Configurations of trivial extensions of Dynkin type $\mathbf{D}_n$

Octavio Mendoza Hernández

Instituto de Matemáticas
Universidad Nacional Autónoma de México.
Circuito Exterior, Ciudad Universitaria
C.P. 04510, México, D.F. MEXICO.

E-mail address: `omendoza@matem.unam.mx`

Abstract. For a given trivial extension Λ of Cartan class $\mathbf{D}_n$, we find a combinatorial algorithm giving the configuration of $\mathbb{Z}\mathbf{D}_n$ associated to Λ; this algorithm is stated in the Theorem 6.8 and illustrated in the Example 6.3. In order to do that, we develop an inductive embedding technique for the Auslander-Reiten quiver Γ_Λ of a trivial extension Λ of type $\mathbf{D}_n$. This technique uses the ordinary quiver Q_Λ, which is the union of oriented cycles, and a well known set of relations ρ such that Λ is the quotient path algebra $kQ_\Lambda/<\rho>$ (see [1], [5] and [6]). Giving a trivial extension Γ of type $\mathbf{D}_N$, we get a trivial extension Λ of type $\mathbf{D}_{N+m-1}$ by inserting a cycle (with m vertices) in one of the so called insertion vertices of Q_Γ. In Section 3, we describe the embedding of Γ_Γ into Γ_Λ. This will allow us to deduce properties of Γ_Λ from Γ_Γ; and since Q_Γ has less cycles that Q_Λ, we will use this embedding to reduce the study of Γ_Λ to the minimal trivial extensions, which were done in Section 2.

Introduction

In this paper, we only consider basic finite dimensional k-algebras with k an algebraically closed field. In such a case, an algebra A of this type is isomorphic to kQ_A/I, where Q_A is the ordinary quiver associated to A and I is an admissible ideal of the path algebra kQ_A. For a quiver Q, the set of vertices will be denoted by Q_0 and the set

of arrows by Q_1.

Given a k-algebra A and a vertex j of Q_A, we denote by S_j the simple A-module corresponding to the vertex j; and so, P_j will denote the projective cover of S_j and I_j the injective envelope of S_j.

Let A be an iterated tilted algebra of Dynkin type Δ (see in [2]); and let $T(A) := A \ltimes D_A(A)$ be the trivial extension of A by its minimal injective cogenerator $D_A(A) = \mathrm{Hom}_k(A, k)$. The set of vertices $(\Gamma_A)_0$ of the Auslander-Reiten quiver Γ_A of A can be embedded in the stable part ${}_S\Gamma_{T(A)}$ of the Auslander-Reiten quiver $\Gamma_{T(A)}$ of $T(A)$. Since $\mathbb{Z}\Delta \to {}_S\Gamma_{T(A)}$ is the universal covering of ${}_S\Gamma_{T(A)}$, the vertices of Γ_A can be embedded in $\mathbb{Z}\Delta$ in such a way that knowing the vertices of $\mathbb{Z}\Delta$ corresponding to A-modules, we can obtain the arrows of Γ_A (see in [8]). Hence, the quiver Γ_A is embedded in $\mathbb{Z}\Delta$; and it is very useful for applications to describe this embedding explicitly. In [8], this problem was solved by assuming that the set of vertices in $\mathbb{Z}\Delta$ corresponding to the radicals of the indecomposable projective $T(A)$-modules is known. So, to have a complete description of this embedding, it is necessary to give an algorithm to determine which subsets of vertices in $\mathbb{Z}\Delta$ represent the radicals of the indecomposable projective modules over a given trivial extension T of Dynkin type Δ. These kind of vertices of $\mathbb{Z}\Delta$ have been considered before by Chr. Riedtmann, who called them configurations, in a more general setting [4, 10, 11, 12]. The configurations of selfinjective algebras of finite type were computed in these works. One could use the results for selfinjective algebras and then decide which configurations correspond to a given trivial extension Λ. With a different approach, we present here a new algorithm giving directly the configuration of a given trivial extension of Dynkin type $\mathbf{D}_n$. The $\mathbf{A}_n$ case was solved in [9]. In what follows, we summarize the content of the paper.

Section 1: By following [5], we give (see Proposition 1.8) a presentation of the trivial extensions Λ of Cartan class $\mathbf{D}_N$; and also we recall the lifting procedure of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ as was settled in [8, 9].

Section 2: Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ and $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ be the universal covering of translation quivers. Consider the subset $\mathcal{C}_\Lambda := \{rP_i \; : \; i \in (Q_\Lambda)_0\}$ of ${}_S\Gamma_\Lambda$ representing the radicals of the indecomposable projective Λ-modules P_i for $i \in (Q_\Lambda)_0$, and the set $\widetilde{\mathcal{C}_\Lambda} = \pi^{-1}(\mathcal{C}_\Lambda)$. In this section, we describe the configuration $\widetilde{\mathcal{C}_\Lambda}$ associated to a given minimal trivial extension Λ of

type $\mathbf{D}_N$. We start by studying, under very special situations, how rP_j can be lifted to $\mathbb{Z}\mathbf{D}_N$ for some $j \in (Q_\Lambda)_0$. We do this first for arbitrary trivial extensions of $\mathbf{D}_N$ type, and then specialize for the different minimal types defined in the previous section.

