\end{aligned}$$

Using that $p_\lambda^{\varepsilon_0}$ is Lipschitz with Lipschitz constant ≤ 1 and $p_\lambda^{\varepsilon_0}(0) = 0$, we obtain

$$\begin{aligned}\theta(t) &\leq \int_0^t K\nu(\lambda)e^{-(t-s)a_-}\Big[\theta(s) + \|p_\lambda^\varepsilon(w_-(s,\varepsilon)) - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|\Big]ds \\ &+ \int_0^t 2Ke^{-(t-s)a_-}C_1(\varepsilon)\|w_-(s,\varepsilon_0)\|ds.\end{aligned}$$

Now, using the same argument for $p^{\varepsilon}_{\lambda}$, we have

$$\begin{aligned}
\|p^{\varepsilon}_{\lambda}(w_-(s,\varepsilon)) - p^{\varepsilon_0}_{\lambda} & (w_-(s,\varepsilon_0))\| \\
& \leq \|p^{\varepsilon}_{\lambda}(w_-(s,\varepsilon)) - p^{\varepsilon}_{\lambda}(w_-(s,\varepsilon_0))\| \\
& + \|p^{\varepsilon}_{\lambda}(w_-(s,\varepsilon_0)) - p^{\varepsilon_0}_{\lambda}(w_-(s,\varepsilon_0))\| \\
& \leq \|w_-(s,\varepsilon) - w_-(s,\varepsilon_0)\| \\
& + \|p^{\varepsilon}_{\lambda}(w_-(s,\varepsilon_0)) - p^{\varepsilon_0}_{\lambda}(w_-(s,\varepsilon_0))\| \\
& = \theta(s) + \|p^{\varepsilon}_{\lambda}(w_-(s,\varepsilon_0)) - p^{\varepsilon_0}_{\lambda}(w_-(s,\varepsilon_0))\| \\
& = \theta(s) + \|w_-(s,\varepsilon_0)\| \frac{\|p^{\varepsilon}_{\lambda}(w_-(s,\varepsilon_0)) - p^{\varepsilon_0}_{\lambda}(w_-(s,\varepsilon_0))\|}{\|w_-(s,\varepsilon_0)\|} \\
& \leq \theta(s) + \|w_-(s,\varepsilon_0)\| \rho\left(p^{\varepsilon}_{\lambda}, p^{\varepsilon_0}_{\lambda}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\theta(t) & \leq \int_0^t K\nu(\lambda) e^{-(t-s)a_-}[\theta(s) + \theta(s) + \|w_-(s,\varepsilon_0)\| \rho\left(p^{\varepsilon}_{\lambda}, p^{\varepsilon_0}_{\lambda}\right)] ds \\
& + \int_0^t 2KC_1(\varepsilon) e^{-(t-s)a_-} \|w_-(s,\varepsilon_0)\| ds \\
& = \int_0^t 2K\nu(\lambda) e^{-(t-s)a_-} \theta(s) ds \\
& + \int_0^t K\nu(\lambda) e^{-(t-s)a_-} \|w_-(s,\varepsilon_0)\| \rho\left(p^{\varepsilon}_{\lambda}, p^{\varepsilon_0}_{\lambda}\right) ds \\
& + \int_0^t 2KC_1(\varepsilon) e^{-(t-s)a_-} \|w_-(s,\varepsilon_0)\| ds.
\end{aligned}$$

Using (3.11), we obtain

$$\begin{aligned}
\theta(t) \leq & \int_0^t 2K\nu(\lambda)e^{-(t-s)a_-}\theta(s)ds \\
&+ \int_0^t K\nu(\lambda)e^{-(t-s)a_-}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})K\|\varphi_-\|e^{-(a_- - 2K\nu(\lambda))s}ds \\
&+ \int_0^t 2KC_1(\varepsilon)e^{-(t-s)a_-}K\|\varphi_-\|e^{-(a_- - 2K\nu(\lambda))s}ds \\
&= \int_0^t 2K\nu(\lambda)e^{-(t-s)a_-}\theta(s)ds \\
&+ K^2\|\varphi_-\|\nu(\lambda)\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{-a_- t}\int_0^t e^{2K\nu(\lambda)s}ds \\
&+ 2K^2\|\varphi_-\|C_1(\varepsilon)e^{-a_- t}\int_0^t e^{2K\nu(\lambda)s}ds.
\end{aligned}$$

