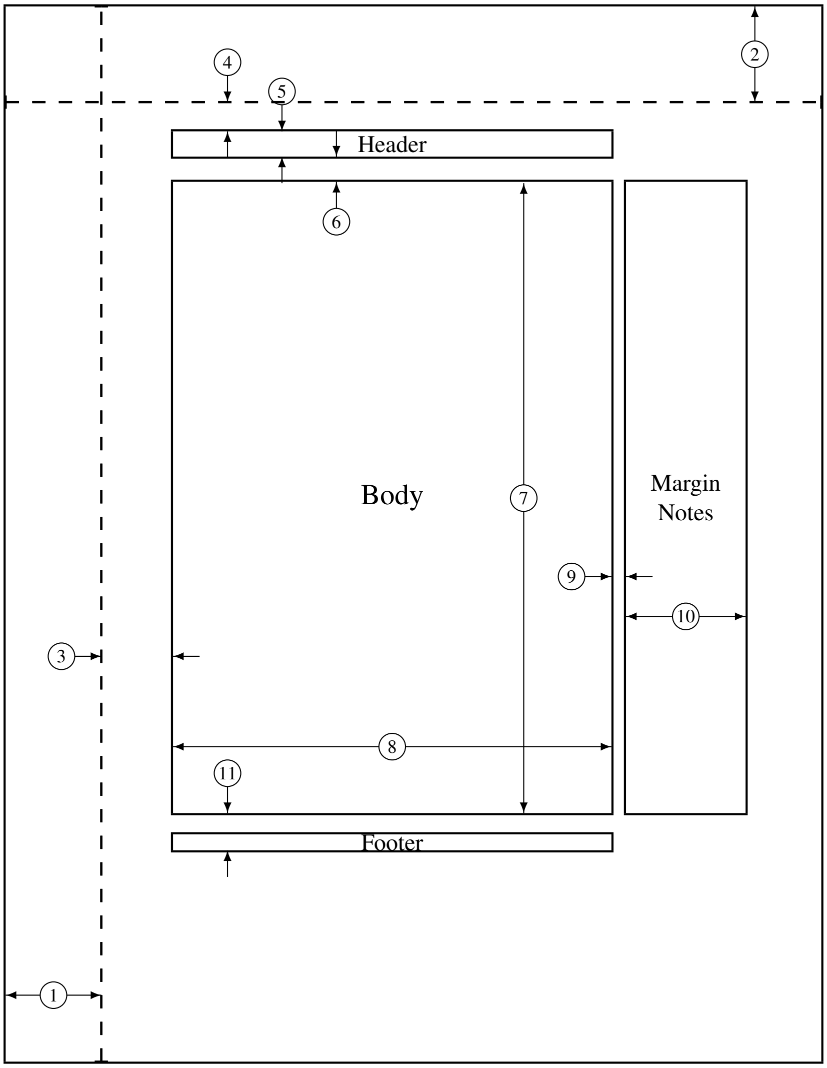

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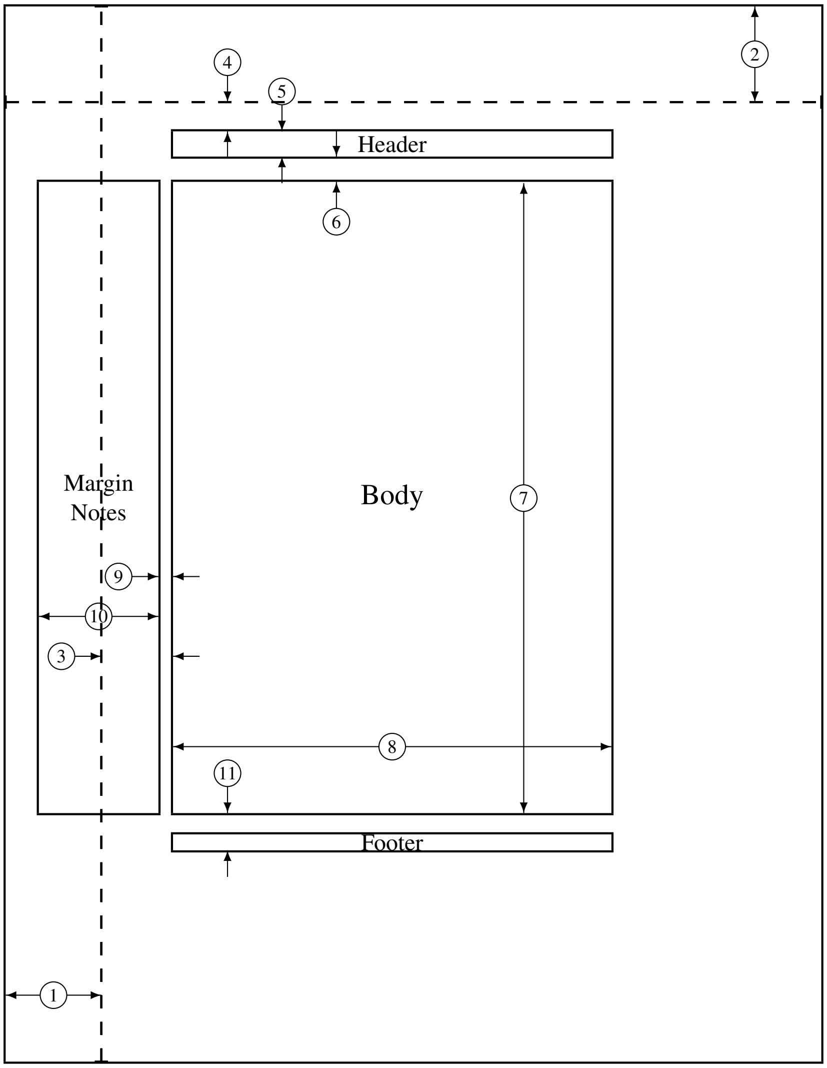