Section 3: Let Γ be a trivial extension of Cartan class $\mathbf{D}_N$, z an insertion vertex of Q_Γ (see Definition 2.1) and Λ be the trivial extension of Cartan class $\mathbf{D}_{N+m-1}$ obtained from Γ by inserting the cycle $C_z = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. In this section, we study the embedding of ${}_S\Gamma_\Gamma$ in ${}_S\Gamma_\Lambda$. We also study the behavior of the irreducible morphisms of $\underline{\mathrm{mod}}\,\Gamma$ inside $\underline{\mathrm{mod}}\,\Lambda$ using the embedding $\imath : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\,\Lambda$.

Section 4: In the previous section, we described an embedding of ${}_S\Gamma_\Gamma$ in ${}_S\Gamma_\Lambda$. In this section, we study the corresponding embedding between their coverings $k(\mathbb{Z}\mathbf{D}_N)$ and $k(\mathbb{Z}\mathbf{D}_{N+m-1})$. Here, the main results are the Theorem 4.11 and its Corollary 4.12.

Section 5: In this section, we extend the Proposition 2.14 to the general case, that is, to a trivial extension Λ of Class II (see Definition 1.2 and Remark 1.9). To do that, we firstly consider the case when Λ is minimal of Class II (see 2.14). Then, by using 4.11, we see how the radicals of the projective modules, associated with the vertices $1, 2$ and 3 of Q_Λ, can be lifted to the universal covering. Furthermore, we also determine some special regions, in the bottom border of $\mathbb{Z}\mathbf{D}_N$, where the radicals of the projective modules associated with the insertion vertices of Q_Λ can be lifted.

Section 6: Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. Consider the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ of ${}_S\Gamma_\Lambda$. In this section, we give an algorithm to determine the configuration $\widetilde{\mathcal{C}_\Lambda}$ of $\mathbb{Z}\mathbf{D}_N$ associated to Λ. We recall that $\widetilde{\mathcal{C}_\Lambda} := \pi^{-1}(\mathcal{C}_\Lambda)$, where $\mathcal{C}_\Lambda$ is the set of vertices of ${}_S\Gamma_\Lambda$ representing the radicals of the indecomposable projective Λ-modules (see [8, 2.7]). We define a subset $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)$ of $(\mathbb{Z}\mathbf{D}_N)_0$ and prove that $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)[\mathbb{Z}] := \cup_{i\in\mathbb{Z}} r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)[i]$ is the desired configuration. In order to do that, we start this section by defining some notations needed to state the algorithm: the sections $\mathcal{S}_x^+$ and $\mathcal{S}_x^-$, the initial vertices, and the functions border and height defined on the set of vertices of Q_Λ. The algorithm is stated in the Theorem 6.8 and illustrated in the Example 6.3. The remainder of the section is devoted to the proof of the Theorem 6.8.

1. **Preliminaries**

We start this section by recalling well known notions about quivers. Let Q be a quiver. We denote by Q_0 the set of vertices and by Q_1 the set of arrows of Q. For any arrow α of Q_1 we have that it starts at $o(\alpha)$ and ends at $e(\alpha)$. A path p in the quiver Q is either an oriented sequence of arrows $\alpha_n \cdots \alpha_1$ with $e(\alpha_t) = o(\alpha_{t+1})$ for $1 \leq t < n$ or the symbol e_i for $i \in Q_0$. We call the paths e_i trivial paths and we define $o(e_i) = e(e_i)$. Given a nontrivial path $p = \alpha_n \cdots \alpha_1$ we define $o(p) = o(\alpha_1)$ and $e(p) = e(\alpha_n)$. If δ is a path in Q, we denote by $\underline{\delta}$ the **support** of δ in Q. Thus, $\underline{\delta}$ is a subquiver of Q having as vertices and arrows those belonging to δ. A nontrivial path p is said to be an **oriented cycle** if $o(p) = e(p)$.

Let $\mathcal{C} = \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1$ be an oriented cycle in Q. We call $\mathcal{C}$ **minimal oriented cycle** if the vertices $o(\alpha_1), o(\alpha_2), \cdots, o(\alpha_n)$ are pairwise different. It is said that Q' is a **full subquiver** of Q in case Q' is a subquiver of Q such that for all vertices $i, j \in Q'$ we have that each arrow $i \xrightarrow{\alpha} j$ of Q is also an arrow of Q'. A full subquiver Q' of Q is called **convex** if for any path $a_0 \to a_1 \to \cdots \to a_t$ in Q, with $a_0, a_t \in Q'_0$, we have $a_i \in Q'_0$ for all $i = 0, 1, \cdots, t$.

1.1. **A presentation of trivial extensions of type $\mathbf{D}_n$.**

Following [5], we know that any trivial extension Λ of type $\mathbf{D}_n$, which is not minimal, can be obtained from one minimal Γ, of type $\mathbf{D}_N$, by inserting to Γ a trivial extension of type $\mathbf{A}_m$ with $n = N + m - 1$. In this section, we explain such a construction and also give a presentation (Q_Λ, I) of Λ, which will be used throughout all the paper.

Proposition 1.1. [5] *Let Λ be a trivial extension. Then, Λ is of type $\mathbf{A}_n$, with $n > 1$, if and only if the ordinary quiver Q_Λ of Λ satisfies the following conditions:*

- (i) *Q_Λ has n vertices;*
- (ii) *Q_Λ is the union of oriented cycles and there are no loops in Q_Λ;*
- (iii) *any two minimal oriented cycles of Q_Λ, having different supports, meet in at most one vertex;*
- (iv) *every vertex $i \in Q_\Lambda$ belongs to at most two different supports of minimal oriented cycles;*

(v) *if* $C_1, C_2, \cdots, C_m$ *are minimal oriented cycles in* Q_Λ *such that* $\underline{C_i} \neq \underline{C_j}$ *for* $i \neq j$, *and* $\underline{C_1} \cap \underline{C_2} \neq \emptyset, \underline{C_2} \cap \underline{C_3} \neq \emptyset, \cdots, \underline{C_{m-1}} \cap \underline{C_m} \neq \emptyset$, *then* $\underline{C_1} \cap \underline{C_m} = \emptyset$.