Thus

$$\begin{aligned}
e^{a_- t}\theta(t) &\leq \int_0^t 2K\nu(\lambda)e^{a_- s}\theta(s)ds + K^2\|\varphi_-\|\nu(\lambda)\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})\int_0^t e^{2K\nu(\lambda)s}ds \\
&+ 2K^2\|\varphi_-\|C_1(\varepsilon)\int_0^t e^{2K\nu(\lambda)s}ds \\
&\leq \int_0^t 2K\nu(\lambda)e^{a_- s}\theta(s)ds + \frac{K\|\varphi_-\|\nu(\lambda)}{2\nu(\lambda)}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{2K\nu(\lambda)t} \\
&+ \frac{K\|\varphi_-\|}{\nu(\lambda)}C_1(\varepsilon)e^{2K\nu(\lambda)t}.
\end{aligned}$$

From the generalized Gronwall's Lemma, (see [12]), it follows that

$$e^{a_- t}\theta(t) \leq e^{2K\nu(\lambda)t}\left[\frac{K\|\varphi_-\|}{2}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{2K\nu(\lambda)t} + \frac{K\|\varphi_-\|C_1(\varepsilon)}{\nu(\lambda)}e^{2K\nu(\lambda)t}\right].$$

Hence

$$\theta(t) \leq \frac{K\|\varphi_-\|}{2}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{-(a_- - 4K\nu(\lambda))t} + \frac{K\|\varphi_-\|C_1(\varepsilon)}{\nu(\lambda)}e^{-(a_- - 4K\nu(\lambda))t}. \tag{3.12}$$

Now

$$\begin{aligned}\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_-)\| &\leq \int_0^\infty \|T(-s)\pi_0[f_\lambda^\varepsilon(w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon)))\\ &- f_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)))]\|ds\\ &+ \int_0^\infty \|T(-s)\pi_+[f_\lambda^\varepsilon(w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon)))\\ &- f_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)))]\|ds\\ &\leq \int_0^\infty Ke^{a_0 s}\|f_\lambda^\varepsilon[w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon))]\\ &- f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\|ds\\ &+ \int_0^\infty Ke^{-a_+ s}\|f_\lambda^\varepsilon[w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon))]\\ &- f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\|ds.\end{aligned}$$

Using that, for $s > 0$, $-a_+ s < a_0 s$, it follows that

$$\begin{aligned}\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_+)\| &\leq \int_0^\infty 2Ke^{a_0 s}\|f_\lambda^\varepsilon[w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon))]\\ &- f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\|ds.\end{aligned}$$

Subtracting and summing the term

$$f_\lambda^\varepsilon[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))],$$

we obtain

$$\begin{aligned}\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_+)\| &\leq \int_0^\infty 2Ke^{a_0 s}\|f_\lambda^\varepsilon[w_-(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon))]\\ &- f_\lambda^\varepsilon[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\|ds\\ &+ \int_0^\infty 2Ke^{a_0 s}\|f_\lambda^\varepsilon[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\\ &- f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))]\|ds.\end{aligned}$$

Using (3.9) and (3.10), it follows that

$$\begin{aligned}
&\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_-)\| \\
&\le \int_0^\infty 2K\nu(\lambda)e^{a_0 s}\Big\{\|w_-(s,\varepsilon) - w_-(s,\varepsilon_0)\| \\
&\quad + \|p_\lambda^\varepsilon(w_-(s,\varepsilon)) - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|\Big\}ds \\
&\quad + \int_0^\infty 2Ke^{a_0 s}C_1(\varepsilon)\|w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|ds \\
&= \int_0^\infty 2K\nu(\lambda)e^{a_0 s}\Big\{\theta(s) + \|p_\lambda^\varepsilon(w_-(s,\varepsilon)) - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|\Big\}ds \\
&\quad + \int_0^\infty 2Ke^{a_0 s}C_1(\varepsilon)\|w_-(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|ds \\
&\le \int_0^\infty 2K\nu(\lambda)e^{a_0 s}\Big\{\theta(s) + \|p_\lambda^\varepsilon(w_-(s,\varepsilon)) - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\|\Big\}ds \\
&\quad + \int_0^\infty 4Ke^{a_0 s}C_1(\varepsilon)\|w_-(s,\varepsilon_0)\|ds.
\end{aligned}$$