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2-Calabi-Yau tilted algebras

Idun Reiten

Introduction

These notes follow closely the series of lectures I gave at the workshop of ICRA 13 in Sao Paulo in July/August 2008, and hence the style is quite informal. The material is centered around the work in [BIRSm] on the connection between mutation of cluster-tilting objects started in [BMRRT] and mutation of quivers with potentials investigated in [DWZ1]. It belongs to the ongoing work on representation theory inspired by the theory of cluster algebras initiated by Fomin-Zelevinsky [FZ].

There is on one hand the mutation of quivers, which is an essential ingredient in the definition of cluster algebras. Then there are two mutations related to two classes of algebras, which both play an important role in the theory: the 2-Calabi-Yau-tilted algebras (2-CY-tilted for short) and the Jacobian algebras. They are defined in completely different ways, but are nevertheless closely related, as the finite dimensional Jacobian algebras are known to be 2-CY-tilted [A1][K3].

We start with discussing the two classes of algebras in section 1. Then we explain the different kinds of mutation in section 2. In section 3 we give examples of triangulated 2-CY categories with cluster-tilting objects, which are the basis for the definition of 2-CY-tilted algebras. Then we discuss some basic properties of 2-CY-tilted algebras in section 4. In section 5 we give the main result on the connection between the two mutations, namely that they "commute." As applications we give classes of examples of 2-CY-tilted algebras which are Jacobian, and we show that certain maps are well defined.

1. The relevant classes of algebras

In this section we discuss the two classes of algebras which play an important role in these notes.

1.1. 2-CY-tilted algebras.

(a) We first need to recall the definitions of 2-CY triangulated categories and their cluster-tilting objects. Let throughout K be an algebraically closed field, and $\mathscr{C}$ a triangulated $K-$category which is Hom-finite, that is, the homomorphism spaces are finite dimensional. Then $\mathscr{C}$ is said to be 2-CY if we have a functorial isomorphism $D\mathrm{Ext}^1_{\mathscr{C}}(A,B) \simeq \mathrm{Ext}^1_{\mathscr{C}}(B,A)$, where $D = \mathrm{Hom}_K(\ ,K)$ is the duality functor.

(b) Important objects in 2-CY triangulated categories are the *cluster-tilting* objects. An object T in $\mathscr{C}$ is cluster-tilting if
- $\mathrm{Ext}^1_{\mathscr{C}}(T,T) = 0$
- $\mathrm{Ext}^1_{\mathscr{C}}(T,X) = 0 \Rightarrow X \in \mathrm{add}T$,

where add$T$ denotes the summands of finite direct sums of copies of T.

The algebras $\Gamma = \mathrm{End}_{\mathscr{C}}(T)$ for a cluster-tilting object T in a Hom-finite triangulated 2-CY category $\mathscr{C}$ are by definition the 2-*CY-tilted algebras*.

When the 2-CY category $\mathscr{C}$ is a cluster category, that is, a certain orbit category of the bounded derived category of a hereditary algebra (see section 3), the corresponding algebras have been called *cluster-tilted* algebras. Their theory was initiated in [BMR2].

1.2. Jacobian algebras.

We start with an example of Jacobian algebras, which were introduced and investigated in [DWZ1].

Example 1.2.1. *Let Q be the quiver*

$$\begin{tikzcd} 4 \arrow[d, "d"'] & 3 \arrow[l, "c"'] \\ 1 \arrow[r, "a"'] & 2 \arrow[u, "b"'] \end{tikzcd}$$

and $W = abcd$ a ***potential****. We take the* ***cyclic derivative*** *∂_a of W with respect to the arrows in this cycle. This is given by $\partial_a W = \partial_a(abcd) = bcd, \partial_b(bcda) = cda$, $\partial_c(cdab) = dab$ and $\partial_d(dabc) = abc$. Then $\mathcal{P}(Q,W) = KQ/\langle \partial_x W; x \in Q_1\rangle = KQ/\langle bcd, cda, dab, abc\rangle$ is the* ***Jacobian algebra*** *associated with the* ***quiver with potential*** (Q,W).

More generally, one starts with a finite quiver with no loops, and denotes by Q_i the paths of length i. Let $KQ = \prod_{i\geq 0} KQ_i$ and $W \in \prod_{i\geq 2} KQ_{i,cyc}$, where $Q_{i,cyc}$ consists of cyclic paths of length i. Then $K\hat{Q}/\overline{\langle \partial_a W; a \in Q_1 \rangle}$ is the associated Jacobian algebra, where $\overline{I}$ denotes the closure of the ideal I (see [DWZ1]).

2. **Mutations**

We discuss mutations of quivers, of cluster-tilting objects in triangulated 2-CY-categories and of quivers with potential. The first mutation was an essential ingredient in the definition of cluster algebras in [FZ]. The second one was developed in [BMRRT] in the context of cluster categories, and was essential for the categorical modelling of acyclic cluster algebras. The mutation of quivers with potentials in [DWZ1] has recently been applied to cluster algebras [DWZ2].