In what follows, we give, following [5], the ordinary quivers Q_Λ corresponding to the smallest trivial extensions Λ of type $\mathbf{D}_N$.

Definition 1.2. [5] *Let* Λ *be a trivial extension of Cartan class* $\mathbf{D}_N$. *We say that* Λ *is minimal if the ordinary quiver* Q_Λ *of* Λ *belongs to one of the following classes of quivers.*
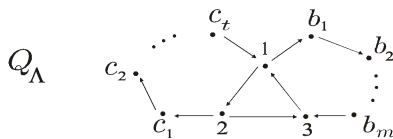
(a) *Class I with* $N = 4 + n$.
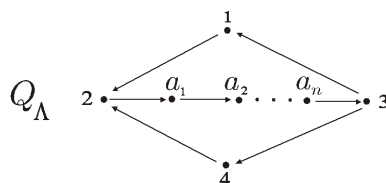
(b) *Class II. This class is divided in 3 subclasses:* II_1, II_2 *and* II_3.
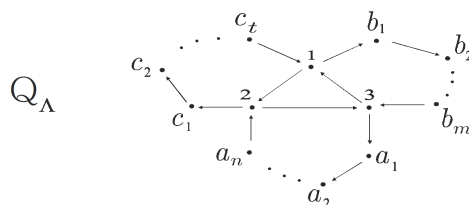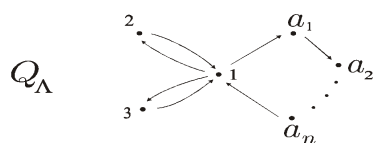(b1) *Subclass* II_1 *with* $N = 3 + n$.
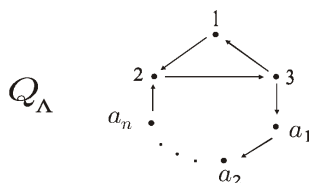
(b2) *Subclass* II_2 *with* $N = 3 + t + m$.

(b3) *Subclass* II_3 *with* $N = 3 + t + m + n$.

Q_Λ

(c) *Class III with $N = 3 + n$.*

Q_Λ

Remark 1.3. *For the trivial extensions of class II, it will be convenient to relax the values of m, n and t to be zero. This means for example that if $m = t = 0$ then Λ is of class II_1. Such a notation allows us to consider the subclasses II_1 and II_2 as particular cases of the subclass II_3.*

Definition 1.4. [5, 7] *Let Λ be a trivial extension of Dynkin Cartan class. By [5] (see also [7]), it is known that Q_Λ is the union of oriented cycles. A minimal oriented cycle C of Q_Λ is said to be* **elemental** *if it is not zero in Λ.*

The elemental cycles are very important to describe the trivial extensions of finite representation type as can be seen in the following result.

Theorem 1.5. [5] *Let Λ be a trivial extension of Dynkin Cartan class. Then $\Lambda \simeq kQ_\Lambda/I$, where the quiver Q_Λ is the union of oriented cycles and the admissible ideal I is generated by:*

   (i) *the paths consisting of $n + 1$ arrows in an elemental cycle of length n;*
  (ii) *the paths whose arrows do not belong to a single elemental cycle;*
 (iii) *the difference $q - q'$, where q and q' are paths starting and ending at the same vertices and such that there exists a path ν with νq and $\nu q'$ elemental cycles.*

Remark 1.6. *Let Λ be a trivial extension of Dynkin Cartan class Δ. Then:*

(1) *To have a presentation (Q_Λ, I) for Λ, it is enough to give the quiver Q_Λ and the set $\{\underline{\mathcal{C}}_1, \cdots, \underline{\mathcal{C}}_m\}$ of supports of the elemental cycles in Q_Λ since the admissible ideal I is given by the above theorem.*

(2) *If $\Delta = \mathbf{A}_n$, then by [5], we have that the elemental cycles of Q_Λ are the minimal oriented cycles.*

(3) *If $\Delta = \mathbf{D}_n$ and Λ is minimal, we have by [5], that the elemental cycles of Q_Λ are the minimal oriented cycles with one exception: the only minimal oriented cycle, which is not allowed to be elemental, is the minimal one formed by all the vertices of Q_Λ for Λ of Subclass II_3.*

In order to get all the trivial extensions of type $\mathbf{D}_N$, by using minimal algebras, we need to add trivial extensions of type $\mathbf{A}_k$ to some special vertices in the minimal algebras. So, we recall from [5] this procedure.