Using once again that

$$\|p_\lambda^\varepsilon(w_-(s,\varepsilon)) - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0))\| \le \theta(s) + \|w_-(s,\varepsilon_0)\|\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}),$$

we obtain

$$\begin{aligned}
\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_-)\| &\le \int_0^\infty 2K\nu(\lambda)e^{a_0 s}\Big[2\theta(s) + \|w_-(s,\varepsilon_0)\|\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})\Big]ds \\
&+ \int_0^\infty 4KC_1(\varepsilon)e^{a_0 s}\|w_-(s,\varepsilon_0)\|ds.
\end{aligned}$$

Now, using (3.11), it follows that

$$\begin{aligned}
\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_-)\| &\le \int_0^\infty 4K\nu(\lambda)e^{a_0 s}\theta(s)ds \\
&+ \int_0^\infty 2K^2\nu(\lambda)\|\varphi_-\|\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{-(a_- - a_0 - 2K\nu(\lambda))s}ds \\
&+ \int_0^\infty 4K^2\|\varphi_-\|C_1(\varepsilon)e^{-(a_- - a_0 - 2K\nu(\lambda))s}ds.
\end{aligned}$$

Thus,

$$\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_-)\| \le I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^\infty 4K\nu(\lambda)e^{a_0 s}\theta(s)ds,$$

$$I_2 = \int_0^\infty 2K^2\nu(\lambda)\|\varphi_-\|\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{-(a_- - a_0 - 2K\nu(\lambda))s}ds$$

and

$$I_3 = \int_0^\infty 4K^2\|\varphi_-\|C_1(\varepsilon)e^{-(a_- - a_0 - 2K\nu(\lambda))s}ds.$$

Using the estimate obtained for $\theta(t)$ in (3.12), we obtain

$$\begin{aligned}
I_1 \leq \int_0^\infty 4K\nu(\lambda)e^{a_0 s}\Big[&\frac{K}{2}\|\varphi_-\|\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{-(a_- - 4K\nu(\lambda))s} \\
&+ \frac{K\|\varphi_-\|}{\nu(\lambda)}C_1(\varepsilon)e^{-(a_- - 4K\nu(\lambda))s}\Big]ds \\
= \int_0^\infty & 2K^2\|\varphi_-\|\nu(\lambda)\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{-(a_- - a_0 - 4K\nu(\lambda))s}ds \\
&+ \int_0^\infty 4K^2\|\varphi_-\|C_1(\varepsilon)e^{-(a_- - a_0 - 4K\nu(\lambda))s}ds \\
&= \frac{2K^2\nu(\lambda)\|\varphi_-\|}{a_- - a_0 - 4K\nu(\lambda)}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) + \frac{4K^2\|\varphi_-\|}{a_- - a_0 - 4K\nu(\lambda)}C_1(\varepsilon).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_2 &= \int_0^\infty 2K^2\nu(\lambda)\|\varphi_-\|\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{-(a_- - a_0 - 2K\nu(\lambda))s}ds \\
&= \frac{2K^2\nu(\lambda)\|\varphi_-\|}{a_- - a_0 - 2K\nu(\lambda)}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_0^\infty 4K^2\|\varphi_-\|C_1(\varepsilon)e^{-(a_- - a_0 - 2K\nu(\lambda))s}ds \\
&= \frac{4K^2\|\varphi_-\|}{a_- - a_0 - 2K\nu(\lambda)}C_1(\varepsilon).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_-)\| \\
&\leq \frac{2K^2\nu(\lambda)\|\varphi_-\|}{a_- - a_0 - 4K\nu(\lambda)}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) + \frac{4K^2\|\varphi_-\|}{a_- - a_0 - 4K\nu(\lambda)}C_1(\varepsilon) \\
&+ \frac{2K^2\nu(\lambda)\|\varphi_-\|}{a_- - a_0 - 2K\nu(\lambda)}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) + \frac{4K^2\|\varphi_-\|}{a_- - a_0 - 2K\nu(\lambda)}C_1(\varepsilon) \\
&= \left[\frac{2K^2\nu(\lambda)\|\varphi_-\|}{a_- - a_0 - 4K\nu(\lambda)} + \frac{2K^2\nu(\lambda)\|\varphi_-\|}{a_- - a_0 - 2K\nu(\lambda)}\right]\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) \\
&+ \left[\frac{4K^2\|\varphi_-\|}{a_- - a_0 - 4K\nu(\lambda)} + \frac{4K^2\|\varphi_-\|}{a_- - a_0 - 2K\nu(\lambda)}\right]C_1(\varepsilon).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_-)\|}{\|\varphi_-\|} \\
&\leq \left[\frac{2K^2\nu(\lambda)}{a_- - a_0 - 4K\nu(\lambda)} + \frac{2K^2\nu(\lambda)}{a_- - a_0 - 2K\nu(\lambda)}\right]\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) \\
&\qquad + \left[\frac{4K^2}{a_- - a_0 - 4K\nu(\lambda)} + \frac{4K^2}{a_- - a_0 - 2K\nu(\lambda)}\right]C_1(\varepsilon),
\end{aligned}$$