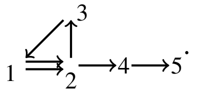
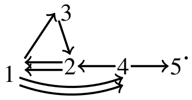
2.1. **Mutation of quivers.** Let Q be a finite quiver with vertices $1, 2, \ldots, n$, and with no loops or 2-cycles. Then for each vertex k in Q there is defined a new quiver $\mu_k(Q)$ as follows.

- Reverse all arrows entering and leaving the vertex k.
- If there are a arrows from i to k and b arrows from k to j, and c arrows from i to j and d arrows from j to i (where $cd = 0$), then there are $ab + c - d$ arrows from i to j in $\mu_k(Q)$ if $ab + c - d \geq 0$, and otherwise $d - c - ab$ arrows from j to i.

We illustrate with two examples

Example 2.1.1. *Let Q be the quiver* $1 \longrightarrow 2 \longrightarrow 3$. *Then $\mu_3(Q)$ is the quiver* $1 \longrightarrow 2 \longleftarrow 3$.

In this case, and more generally, when we mutate at a sink or a source of the quiver, then mutation coincides with Bernstein-Gelfand-Ponomarev reflection.

Example 2.1.2. *Let Q be the quiver*  *Then $\mu_2(Q)$ is the quiver* 

It is easy to see that $\mu_2(\mu_2(Q)) = Q$, which is a general fact.

2.2. **Mutation of cluster-tilting objects.** Let $\mathscr{C}$ be a Hom-finite triangulated 2-CY-category, and $T = T_1 \oplus \ldots \oplus T_n$ a cluster-tilting object in $\mathscr{C}$, where the T_i are indecomposable and $T_i \not\simeq T_j$ for $i \neq j$. Then we have the following.

Theorem 2.1. **(a)** *For each $k = 1, \ldots, n$, there is a unique indecomposable object T_k^* with $T_k^* \not\simeq T_k$, such that $T^* = T/T_k \oplus T_k^*$ is a cluster-tilting object.*

(b) *There are triangles $T_k \xrightarrow{f'} B'_k \xrightarrow{g'} T_k^* \to$ and $T_k \xrightarrow{f} B_k \xrightarrow{g} T_k \to$ in $\mathscr{C}$, where g, g' are minimal right $add(T/T_k)$−approximations and f, f' are minimal left $add(T/T_k)$−
approximations.*

This was first proved in [BMRRT] for cluster categories, and then in [GLS1] in the context of preprojective algebras of Dynkin type. In the general case it is proved in [IY]. The extra information needed was a new approach to part (a).

This mutation procedure gives rise to a *cluster tilting graph* as follows. The vertices correspond to cluster-tilting objects up to isomorphism, and there is an edge between two objects related by one mutation. Then each object has exactly n neighbours. It is not known in general if this graph is connected when $\mathscr{C}$ is connected.

Assume now that the 2-CY triangulated category $\mathscr{C}$ has *no loops or* 2-*cycles*, in the sense that the associated 2-CY-tilted algebras have no loops or 2-cycles in their quiver. Consider the following diagram.

$$\begin{tikzcd}
T \arrow[r, "\mu_k"] \arrow[d] & T^* = \mu_k(T) \arrow[d] \\
\mathrm{End}(T) \arrow[r, dashed] \arrow[d] & \mathrm{End}(T^*) \arrow[d] \\
Q_T \arrow[r, "\mu_k"] & \overset{Q_{T^*}}{\mu_k(Q_T)}
\end{tikzcd}$$

where Q_T and Q_{T^*} denote the quivers of $\mathrm{End}(T)$ and $\mathrm{End}(T^*)$ respectively.

Then we have the following connection between the two mutations

Theorem 2.2. *Let $\mathscr{C}$ be 2-CY triangulated with no loops or 2-cycles, and let $T = T_1 \oplus \ldots \oplus T_n$ be a cluster-tilting object in $\mathscr{C}$. Then for $k = 1, \ldots, n$, we have $\mu_k(Q_T) = Q_{\mu_k(T)}$.*

This was first proved in [BMR3] in the context of cluster categories, and it was an important step in the categorification of acylic cluster algebras. It was shown in [GLS1] in the case of preprojective algebras of Dynkin type, and in the general case in [BIRSc]. Actually, in the terminology of [BIRSc], a 2-CY triangulated category with no loops or 2-cycles has a *cluster structure*, which means that Theorems 2.1 and 2.2 hold.

A natural question is whether $\mathrm{End}(T^*)$ is uniquely determined by $\mathrm{End}(T)$. We will return to this problem in the last section.

2.3. **Mutation of quivers with potentials.** Assume now that Q is a finite quiver with no loops, and vertices $1, \ldots, n$. Assume also that the vertex k does not lie on a 2-cycle, but otherwise the quiver Q may have 2-cycles. Let W be a potential on Q.

The definition of quivers with potentials (Q, W) in [DWZ1] goes in two steps:

$$(Q, W) \longmapsto \tilde{\mu}_k(Q, W) = (\tilde{Q}, \tilde{W}) \longmapsto \mu_k(Q, W) = (\bar{Q}, \bar{W}).$$

We illustrate the definitions on examples, and refer to [DWZ1] for the general definition.