Definition 1.7. [5] *Let $\Gamma = kQ_\Gamma/I$ be a trivial extension with I an admissible ideal of kQ_Γ. Let $a_1, a_2, \cdots, a_m$ be different vertices in Q_Γ and $n_1, n_2, \cdots, n_m$ be natural numbers. Let $\Gamma' := kQ'/(I \cap kQ')$ be a trivial extension, where Q' is a full subquiver of Q_Γ. For each $j = 1, 2, \cdots, m$, we consider a full subquiver Q_{a_j} of Q_Γ and set $\Gamma_{a_j} := kQ_{a_j}/(I \cap kQ_{a_j})$. We say that Γ is obtained from Γ' by* **adding trivial extensions** Γ_{a_j} *of Cartan type $\mathbf{A}_{n_j}$ to a_j for $j = 1, 2, \cdots, m$, if the following conditions hold:*
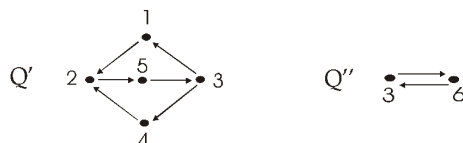
(i) *for each j, Γ_{a_j} is a trivial extension of Cartan type $\mathbf{A}_{n_j}$ with $n_j \geq 2$;*

(ii) *$(Q_\Gamma)_0 = Q'_0 \cup (Q_{a_1})_0 \cup \cdots \cup (Q_{a_m})_0$; moreover, for each j, we have that*

$$Q'_0 \cap (Q_{a_j})_0 = \{a_j\} \quad and \quad (Q_{a_i})_0 \cap (Q_{a_j})_0 = \emptyset \quad if \quad i \neq j;$$

(iii) *I is generated by $I \cap kQ'$ and $I \cap kQ_{a_j}$ for $j = 1, 2, \cdots, m$.*

Example. Let Γ be the trivial extension of Cartan class $\mathbf{D}_6$ with Q_Γ the following quiver and I the ideal considered in 1.5.
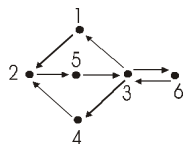
Consider the following full subquivers of Q_Γ,

We have that $\Gamma' := kQ'/(I \cap kQ')$ and $\Gamma'' := kQ''/(I \cap kQ'')$ are trivial extension of Cartan class $\mathbf{D}_5$ and $\mathbf{A}_2$, respectively. So, Γ is obtained from Γ' by adding the trivial extension Γ'' of Cartan class $\mathbf{A}_2$ to the vertex 3 in Q'.

The description of the trivial extensions of Cartan class $\mathbf{D}_N$ is given in the following result. Here we use the notation discussed in 1.3.

Proposition 1.8. [5] *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. Then, Λ is either minimal or can be obtained from a minimal one by adding trivial extensions of Cartan class $\mathbf{A}_{n_j}$ to some of the following vertices x_j indicated with $\square$*
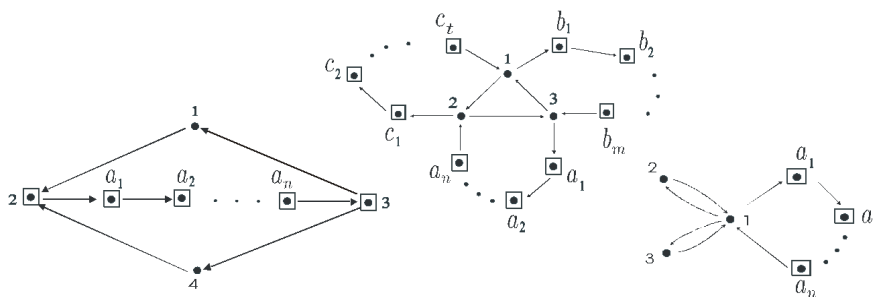
Remark 1.9. (1) *We will denote by Λ_{min} to the minimal trivial extension corresponding to Λ in 1.8.*

(2) *Any trivial extension Λ of Cartan class $\mathbf{D}_N$ will be represented by one of the pictures given in 1.8. The little square on the vertex x_j*

represents a quiver Q_{x_j}, which will be not drawn, where Q_{x_j} is either $\{x_j\}$ or the ordinary quiver of a trivial extension Γ_{x_j} of Cartan class $\mathbf{A}_{n_j}$ with $n_j \geq 2$. Moreover, we say that Λ is of class I, II_1, II_2, II_3 or III if Λ_{min} is so.

1.2. Lifting $_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$ having some $M \in \underline{\mathrm{ind}}\,\Lambda$ as a source.

Throughout this paper, we will use freely the notations that were settled in [8, 9]. In particular, the lifting process defined in [8], will be crucial in the paper. For a given trivial extension Λ of Dynkin Cartan class Δ, a fixed object M in $\underline{\mathrm{ind}}\,\Lambda$ and the universal covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ of $_S\Gamma_\Lambda$, where $_S\Gamma_\Lambda$ is the stable AR-quiver of Λ, it was introduced in [8, Section 3] the notion of the **lifting of $_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$ at the vertex M.** Recall that this lifting procedure starts by fixing an element $M[0]$ in the fiber $\pi^{-1}(M)$; afterwards, we take a slice of $_S\Gamma_\Lambda$ starting at M and lift such slice through the universal covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ to a unique slice of $\mathbb{Z}\Delta$ starting at $M[0]$. Analogously, the slice of $_S\Gamma_\Lambda$ starting at $\tau^{-1}(M)$ can be lifted to a slice of $\mathbb{Z}\Delta$ starting at $\tau^{-1}(M[0])$. We iterate this procedure, the necessary number of times, lifting all the vertices of $_S\Gamma_\Lambda$. The minimal connected subquiver of $\mathbb{Z}\Delta$, which contains all the lifted slices, is denoted by $_S\Gamma_\Lambda[0]$ and will be called **the lifting of $_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$ at M.** Then $\pi|_{{}_S\Gamma_\Lambda[0]} : {}_S\Gamma_\Lambda[0] \to {}_S\Gamma_\Lambda$ is a quiver morphism which is a bijection on the vertices of $_S\Gamma_\Lambda[0]$. The inverse $\varphi_M : ({}_S\Gamma_\Lambda)_0 \to (\mathbb{Z}\Delta)_0$ of this bijection, defines an embedding of $_S\Gamma_\Lambda$ into $\mathbb{Z}\Delta$. For $X \in \underline{\mathrm{ind}}\,\Lambda$ and $i \in \mathbb{Z}$, we denote by $X[i]$ the vertex $\tau^{-i\,m_\Delta}X[0]$ of $\mathbb{Z}\Delta$, where $X[0] := \varphi_M(X)$ and m_Δ is the Loewy Length of the mesh category $k(\mathbb{Z}\Delta)$ (see [8, Section 3]). Furthermore, if we do not want to state precisely the lifting vertex, we will just say that $_S\Gamma_\Lambda[0]$ is a lifting of $_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$. We recall that $m_\Delta = 2N - 3$ if $\Delta = \mathbf{D}_N$.