which implies

$$\begin{aligned}
&\sup_{\substack{\varphi\in X\\ \varphi_-\neq 0}} \frac{\|p_\lambda^\varepsilon(\varphi_-) - p_\lambda^{\varepsilon_0}(\varphi_-)\|}{\|\varphi_-\|} \\
&\leq \left[\frac{2K^2\nu(\lambda)}{a_- - a_0 - 4K\nu(\lambda)} + \frac{2K^2\nu(\lambda)}{a_- - a_0 - 2K\nu(\lambda)}\right]\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) \\
&\qquad + \left[\frac{4K^2}{a_- - a_0 - 4K\nu(\lambda)} + \frac{4K^2}{a_- - a_0 - 2K\nu(\lambda)}\right]C_1(\varepsilon).
\end{aligned}$$

Therefore

$$\begin{aligned}
\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) &\leq \left[\frac{2K^2\nu(\lambda)}{a_- - a_0 - 4K\nu(\lambda)} + \frac{2K^2\nu(\lambda)}{a_- - a_0 - 2K\nu(\lambda)}\right]\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) \\
&+ \left[\frac{4K^2}{a_- - a_0 - 4K\nu(\lambda)} + \frac{4K^2}{a_- - a_0 - 2K\nu(\lambda)}\right]C_1(\varepsilon).
\end{aligned}$$

Choosing λ sufficiently small, we have

$$\left[\frac{2K^2\nu(\lambda)}{a_- - a_0 - 4K\nu(\lambda)} + \frac{2K^2\nu(\lambda)}{a_- - a_0 - 2K\nu(\lambda)}\right] < \frac{1}{2}.$$

Thus

$$\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) < \frac{1}{2}\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) + C_2(\varepsilon),$$

where $C_2(\varepsilon) = \left[\frac{4K^2}{a_- - a_0 - 4K\nu(\lambda)} + \frac{4K^2}{a_- - a_0 - 2K\nu(\lambda)}\right] C_1(\varepsilon)$.

Therefore

$$\rho(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) < 2C_2(\varepsilon),$$

where $C_2(\varepsilon) \to 0$, as $\varepsilon \to \varepsilon_0$, concluding the proof. □

3.2. **Continuity of the center-stable (center-unstable) manifolds.** In this section we prove the continuity of the center-stable and center-unstable manifolds.