Example 2.3.1. *Let Q be the quiver* $\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \arrow[ll, bend left, "c"] \end{tikzcd}$ *and $W = abc$ a potential. We consider first $\tilde{\mu}_2(Q, W) = (\tilde{Q}, \tilde{W})$. Then $\tilde{Q}$ is the quiver* $\begin{tikzcd} 1 \arrow[r, bend right, "{[ab]}"'] & 2^* \arrow[l, "a^*"'] & 3 \arrow[l, "b^*"'] \arrow[ll, bend left, "c"] \end{tikzcd}$ *and $\tilde{W} = [ab]c + [ab]b^*a^*$. We have here replaced the vertex 2 by another vertex 2^*, and we have reversed the direction of the arrows involving the vertex 2. We have further replaced the path ab from 1 to 3, through 2, by a new arrow $[ab]$ from 1 to 3. In the new potential $\tilde{W}$ we have replaced ab by $[ab]$ and added a new term $[ab]b^*a^*$.*

Since $\tilde{W}$ has a term $[ab]c$ of length 2, the potential of $\tilde{W}$ is by definition ***not reduced.*** *Then we can get rid of the 2-cycle $[ab]c$ in the next step, and we end up with $\mu_2(Q, W) = (\bar{Q}, \bar{W})$, where $\bar{Q}$ is* $1 \xleftarrow{a^*} 2^* \xleftarrow{b^*} 3$ *and $\bar{W} = 0$.*

Example 2.3.2. *Let Q be as in Example 2.3.1 and $W = abcabc$. Then $\tilde{Q}$ is also as in Example 2.3.1, and $\tilde{W} = [ab]c[ab]c + [ab]b^*a^*$. In this case there are no terms of length two, so the potential $\tilde{W}$ is reduced, and hence $(\bar{Q}, \bar{W}) = (\tilde{Q}, \tilde{W})$.*

Assume now that Q is a finite quiver, with vertices $1, \ldots, n$ and with no loops or 2-cycles. Then we have the following diagram, similar to the one discussed in part (2.2).

$$\begin{tikzcd}
(Q, W) \arrow[r, "\mu_k"] \arrow[d] & (\bar{Q}, \bar{W}) \arrow[d] \\
\mathcal{P}(Q, W) \arrow[r, dashed] \arrow[d] & \mathcal{P}(\bar{Q}, \bar{W}) \arrow[d, "g"] \\
Q \arrow[r, "\mu_k"] & \mu_k(Q) = \bar{Q}'
\end{tikzcd}$$

Here g is defined by dropping all 2-cycles in the quiver $\bar{Q}$ to get the quiver $\bar{Q}'$. Then we have $\mu_k(Q) = \bar{Q}'$ by the definition of mutation of quivers with potentials.

Similar as before we have the following question: Is $\mathcal{P}(\bar{Q}, \bar{W})$ uniquely determined by $\mathcal{P}(Q, W)$? We will get back to this question in the last section.

3. **Classes of 2-CY categories**

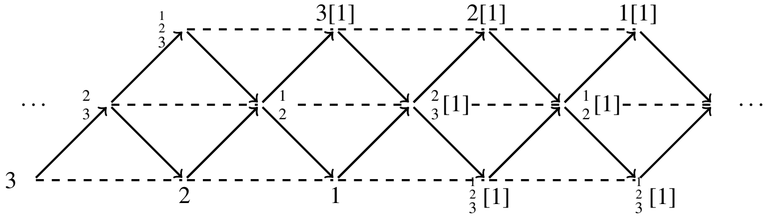
In this section we discuss some main sources of examples of Hom-finite triangulated 2-CY categories, starting with the cluster categories (see [BMRRT]).

3.1. **Cluster categories.**

3.1.1. *Definition:* Let Q be a finite quiver without oriented cycles, and let KQ be the associated path algebra over the field K. Let $D^b(KQ)$ be the bounded derived category of the finitely generated KQ-modules. Then the cluster category $\mathcal{C}_Q$ is by definition the orbit category $\mathcal{C}_Q = D^b(KQ)/\tau^{-1}[1]$. Here τ is the AR-translation functor on $D^b(KQ)$, which restricts to the AR-translation in mod KQ for the indecomposable nonprojective KQ-modules.

The category $\mathcal{C}_Q$ is Hom-finite and triangulated [K1], and we have the functorial isomorphism $D\mathrm{Ext}^1_{\mathcal{C}_Q}(A, B) \simeq \mathrm{Ext}^1_{\mathcal{C}_Q}(B, A)$, which says that $\mathcal{C}_Q$ is 2-CY.

3.1.2. *Example:* We illustrate with the following example.
Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. Then the AR-quiver of $D^b(KQ)$ is as follows



Here the indecomposable KQ-modules, together with the objects $3[1]$, $\begin{smallmatrix}2\\3\end{smallmatrix}[1]$ and $\begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$ (the shifts of the indecomposable projectives) form a set of representatives of the indecomposable objects in $\mathscr{C}_Q$, namely a so-called *fundamental domain.* In $\mathscr{C}_Q$ the objects $3, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ are identified with the objects $2[1], \begin{smallmatrix}1\\2\end{smallmatrix}[1], \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$ respectively.