We introduce now the notions of height function and borders in $\mathbb{Z}\mathbf{D}_N$. To do that, we label the vertices of $\mathbf{D}_N$ as follows:
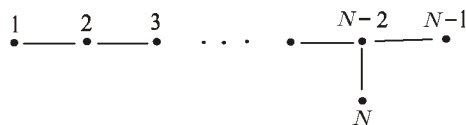
Definition 1.10. *The* **height** *in $\mathbb{Z}\mathbf{D}_N$ is the function $h = h_{\mathbb{Z}\mathbf{D}_N}$: $(\mathbb{Z}\mathbf{D}_N)_0 \to \{1, 2, \cdots, N\}$ defined by $h_{\mathbb{Z}\mathbf{D}_N}(p, q) := q$. We distinguish 3 types of borders in $\mathbb{Z}\mathbf{D}_N$: (a) the* **bottom border** *given by all the vertices of $\mathbb{Z}\mathbf{D}_N$ with height 1, (b) the* **top border** *corresponds to the vertices of $\mathbb{Z}\mathbf{D}_N$ with height $N-1$ and (c) the* **middle border** *consists of the vertices having height N.*
The next picture shows which vertices of the section of $\mathbb{Z}\mathbf{D}_N$, starting at $(0, 1)$, belong to the borders of $\mathbb{Z}\mathbf{D}_N$.
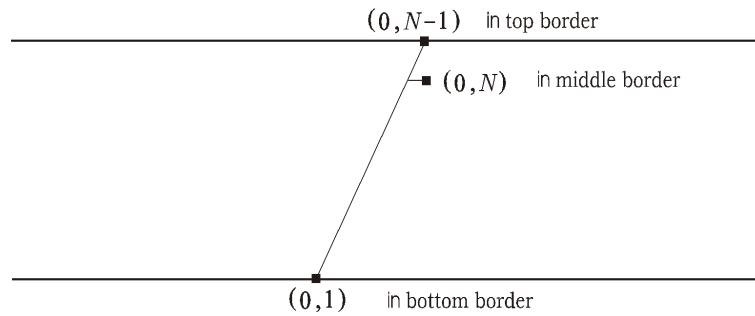

Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ and $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ the universal covering of translation quivers. Consider the subset $\mathcal{C}_\Lambda := \{rP_i \ : \ i \in (Q_\Lambda)_0\}$ of ${}_S\Gamma_\Lambda$ representing the radicals of the indecomposable projective Λ-modules P_i for $i \in (Q_\Lambda)_0$, and set $\widetilde{\mathcal{C}_\Lambda} = \pi^{-1}(\mathcal{C}_\Lambda)$. From [10, 2.3 and 2.2], we know that $\widetilde{\mathcal{C}_\Lambda}$ is a configuration of $\mathbb{Z}\Delta$ and $\mathcal{C}_\Lambda$ is a configuration of ${}_S\Gamma_\Lambda$. In this paper, we will define an algorithm to compute the vertices of $\mathbb{Z}\mathbf{D}_N$ corresponding to the configuration $\widetilde{\mathcal{C}_\Lambda}$ associated with Λ. The fundamental question to start working with is the following. Let $i \to j$ be an arrow of Q_Λ and suppose that we know the position on $\mathbb{Z}\mathbf{D}_N$ of some element $rP_j[0]$ belonging to the fiber $\pi^{-1}(rP_j)$. Then, how does the radical rP_i lifts to $\mathbb{Z}\mathbf{D}_N$? To answer this question, we will need to compute the support of the functors $k(\mathbb{Z}\mathbf{D}_N)(x, -)$ and $k(\mathbb{Z}\mathbf{D}_N)(-, x)$ for any vertex $x \in \mathbb{Z}\mathbf{D}_N$. We will denote by $\operatorname{Supp} F$ the support of a given functor F. Also, for the sake of simplicity, we set $\operatorname{Supp}(x, -) := \operatorname{Supp} k(\mathbb{Z}\mathbf{D}_N)(x, -)$ and $\operatorname{Supp}(-, x) := \operatorname{Supp} k(\mathbb{Z}\mathbf{D}_N)(-, x)$ whenever it is clear from the