Theorem 3.2. *Assume the same hypotheses of Theorem 3.1. Then the center stable, center unstable manifolds, W_ε^{*s}, W_ε^{*u} are continuous at ε_0.*

Proof We prove the continuity of W_ε^{*s}. The continuity of W_ε^{*u} follows in a similar way. By Theorem 2.6, W_ε^{*s} is the graph of a Lipschitz function $p_\lambda^{*\varepsilon}$, where $p_\lambda^{*\varepsilon}$ satisfies

$$w_-(t,\varepsilon) = T(t)\varphi_- + \int_0^t T(t-s)\pi_- f_\lambda^\varepsilon(w_-(s,\varepsilon) + w_0(s,\varepsilon) \\ + p_\lambda^{*\varepsilon}(w_-(s,\varepsilon) + w_0(s,\varepsilon)))ds,$$

$$w_0(t,\varepsilon) = T(t)\varphi_0 + \int_0^t T(t-s)\pi_0 f_\lambda^\varepsilon(w_-(s,\varepsilon) + w_0(s,\varepsilon) \\ + p_\lambda^{*\varepsilon}(w_-(s,\varepsilon) + w_0(s,\varepsilon)))ds,$$

$$p_\lambda^{*\varepsilon}(\varphi_- + \varphi_0) = \int_\infty^0 T(-s)\pi_+ f_\lambda^\varepsilon(w_-(s,\varepsilon) + w_0(s,\varepsilon) \\ + p_\lambda^{*\varepsilon}(w_-(s,\varepsilon) + w_0(s,\varepsilon)))ds.$$

Furthermore, p_λ^* has Lipschitz constant equal or smaller than 1 and $p_\lambda^*(0) = 0$.

To facilitate the notation, from now on, in this subsection, we write p_λ^ε in the place of $p_\lambda^{*\varepsilon}$.

Using (3.10), we have

$$\begin{aligned}\|w_-(s,\varepsilon)+w_0(s,\varepsilon)\| \leq\, & Ke^{-a_-t}\|\varphi_-\| + \int_0^t K\nu(\lambda)e^{-(t-s)a_-}\|w_-(s,\varepsilon) \\ & + w_0(s,\varepsilon) + p_\lambda^\varepsilon(w_-(s,\varepsilon)+w_0(s,\varepsilon))\|ds \\ & + Ke^{a_0t}\|\varphi_0\| + \int_0^t K\nu(\lambda)e^{(t-s)a_0}\|w_-(s,\varepsilon)+w_0(s,\varepsilon) \\ & + p_\lambda^\varepsilon(w_-(s,\varepsilon)+w_0(s,\varepsilon))\|ds \\ \leq\, & Ke^{-a_-t}\|\varphi_-\| + \int_0^t 2K\nu(\lambda)e^{-(t-s)a_-}\|w_-(s,\varepsilon) \\ & + w_0(s,\varepsilon)\|ds \\ & + Ke^{a_0t}\|\varphi_0\| + \int_0^t 2K\nu(\lambda)e^{(t-s)a_0}\|w_-(s,\varepsilon) \\ & + w_0(s,\varepsilon)\|ds.\end{aligned}$$

Since $-a_- < a_0$, it follows that

$$\begin{aligned}&\|w_-(s,\varepsilon)+w_0(s,\varepsilon)\| \\ &\leq Ke^{a_0t}\|\varphi_0\| + \int_0^t 4K\nu(\lambda)e^{(t-s)a_0}\|w_-(s,\varepsilon)+w_0(s,\varepsilon)\|ds.\end{aligned}$$

By Gronwall's Lemma, we obtain

$$\|w_-(s,\varepsilon)+w_0(s,\varepsilon)\| \leq K\|\varphi_-+\varphi_0\|e^{(4K\nu(\lambda)+a_0)t}. \tag{3.13}$$