In comparison with KQ the main differences are the following

(i) There are additional indecomposable objects: $3[1], \begin{smallmatrix}2\\3\end{smallmatrix}[1], \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$.
(ii) There are additional maps: $\mathrm{Hom}_{KQ}(1,3) = 0$, whereas $\mathrm{Hom}_{\mathscr{C}_Q}(1,3) = \mathrm{Hom}_{KQ}(1,3) \oplus \mathrm{Hom}_{D^b(KQ)}(1, \tau^{-1}(3[1])) \neq 0$.

3.1.3. *Motivation:* The introduction of the cluster categories was motivated by the Fomin-Zelevinsky theory of cluster algebras [FZ], via [MRZ] (see also [CCS] for another approach to the A_n case). The development of the theory of cluster categories was an attempt to "categorify" some of the essential notions and results appearing for cluster algebras, in particular for the *acyclic* cluster algebras. These are the ones associated with finite quivers with no oriented cycles.

One question connected with the categorification of acyclic cluster algebras was to find an appropriate category with a class of objects being the analogs of the n-element subsets called clusters, where n is the number of vertices of the associated quiver. The tilting KQ-modules seemed to be natural candidates, but the problem was that if $T = T_1 \oplus \ldots \oplus T_n$ is a tilting module, where T_i are the indecomposable modules and $T_i \not\simeq T_j$ for $i \neq j$, then there may not be some $T_k^* \not\simeq T_k$ such that $T/T_k \oplus T_k^*$ is a tilting module (see [HU1]). For example for KQ with Q being the quiver $1 \longrightarrow 2 \longrightarrow 3$, then $T = 3 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 1$ is a tilting module. Then there is no indecomposable module

$T_2 \not\simeq \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ such that $3 \oplus T_2^* \oplus 1$ is a tilting module. However, in $\mathscr{C}_Q$ we have the candidate $T_2^* = \begin{smallmatrix}2\\3\end{smallmatrix}[1]$, provided we have an appropriate definition.

3.1.4. *The relevant objects:* What turned out to be the appropriate objects to consider as analogs of clusters in $\mathscr{C}_Q$ are the union of the tilting KQ'-modules, viewed as objects in $\mathscr{C}_Q$, for all path algebras KQ' derived equivalent to KQ. An equivalent description is given by the cluster-tilting objects defined in section 1, which has the advantage that it can be formulated in a more general setting. With the same definition, these were called Ext-configurations in [BMRRT]. Another equivalent description in the context of cluster categories is T being *maximal rigid,* used in [GLS1], that is, $\operatorname{Ext}^1(T,T) = 0$ and T is maximal with this property. The concept of cluster-tilting also coincides with Iyama's definition of maximal 1-orthogonal, which was defined in a more general context.

About the cluster-tilting graph, it is known to be connected in the case of cluster categories of a connected quiver ([BMRRT], using [HU2]). With the same definition one can more generally consider cluster categories $\mathscr{C}_{\mathscr{H}}$ of a hereditary abelian category $\mathscr{H}$ with tilting objects. Then there are additional situations (tubular case) where the cluster-tilting graph is connected [BKL].

Since the cluster categories have no loops or 2-cycles [BMRRT][BMR3], they have a cluster structure by the definition given in section 2.

3.2. **Preprojective algebras of Dynkin type.** To any finite quiver Q without oriented cycles we have associated a preprojective algebra $\prod(Q)$, whose quiver $\bar{Q}$ is obtained by adding an arrow $i \overset{a^*}{\to} j$ to the quiver Q whenever there is an arrow $j \overset{a}{\to} i$. Then by definition, $\prod(Q) = K\bar{Q}/(\sum\limits_{a \in Q_1} aa^* - a^*a)$. Let Λ be the completion of $\prod(Q)$ with respect to the Jacobson radical. If Q is Dynkin, then $\Lambda \simeq \prod(Q)$ since $\prod(Q)$ is then a finite dimensional algebra.

For example, if Q is the quiver $\xrightarrow{a}\xrightarrow{b}$, then $\bar{Q}$ is the quiver $\underset{a^*}{\overset{a}{\rightleftarrows}}\underset{b^*}{\overset{b}{\rightleftarrows}}$, and $\Lambda = KQ/\langle bb^* - a^*a, aa^*, b^*b\rangle$.

For Q Dynkin, the stable category $\underline{\operatorname{mod}\Lambda}$ is known to be a Hom-finite triangulated 2-CY-category. There are similar results here, formulated in the setting of the module category $\operatorname{mod}\Lambda$ in [GLS1]. In particular, $\underline{\operatorname{mod}\Lambda}$ has a cluster structure, and the cluster tilting and the maximal rigid objects coincide.

3.3. **Examples coming from Coxeter groups.** We discuss examples of 2-CY categories coming from reduced expressions in Coxeter groups (see [BIRSc] [GLS1] [GLS2]), and we start with some basic definitions.

3.3.1. *Reduced expressions:* Let Q be a finite quiver with vertices $1, \dots, n$ and no oriented cycles. Denote by W_Q the associated Coxeter group. It has generators $s_1, \dots, s_n$ which are in bijection with the vertices of the quiver, and the relations are given by $s_i^2 = 1$, $s_i s_j = s_j s_i$ if there is no arrow between i and j, and $s_i s_j s_i = s_j s_i s_j$ if there is exactly one arrow between i and j.