context which functor is being considered. We recall that (see [8, Section 3]) the additive function $f_x : (\mathbb{Z}\mathbf{D}_N)_0 \to \mathbb{Z}$, which has value 1 on the slice starting at x, determines the support of the functor $k(\mathbb{Z}\mathbf{D}_N)(x, -)$ since $dim_k\, k(\mathbb{Z}\mathbf{D}_N)(x, y) = f_x(y)$. Moreover, it is known that $\text{Supp}\, k(\mathbb{Z}\mathbf{D}_N)(x, -) = \text{Supp}\, k(\mathbb{Z}\mathbf{D}_N)(-, \nu_{\mathbf{D}_N}(x))$, where $\nu_{\mathbf{D}_N}(x)$ is the Nakayama permutation. So, by using the additive function f_x, it can be seen that the support of the functors we are interested in has the following shape.
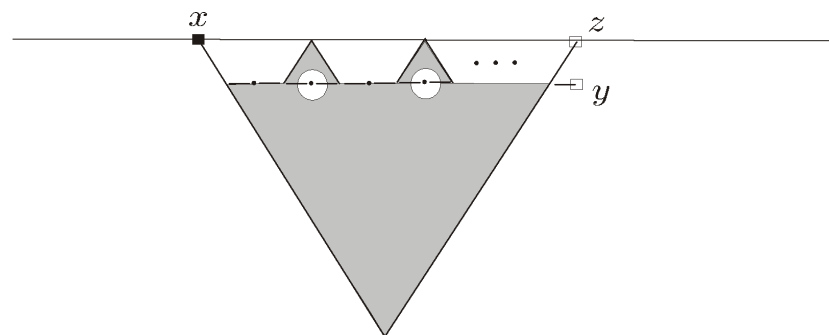
Remark 1.11. *The shaded regions of the pictures below represent the support of the functor $k(\mathbb{Z}\mathbf{D}_N)(x, -)$.*
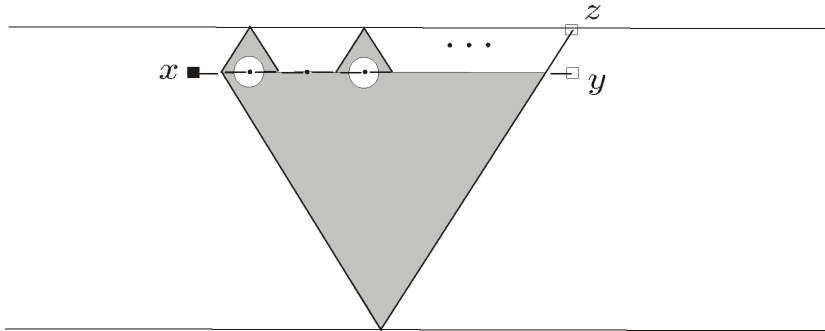
(a) *For $1 \leq h_{\mathbb{Z}\mathbf{D}_N}(x) \leq N - 2$:*
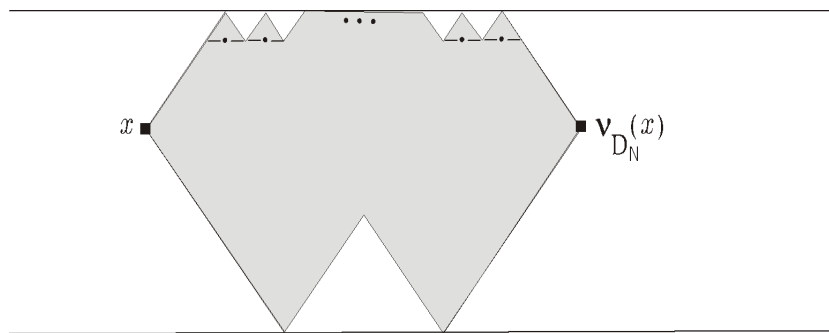
In particular, if x is in the bottom border of $\mathbb{Z}\mathbf{D}_N$, we obtain a triangle with x and $\nu_{\mathbf{D}_N}(x)$ in the bottom border.

(b) *For $h_{\mathbb{Z}\mathbf{D}_N}(x) = N - 1$:*

$$\text{where } \nu_{\mathbf{D}_N}(x) = \begin{cases} z & \text{if } N \text{ is even} \\ y & \text{if } N \text{ is odd.} \end{cases}$$

(c) *For* $h_{\mathbb{Z}\mathbf{D}_N}(x) = N$:

$$\text{where } \nu_{\mathbf{D}_N}(x) = \begin{cases} y & \text{if } N \text{ is even} \\ z & \text{if } N \text{ is odd.} \end{cases}$$

Proposition 1.12. *Let* Λ *be a trivial extension of Cartan class* $\mathbf{D}_N$, $i \to j$ *an arrow in* Q_Λ *and* $_S\Gamma_\Lambda[0]$ *a lifting of* $_S\Gamma_\Lambda$ *to* $\mathbb{Z}\mathbf{D}_N$ *at* rP_j. *Then*

(a) $rP_i[0] \in \text{Supp}(\tau^{-1}rP_j[0], -) \setminus \text{Supp}(rP_j[0], -)$;

(b) *if* $rP_i \simeq P_j/\text{soc}\, P_j$, *then* $\tau^{-1}rP_j[0] = rP_i[0]$ *and* $rP_j[0]$ *belongs to some border of* $\mathbb{Z}\mathbf{D}_N$.

Proof. (a) By [9, Prop. 3.4], we get that $rP_i[0] \in \text{Supp}(\tau^{-1}rP_j[0], -)$. On the other hand $k(\mathbb{Z}\mathbf{D}_N)(rP_j[0], rP_i[0]) = 0$ since Q_Λ has no loops (see 1.8 and [9, Proposition 1.1]).

(b) Let $rP_i \simeq P_j/\text{soc}\, P_j$. Thus $\tau^{-1}rP_j \simeq rP_i$. On the other hand, in order to see that $rP_j[0]$ belongs to some border of $\mathbb{Z}\mathbf{D}_N$, it is enough to prove that $rP_j/\text{soc}\, P_j$ is indecomposable. To do that, we will use; firstly, the description of Q_Λ given in Subsection 1.1; and secondly, a combinatorial description of the number of the indecomposable direct summands of $rP_j/\text{soc}\, P_j$ which is given in [6, Corollary 4.12].