Let ρ^* be the metric given by

$$\rho^*(h_1,h_2) = \sup_{\varphi\in X,\ \varphi_-+\varphi_0\neq 0} \frac{\|h_1(\varphi_-+\varphi_0) - h_2(\varphi_-+\varphi_0)\|}{\|\varphi_-+\varphi_0\|},$$

equipped with which, the set

$$\begin{aligned}G^* =\{& h:\pi_-X\oplus\pi_0X\to\pi_+X\oplus\pi_0X, \|h(\varphi_-+\varphi_0)-h(\psi_++\psi_0)\| \\ & \leq \|\varphi_-+\varphi_0-\psi_-+\psi_0\|,\ \forall\,\varphi,\,\psi\in X,\ h(0)=0\}\end{aligned}$$

becomes a complete metric space. Let $\theta^*(t) = \|w_-(t,\varepsilon) + w_0(t,\varepsilon) - w_-(t,\varepsilon_0) - w_0(t,\varepsilon_0)|$, $t \geq 0$. Then

$$\begin{aligned} \theta^*(t) &\leq \int_0^t Ke^{-(t-s)a_-}\|f_\lambda^\varepsilon[w_-(s,\varepsilon)+w_0(s,\varepsilon)+p_\lambda^\varepsilon(w_-(s,\varepsilon)+w_0(s,\varepsilon))] \\ &- f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0)+p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0))]\|ds \\ &+ \int_0^t Ke^{(t-s)a_0}\|f_\lambda^\varepsilon[w_-(s,\varepsilon)+w_0(s,\varepsilon)+p_\lambda^\varepsilon(w_-(s,\varepsilon)+w_0(s,\varepsilon))] \\ &- f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0)+p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0))]\|ds. \end{aligned}$$

Using that $-(t-s)a_- \leq (t-s)a_0$, we obtain

$$\begin{aligned} \theta^*(t) &\leq \int_0^t 2Ke^{(t-s)a_0}\|f_\lambda^\varepsilon[w_-(s,\varepsilon)+w_0(s,\varepsilon)+p_\lambda^\varepsilon(w_-(s,\varepsilon)+w_0(s,\varepsilon))] \\ &- f_\lambda^{\varepsilon_0}[w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0)+p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0))]\|ds. \end{aligned}$$

Summing and subtracting the term

$$f_\lambda^\varepsilon[w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0)+p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0))],$$

and using (3.9), (3.10), we have

$$\begin{aligned} \theta^*(t) &\leq \int_0^t 2Ke^{(t-s)a_0}\nu(\lambda)\big[\theta^*(s)+\|p_\lambda^\varepsilon(w_-(s,\varepsilon)+w_0(s,\varepsilon)) \\ &- p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0))\|\big]ds \\ &+ \int_0^t 2Ke^{(t-s)a_0}C_1(\varepsilon)\|w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0)\|ds. \end{aligned}$$

Now, using that p_λ^ε is Lipschitzian with Lipschitz constant smaller than 1 and $p_\lambda^{\varepsilon_0}(0)=0$, it follows that

$$\begin{aligned} &\|p_\lambda^\varepsilon(w_-(s,\varepsilon)+w_0(s,\varepsilon))-p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0))\| \\ &\leq \|w_-(s,\varepsilon)+w_0(s,\varepsilon)-w_-(s,\varepsilon_0)-w_0(s,\varepsilon_0)\|+\|w_-(s,\varepsilon_0) \\ &- w_0(s,\varepsilon_0)\|\frac{\|p_\lambda^\varepsilon(w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0))-p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0))\|}{\|w_-(s,\varepsilon_0)-w_0(s,\varepsilon_0)\|} \\ &\leq \theta^*(s)+\rho^*(p_\lambda^\varepsilon,p_\lambda^{\varepsilon_0})\|w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0)\|. \end{aligned}$$

Then

$$\begin{aligned} \theta^*(t) &\leq \int_0^t 2Ke^{(t-s)a_0}\nu(\lambda)\big[2\theta^*(s)+\|w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0)\|\rho^*(p_\lambda^\varepsilon,p_\lambda^{\varepsilon_0})\big]ds \\ &+ \int_0^t 2KC_1(\varepsilon)e^{(t-s)a_0}\|w_-(s,\varepsilon_0)+w_0(s,\varepsilon_0)\|ds. \end{aligned}$$