Let $w \in W_Q$. Then $w = s_{i_i} \dots s_{i_t}$ is a *reduced* expression (or reduced word) if t is smallest possible. In that case $t = l(w)$ is the *length* of w.

3.3.2. *Associated* 2*-CY-category:* Let the notation be as above, and let $w = s_{i_1}$
$\dots s_{i_t}$ be a reduced expression. For each $j = 1, \dots, n$, consider the ideal $I_j = \Lambda(1 - e_j)\Lambda$ in the associated completion of the projective algebra, where e_i denotes the trivial path at the the vertex i. Define $I_{\{i_1,\dots,i_t\}} = I_{i_1} \dots I_{i_t}$. This is an ideal which does not depend on the reduced expression of w, and hence we denote it by I_w. Then $\Lambda_w = \Lambda / I_w$ is a finite dimensional K-algebra, and it is known to be Gorenstein of dimension at most 1. Denote by $\mathrm{Sub}\Lambda_w$ the subcategory of finitely generated Λ_w-modules which are contained in a finite direct sum of copies of Λ_w. Then the stable category $\underline{\mathrm{Sub}\Lambda_w}$ is Hom-finite triangulated 2-CY.

3.3.3. *Cluster-tilting objects:* Denote by P_i the indecomposable projective Λ-module associated with the vertex i. For each reduced expression $w = s_{i_1} \dots s_{i_t}$ we define $T_{\{i_1,\dots,i_t\}} = P_{i_1}/I_{i_1}P_{i_1} \oplus P_{i_2}/I_{i_1}I_{i_2}P_{i_2} \oplus \dots P_{i_t}/I_{i_1} \dots I_{i_t}P_{i_t}$, which is a cluster-tilting object in $\underline{\mathrm{Sub}\Lambda_w}$. Note that we may get many non-isomorphic cluster-tilting objects for different reduced expressions. We call them all *standard* cluster-tilting objects.

Also note that this class of triangulated 2-CY categories contains the cluster categories $\mathscr{C}_Q$ and the stable categories $\mathrm{mod}\Lambda$, where Λ is the preprojective algebra of a Dynkin quiver, as special cases (see [BIRSc][GLS2][A1]).

3.4. **Generalized cluster categories.** Let A be a finite dimensional K-algebra of global dimension at most 2. Associated with A is the orbit category $D^b(A)/\tau^{-1}[1]$ formed the same way as when A is the path algebra KQ, which has global dimension at most 1. This category is not necessarily triangulated. Then in [A1] the *generalized cluster category* of A is defined to be the triangulated hull $\mathscr{C}_Q$ of the orbit category. It is a triangulated category which is

2-CY when it is Hom-finite [A1]. Further, the algebra A is a cluster-tilting object in $\mathscr{C}_Q$ in this case. This clearly generalizes the class of cluster categories, but it even turns out to contain all the 2-CY triangulated categories $\underline{\mathrm{Sub}\Lambda_w}$. associated with reduced words [A1][ART].

3.5. **2-CY categories associated with Jacobian algebras.** For any finite dimensional Jacobian K-algebra there is constructed in [A1][K2] a triangulated 2-CY-category $\mathscr{C}$ with a cluster-tilting object T, where $\mathrm{End}_{\mathscr{C}}(T)$ is isomorphic to the given Jacobian algebra.

In particular, any finite dimensional Jacobian algebra is 2-CY-tilted.

3.6. **Cohen-Macaulay modules.** There are also interesting examples of 2-CY triangulated categories arising from stable categories of maximal Cohen-Macaulay modules over Gorenstein rings, in particular for 1-dimensional hypersurfaces. In this case loops and 2-cycles occur frequently (see [BIKR]).

3.7. **2-CY categories as subfactors.** There is a general method for constructing new 2-CY triangulated categories from given ones [IY] (see also [BIRSc]).

We state a special case of this procedure, and refer to [IY, 4.9] for a more general statement.

Let $\mathscr{C}$ he a Hom-finite triangulated 2C-category and D an object in $\mathscr{C}$ with $\mathrm{Ext}^1(D, D) = 0$. Then $\mathcal{B} = {}^{\perp}D[1] = \{X \in \mathscr{C} : \mathrm{Hom}(X, D[1]) = 0\}$ is a functorially finite extension closed subcategory of $\mathscr{C}$, and the factor category $\mathcal{B}/D$ is a Hom-finite triangulated 2-CY category.

Further, there is a $1 - 1$ correspondence between cluster-tilting objects in $\mathscr{C}$ having D as a summand, and cluster-tilting objects in $\mathcal{B}/D$. It is given by $T \mapsto T/D$.

4. Properties of 2-CY-tilted algebras

In this section we discuss some properties which hold for endomorphism algebras of cluster-tilting objects in arbitrary Hom-finite triangulated 2-CY-categories, and some additional properties of cluster-tilted algebras.

4.1. **Connection between 2-CY-categories and 2-CY-tilted algebras.** The following result shows that the category of finitely generated modules over a

2-CY-tilted algebra can be obtained directly from the triangulated 2-CY category it came from. This was first proved for cluster categories in [BMR2], and in the general case in [KR1].