Suppose that $rP_j/\text{soc}\, P_j$ has at least two indecomposable direct summands. Then, from [6, Corollary 4.12], we get the existence of two elemental cycles C and C' in Q_Λ such that: (i) $\alpha : i \to j$ is in

C but not in C' and (ii) there is an arrow $\beta : j \to t$ which is in C' but not in C. Moreover, by [9, Proposition 1.1] and 1.8, we have that the cycles C and C' have exactly one common vertex: the vertex j. Therefore, from 1.5, we conclude that the path $\beta\alpha$ is zero in Λ. Hence, the simple S_t, associated with the vertex t, is a composition factor of $P_j/\mathrm{soc}\, P_j$ which is not a composition factor of rP_i; giving a contradiction since $P_j/\mathrm{soc}\, P_j \simeq rP_i$. $\square$

We will need to know the vertices of $\mathrm{Supp}\,(\tau^{-1}rP_j[0], -) \setminus$ $\mathrm{Supp}\,(rP_j[0], -)$ in case $rP_j[0]$ is either on the top or in the middle border of $\mathbb{Z}\mathbf{D}_N$. These vertices can be easily computed by 1.11.

Remark 1.13. *Suppose that $rP_j[0]$ is either on the top or in the middle border of $\mathbb{Z}\mathbf{D}_N$. In the picture below, we have indicated with $\square$ the vertices of*

$$\mathrm{Supp}\,(\tau^{-1}rP_j[0], -) \setminus \mathrm{Supp}\,(rP_j[0], -)$$

in case $h_{\mathbb{Z}\mathbf{D}_N}(rP_j[0]) = N - 1$ (for $h_{\mathbb{Z}\mathbf{D}_N}(rP_j[0]) = N$, the pictures to be considered are essentially the same).
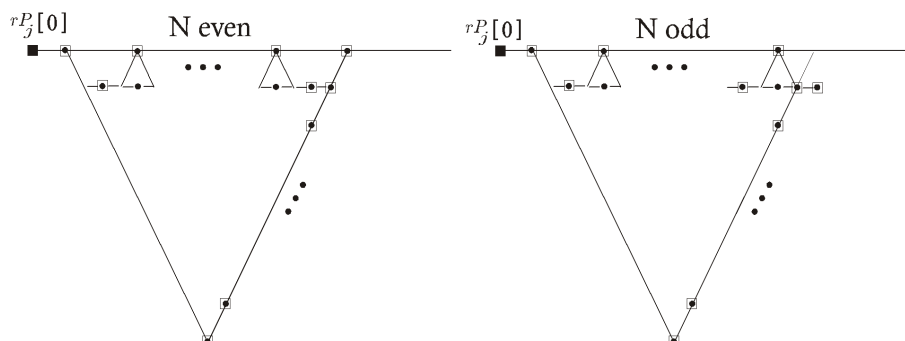
2. **Lifting some vertices of $_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ for trivial extensions Λ**

Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ and $\pi : \mathbb{Z}\mathbf{D}_N \to$ $_S\Gamma_\Lambda$ be the universal covering of translation quivers. Consider the subset $\mathcal{C}_\Lambda := \{rP_i \; : \; i \in (Q_\Lambda)_0\}$ of $_S\Gamma_\Lambda$ representing the radicals of the indecomposable projective Λ-modules P_i for $i \in (Q_\Lambda)_0$, and the

set $\widetilde{\mathcal{C}_\Lambda} = \pi^{-1}(\mathcal{C}_\Lambda)$. In this section, we will describe the configuration $\widetilde{\mathcal{C}_\Lambda}$ associated to a given minimal trivial extension Λ of type $\mathbf{D}_N$. We start by studying, under very special situations, how rP_j can be lifted to $\mathbb{Z}\mathbf{D}_N$ for some $j \in (Q_\Lambda)_0$. We do this first for arbitrary trivial extensions of $\mathbf{D}_N$ type, and then specialize for the different minimal types defined in the previous section.

2.1. **Insertion vertices in Q_Λ.**

Definition 2.1. [8] *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ and Γ be a trivial extension of Cartan class $\mathbf{D}_k$. If C is an elemental cycle of Q_Λ, Q_Γ is the union of the remaining cycles of Q_Λ and $C \cap Q_\Gamma$ is a single vertex z, we say that C is an* **elimination cycle** *of Q_Λ and that Γ is obtained from Λ by* **eliminating the cycle** *C. Also we say that Λ is obtained from Γ by* **inserting the cycle** *C at z.*
A vertex x of Q_Γ, where a minimal oriented cycle C can be inserted in order to obtain a trivial extension Λ of Cartan class $\mathbf{D}_N$ with $N > k$, is called **insertion vertex**.

Lemma 2.2. *For a trivial extension Λ of Cartan class $\mathbf{D}_N$, the following statements hold.*

(a) *Let $\gamma = z_1 \to z_2 \to \cdots \to z_{\ell-1} \to z_\ell$ be a path belonging to a single elemental cycle of Q_Λ. If $z_1, z_2, \cdots, z_\ell$ are insertion vertices, then*
$$P_{z_\ell}/\operatorname{soc} P_{z_\ell} \simeq rP_{z_{\ell-1}}, \cdots, P_{z_2}/\operatorname{soc} P_{z_2} \simeq rP_{z_1}.$$

(b) *If Λ is of class I then $rP_1/\operatorname{soc} P_1 \simeq rP_4/\operatorname{soc} P_4$.*

(c) *If Λ is of class II then $rP_i/\operatorname{soc} P_i$ is indecomposable for any $i = 1, 2, 3$. Moreover,*

    (c1) *For Λ of class II_1, we have that $P_3/\operatorname{soc} P_3 \simeq rP_2$. Furthermore,*
    $rP_1/\operatorname{soc} P_1 \simeq rP_{a_1}/\operatorname{soc} P_{a_1}$ if $N = 4$; and in case $N > 4$, $rP_1/\operatorname{soc} P_1 \not\simeq rP_{a_n}/\operatorname{soc} P_{a_n}$.