Using (3.13), we obtain

$$\begin{aligned}\theta^*(t) &\leq \int_0^t 4K\nu(\lambda)e^{(t-s)a_0}\theta^*(s)ds \\ &+ \int_0^t 2Ke^{(t-s)a_0}\nu(\lambda)K\|\varphi_- + \varphi_0\|e^{(4K\nu(\lambda)+a_0)s}ds\rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})ds \\ &+ \int_0^t 2KC_1(\varepsilon)e^{(t-s)a_0}K\|\varphi_- + \varphi_0\|e^{(4K\nu(\lambda)+a_0)s}ds.\end{aligned}$$

Thus

$$\begin{aligned}e^{-a_0t}\theta^*(t) &\leq \int_0^t 4K\nu(\lambda)e^{-a_0s}\theta^*(s)ds \\ &+ 2K^2\nu(\lambda)\|\varphi_- + \varphi_0\|\rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})\int_0^t e^{4K\nu(\lambda)s}ds \\ &+ 2K^2C_1(\varepsilon)\|\varphi_- + \varphi_0\|\int_0^t e^{4K\nu(\lambda)s}ds.\end{aligned}$$

Since

$$\int_0^t e^{4K\nu(\lambda)s}ds \leq \frac{e^{4K\nu(\lambda)t}}{4K\nu(\lambda)},$$

we obtain

$$\begin{aligned}e^{-a_0t}\theta^*(t) &\leq \int_0^t 4K\nu(\lambda)e^{-a_0s}\theta^*(s)ds + \frac{K}{2}\|\varphi_- + \varphi_0\|\rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})e^{4K\nu(\lambda)t} \\ &+ \frac{K}{2\nu(\lambda)}C_1(\varepsilon)\|\varphi_- + \varphi_0\|e^{4K\nu(\lambda)t}.\end{aligned}$$

From Gronwall's Lemma, it follows that

$$\theta^*(t) \leq \left[\frac{K}{2}\rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) + \frac{K}{2\nu(\lambda)}C_1(\varepsilon)\right]e^{(a_0+8K\nu(\lambda))t}\|\varphi_- + \varphi_0\|. \qquad (3.14)$$

Now

$$\begin{aligned}\|p_\lambda^\varepsilon(\varphi_- + \varphi_0) - p_\lambda^{\varepsilon_0}(\varphi_- + \varphi_0)\| &\leq \int_0^\infty Ke^{-a_+s}\|f_\lambda^\varepsilon(w_-(s,\varepsilon) + w_0(s,\varepsilon) \\ &+ p_\lambda^\varepsilon(w_-(s,\varepsilon) + w_0(s,\varepsilon))) \\ &- f_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0) \\ &+ p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0)))\|ds.\end{aligned}$$

Using (3.9) and (3.10), after adding and subtracting the term

$$f_\lambda^\varepsilon(w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0) + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0))),$$

we have

$$\begin{aligned}\|p_\lambda^\varepsilon(\varphi_- + \varphi_0) - p_\lambda^{\varepsilon_0}(\varphi_- + \varphi_0)\| \leq & \int_0^\infty K\nu(\lambda)e^{-a_+ s}\theta^*(s)ds \\ & + \int_0^\infty K\nu(\lambda)e^{-a_+ s}\|p_\lambda^\varepsilon(w_-(s,\varepsilon) + w_0(s,\varepsilon)) \\ & - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0))\|ds \\ & + \int_0^\infty Ke^{-a_+ s}C_1(\varepsilon)\|w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0) \\ & + p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0))\|ds.\end{aligned}$$

Using that

$$\begin{aligned}&\|p_\lambda^\varepsilon(w_-(s,\varepsilon) + w_0(s,\varepsilon)) - p_\lambda^{\varepsilon_0}(w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0))\| \\ &\qquad \leq \theta^*(s) + \rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})\|w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0)\|,\end{aligned}$$

we obtain

$$\begin{aligned}&\|p_\lambda^\varepsilon(\varphi_- + \varphi_0) - p_\lambda^{\varepsilon_0}(\varphi_- + \varphi_0)\| \\ &\leq \int_0^\infty 2K\nu(\lambda)e^{-a_+ s}\theta^*(s)ds + \int_0^\infty K\nu(\lambda)e^{-a_+ s}\rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0})\|w_-(s,\varepsilon_0) \\ &+ w_0(s,\varepsilon_0)\|ds + \int_0^\infty 2KC_1(\varepsilon)e^{-a_+ s}\|w_-(s,\varepsilon_0) + w_0(s,\varepsilon_0)\|ds.\end{aligned}$$