Theorem 4.1. *Let T be the cluster-tilting object in a Hom-finite triangulated* 2*-CY catergory $\mathscr{C}$. Then we have an equivalence of categories*

$$\mathscr{C}/add(\tau T)\xrightarrow{Hom_{\mathscr{C}}(T,\ )}modEnd_{\mathscr{C}}(T).$$

It is however not known in general if the 2-CY-tilted algebras $\mathrm{End}_{\mathscr{C}}(T)$ determine the category $\mathscr{C}$ back again. If the quiver Q of $\mathrm{End}_{\mathscr{C}}(T)$ has no oriented cycles, then it is known that it comes from the cluster category $\mathscr{C}_Q$ [KR2].

4.2. **Homological properties.** We list some properties of a homological nature which are taken from [KR1].

Theorem 4.2. *Let Γ be a* 2*-CY-tilted algebra.*

- **(a)** *Then Γ is Gorenstein of dimension at most* 1 *(that is, the injective dimension of Γ is at most* 1*).*
- **(b)** $gl.dim.\Gamma \leq 1$ *or* $gl.dim.\Gamma = \infty$.

Theorem 4.3. *The triangulated category $\underline{Sub\Gamma}$ is* 3*-CY.*

Here a Hom-finite triangulated K-category is said to be n-CY if we have a functorial isomorphism $D\mathrm{Hom}(A,B) \simeq \mathrm{Hom}(B,A[n])$. In particular, since any finite dimensional Jacobian algebra is 2-CY-tilted (see 3.5), the same result holds for finite dimensional Jacobian algebras. There is also a short direct proof of this fact for Jacobian algebras [BIRSm].

4.3. **Cluster-titled algebras.** We point out two important properties of cluster-tilted algebras.

Theorem 4.4. **(a)** *We obtain all the quivers of cluster-tilted algebras Γ by starting with the quiver of a tilted algebra Λ and adding an arrow from i to j for each relation r from j to i in a minimal set of relations for the tilted algebra.*

- **(b)** *Actually, we have* $\Gamma \simeq \Lambda \ltimes Ext^2(D\Lambda,\Lambda)$.

This was proved in [ABS], with part (1) first proved for finite type in [BR], and also in some cases in [BRS].

The second property was proved in [BMR1] for finite type, and in [BIRSm] in the general case.

Theorem 4.5. *The cluster-tilted algebras are uniquely determined by their quiver.*

Note that the corresponding result is not true for tilted algebras since the algebras given by the quivers with relations $\longrightarrow\longrightarrow$ and $\overset{\text{- - - -}}{\longrightarrow\longrightarrow}$ are both tilted algebras.

5. Some 2-CY-tilted algebras which are Jacobian

An interesting problem is whether any 2-CY-tilted algebra is Jacobian. In particluar, it would be nice to know if this is the case for all the examples of triangulated 2-CY categories which are discussed in section 3. This is also not known in general. But we shall give some such examples in this section, and show that we get a much larger class by applying the results discussed in the next section.

5.1. 2-CY-tilted algebras associated with reduced expressions.

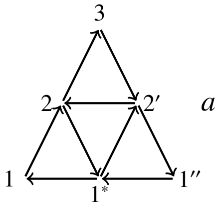
We have the following information for some of the special cluster-tilting objects which we have called standard [BIRSc][BIRSm]. Note that these all lie in the same component of the cluster-tilting graph [BIRSc].

Theorem 5.1. *If T is a standard cluster-tilting object in $\underline{Sub\Lambda_w}$, where w is a word in the Coxeter group, then End(T) is Jacobian, and we have an explicit description of the quivers and the potentials.*

We do not give the general definitions, but illustrate with two examples.

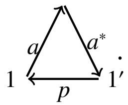
Example 5.1.1. *Let Q be the quiver $\underset{1}{\cdot}\xrightarrow{a}\underset{2}{\cdot}\xrightarrow{b}\underset{3}{\cdot}$ and $w = s_1s_2s_3s_1s_2s_1$.*

Let $T = T_{\{123121\}}$. Then End$_{Sub\Lambda_w}(T)$ has the quiver



and

End$_{\underline{Sub\Lambda_w}}(T)$ has the quiver

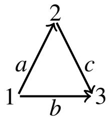


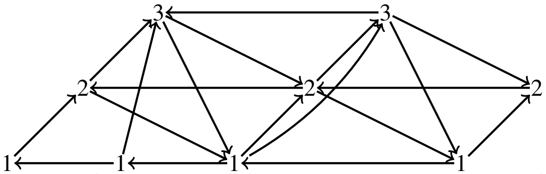
. Here we write the vertices in order $1231'2'1''$, but on different levels to make it easier to draw the picture. For each edge in the underlying graph of Q we draw corresponding arrows between the vertices. For 1——2 we draw the arrows $1\longrightarrow 2$, $2\longrightarrow 1'$, $1'\longrightarrow 2'$, $2'\longrightarrow 1''$, Here we start with the vertex of type 1 which occurs first, and draw an arrow

from the last vertex of type 1 *appearing before some vertex of type* 2. *This arrow should end in the last vertex of type* 2 *appearing before the next vertex of type* 1. *Then we go on this way for each edge in the underlying graph of* Q *to get the first quiver above. To get the second quiver above we drop the last vertex of each kind in the first quiver.*

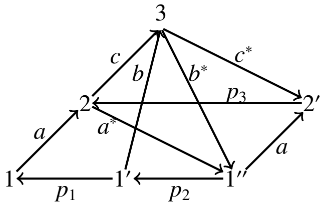
The potential for the last quiver is a sum of "triangles" obtained by labelling the arrow from 2 *to* $1'$ *following the arrow* $1 \overset{a}{\to} 2$ *by* a^*. *Then we start with* aa^*, *and go back, here by* p, *to the start of* a, *going through vertices of type* 1. *In this case* $W = aa^*p$.