Using (3.13) and (3.14), it follows that

$$
\begin{aligned}
&\|p_\lambda^\varepsilon(\varphi_- + \varphi_0) - p_\lambda^{\varepsilon_0}(\varphi_- + \varphi_0)\| \\
&\leq \left(K^2\nu(\lambda)\|\varphi_- + \varphi_0\| \int_0^\infty e^{-(a_+ - a_0 - 8K\nu(\lambda))s} ds\right) \rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) \\
&+ \left(K^2\nu(\lambda)\|\varphi_- + \varphi_0\| \int_0^\infty e^{-(a_+ - a_0 - 4K\nu(\lambda))s} ds\right) \rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) \\
&+ \left(K^2\|\varphi_- + \varphi_0\| \int_0^\infty e^{-(a_+ - a_0 - 8K\nu(\lambda))s} ds\right) C_1(\varepsilon) \\
&+ \left(2K\|\varphi_- + \varphi_0\| \int_0^\infty e^{-(a_+ - a_0 - 4K\nu(\lambda))s} ds\right) C_1(\varepsilon) \\
&= \left[\frac{K^2\nu(\lambda)}{a_+ - a_0 - 8K\nu(\lambda)}\|\varphi_- + \varphi_0\| \right. \\
&+ \left. \frac{K^2\nu(\lambda)}{a_+ - a_0 - 4K\nu(\lambda)}\|\varphi_- + \varphi_0\|\right] \rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) \\
&+ \left[\frac{K^2}{a_+ - a_0 - 8K\nu(\lambda)}\|\varphi_- + \varphi_0\| + \frac{2K}{a_+ - a_0 - 4K\nu(\lambda)}\|\varphi_- + \varphi_0\|\right] C_1(\varepsilon).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\|p_\lambda^\varepsilon(\varphi_- + \varphi_0) - p_\lambda^{\varepsilon_0}(\varphi_- + \varphi_0)\|}{\|\varphi_- + \varphi_0\|} \\
&\qquad \leq \left[\frac{K^2\nu(\lambda)}{a_+ - a_0 - 8K\nu(\lambda)} + \frac{K^2\nu(\lambda)}{a_+ - a_0 - 4K\nu(\lambda)}\right] \rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) \\
&\qquad\qquad + \left[\frac{K^2}{a_+ - a_0 - 8K\nu(\lambda)} + \frac{2K}{a_+ - a_0 - 4K\nu(\lambda)}\right] C_1(\varepsilon).
\end{aligned}
$$

Choosing λ small enough as to have

$$
\left[\frac{K^2\nu(\lambda)}{a_+ - a_0 - 8K\nu(\lambda)} + \frac{K^2\nu(\lambda)}{a_+ - a_0 - 4K\nu(\lambda)}\right] < \frac{1}{2}
$$

and letting

$$
C_2^*(\varepsilon) = \left[\frac{K^2}{a_+ - a_0 - 8K\nu(\lambda)} + \frac{2K}{a_+ - a_0 - 4K\nu(\lambda)}\right] C_1(\varepsilon),
$$

we obtain

$$
\rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) < \frac{1}{2}\rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) + C_2^*(\varepsilon).
$$

Therefore

$$
\rho^*(p_\lambda^\varepsilon, p_\lambda^{\varepsilon_0}) < 2C_2^*(\varepsilon),
$$

where $C_2^*(\varepsilon) \to 0$, as $\varepsilon \to \varepsilon_0$, concluding the proof. □

Remark 3.3. *It follows from* [15] *that the center manifold,* W_ε^c *is given by*

$$W_\varepsilon^c = W_\varepsilon^{*s} \cap W_\varepsilon^{*u}.$$

Therefore, the the continuity of the center manifold, W_ε^c *also follows from Theorem 3.2 .*

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