Then one can show that $\mathrm{End}_{\underline{\mathrm{Sub}\Lambda_w}}(T) \simeq \mathcal{P}(Q, W)$.

Example 5.1.2. *Let* Q *be the quiver* , $w = s_1s_2s_1s_3s_1s_2s_2s_3s_1s_2$. *Then we get the quiver*



for $\mathrm{End}_{\mathrm{Sub}\Lambda_w}(T)$, *where* $T = T_{\{121312312\}}$, *and the quiver*



for $\mathrm{End}_{\underline{\mathrm{Sub}\Lambda_w}}(T)$. *The potential* W *is* $aa^*p_2p_1 + a^*ap_3 + bb^*p_2 + cc^*p_3$. *Then one can show that* $\mathrm{End}_{\underline{\mathrm{Sub}\Lambda_w}}(T) \simeq \mathcal{P}(Q, W)$.

5.2. **Generalized cluster categories.** Let A be a finite dimensional algebra of global dimension 2, with quiver Q, and let $\mathscr{C}_A$ be the associated generalized cluster category. Then the 2-CY-tilted algebra $\mathrm{End}_{\mathscr{C}_A}(A)$ is Jacobian. More specifically, for each relation r in a minimal set of relations for A, draw an arrow α_r from the end vertex to the start vertex of r. If $\tilde{Q}$ is the new quiver obtained this way, and $W = \sum_r r\alpha_r$, then $A \cong \mathcal{P}(\tilde{Q}, W)$ [K3].

So this way we get one 2-CY-tilted algebra which is Jacobian. Using the results of the next section [BIRSc], we get that all algebras in the same component are Jacobian.

6. **Connection between the mutations**

In this section we state the result on the commuting of the mutation of cluster-tilting objects and of quivers with potentials. We discuss some interesting consequences of this fact. All this is taken from [BIRSc].

We have the following main result.

Theorem 6.1. *Assume that there are no loops in the quiver Q, and no 2-cycle at the vertex k. Let T be a cluster-tilting object in a Hom-finite triangulated 2-CY category $\mathscr{C}$, and let (Q, W) be a quiver with potential.*

If $End_{\mathscr{C}}(T) \simeq \mathcal{P}(Q, W)$, then $End_{\mathscr{C}}(\mu_k(T)) \simeq \mathcal{P}(\mu_k(Q, W))$.

In other words, we have the following diagram

$$\begin{tikzcd}
(Q, W) \arrow[r, "\mu_k"] \arrow[d] & T^* = \mu_k(Q, W) \arrow[d] \\
\mathcal{P}(Q, W) \arrow[d, phantom, "\wr"] & \mathcal{P}(\mu_k(Q, W)) \arrow[d, dashed, "\alpha"] \\
\mathrm{End}_{\mathscr{C}}(T) & \mathrm{End}_{\mathscr{C}}(\mu_k(T)) \\
T \arrow[u] \arrow[r, "\mu_k"] & \mu_k(T) \arrow[u]
\end{tikzcd}$$

where we obtain an isomorphism α.

As a first application we can now answer the questions about existence of well defined maps from section 2.

Corollary 6.2. *Let the notation be as above. Then we have the following*

- **(a)** *The algebra $End_{\mathscr{C}}(\mu_k(T))$ is determined by the algebra $End_{\mathscr{C}}(T)$, and does not depend on the choice of cluster-tilting object T.*
- **(b)** *The Jacobian algebra $\mathcal{P}(\mu_k(Q, W))$ is determined by the Jacobian algebra $\mathcal{P}(Q, W)$, and does not depend on the choice of potential (Q, W).*

Another application is the following

Corollary 6.3. *Let T be a cluster-tilting object in a Hom-finite triangulated 2-CY category $\mathscr{C}$. If $\mathrm{End}_{\mathscr{C}}(T)$ is a Jacobian algebra, then $\mathrm{End}_{\mathscr{C}}(T')$ is a Jacobian algebra for any cluster-tilting object T' lying in the same component as T in the cluster-tilting graph for $\mathscr{C}$.*

Since there are situations where some 2-CY-tilted algebra is Jacobian, we can apply Corollary 6.3 to get a much larger class of such examples. We do not however have an explicit description of the quivers and the potential for the new 2-CY-tilted algebras.

Corollary 6.4. **(a)** *The 2-CY-tilted algebras belonging to the same cluster-tilting graph as the algebras $\mathrm{End}_{\underline{\mathrm{Sub}\Lambda_w}}(T)$, where T is a standard cluster-tilting object in $\mathrm{Sub}\Lambda_w$, are Jacobian.*

(b) *The 2-CY-tilted algebras belonging to the same cluster-tilting graph as the 2-CY-tilted algebra $\mathrm{End}_{\mathscr{C}_A}(A)$, where A is an algebra of global dimension 2 and $\mathscr{C}_A$ is the associated generalized cluster category.*

Since the cluster-tilting graph for the cluster category of a connected quiver Q is connected, and KQ is clearly Jacobian, it follows from Corollary 6.4 that any cluster-tilted algebra is Jacobian.

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