    (c2) *If Λ is either of class II_2 or II_3, then*
        (i) *$rP_{c_i}/\operatorname{soc} P_{c_i} \not\simeq rP_3/\operatorname{soc} P_3$ for any $i = 1, 2, \cdots, t$;*

        (ii) *$rP_{b_j}/\operatorname{soc} P_{b_j} \not\simeq rP_2/\operatorname{soc} P_2$ for any $j = 1, 2, \cdots, m$.*

(d) *For Λ of class III, we have that $rP_2/\operatorname{soc} P_2 \simeq S_1 \simeq rP_3/\operatorname{soc} P_3$ and $rP_1/\operatorname{soc} P_1 \simeq S_2 \oplus S_3 \oplus M$, where M is a uniserial Λ-module which has $S_{a_1}, S_{a_2}, \cdots, S_{a_n}$ as composition factors.*

Proof. It can be proven by using the description of Q_Λ given in 1.8 and the generators of the admissible ideal I given in 1.5. $\square$

Lemma 2.3. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$, and let i, j be different vertices of Q_Λ. In the case in which $rP_i/\operatorname{soc} P_i$ is indecomposable, we consider a lifting $_S\Gamma_\Lambda[0]$ of $_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at $rP_i/\operatorname{soc} P_i$.*

(a) *If $rP_i/\operatorname{soc} P_i \simeq rP_j/\operatorname{soc} P_j$ then $rP_i/\operatorname{soc} P_i$ is indecomposable. Moreover, there exist arrows $rP_j[-1] \to rP_i/\operatorname{soc} P_i\,[0] \leftarrow rP_i[-1]$ in $\mathbb{Z}\mathbf{D}_N$. Furthermore, $h_{\mathbb{Z}\mathbf{D}_N}(rP_i/\operatorname{soc} P_i\,[0]) = N - 2$; and, in case $N > 4$, we have that $h_{\mathbb{Z}\mathbf{D}_N}(rP_t[-1]) > N - 2$ for $t = i, j$.*

(b) *If $rP_i/\operatorname{soc} P_i$ and $rP_j/\operatorname{soc} P_j$ are indecomposable and there is an arrow $rP_j[-1] \to rP_i/\operatorname{soc} P_i\,[0]$ in $\mathbb{Z}\mathbf{D}_N$, then $rP_i/\operatorname{soc} P_i \simeq rP_j/\operatorname{soc} P_j$.*

Proof. We recall that, if P is a projective and injective indecomposable Λ-module then $0 \to rP \to rP/\operatorname{soc} P \oplus P \to P/\operatorname{soc} P \to 0$ is an Auslander-Reiten sequence in $\operatorname{mod}\Lambda$.

(a) Since $rP_i/\operatorname{soc} P_i \simeq rP_j/\operatorname{soc} P_j$, we obtain that there exist irreducible morphisms $rP_i \to rP_i/\operatorname{soc} P_i$ and $rP_j \to rP_i/\operatorname{soc} P_i$. Suppose that $rP_i/\operatorname{soc} P_i$ has at least two non-isomorphic indecomposable direct summands X and Y. Then, the quiver of Figure 1 below is a full subquiver of $_S\Gamma_\Lambda$.

Fig. 1 Fig. 2
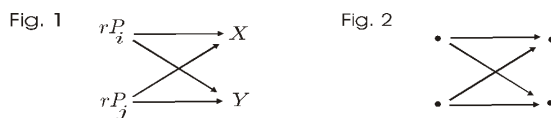
On the other hand, it is known that $_S\Gamma_\Lambda \simeq \mathbb{Z}\mathbf{D}_N/ < \tau^{2N-3} >$ (see [1] and [4]). Hence, $rP_i/\operatorname{soc} P_i$ is indecomposable since the quiver of the Figure 2 is not a full subquiver of $\mathbb{Z}\mathbf{D}_N$.

Using that there is an irreducible morphism $rP_j \to rP_i/\operatorname{soc} P_i$, we get an arrow $rP_t[-1] \to rP_i/\operatorname{soc} P_i\,[0]$ in $\mathbb{Z}\mathbf{D}_N$ for $t = i, j$. On the other hand, since $rP_i/\operatorname{soc} P_i \simeq rP_j/\operatorname{soc} P_j$, by using the shape of $\mathbb{Z}\mathbf{D}_N$, it follows that $h_{\mathbb{Z}\mathbf{D}_N}(rP_i/\operatorname{soc} P_i\,[0]) = N-2$ and $h_{\mathbb{Z}\mathbf{D}_N}(rP_t[-1]) > N-2$ for $t = i, j$; proving (a).

(b) Let $F : k(\mathbb{Z}\mathbf{D}_N) \to \underline{\mathrm{ind}}\, \Lambda$ be a well-behaved functor which is induced by the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ (see [10, 1.5]). Suppose that $rP_j/\mathrm{soc}\, P_j$ is indecomposable and that there exists an arrow $\alpha : rP_j[-1] \to rP_i/\mathrm{soc}\, P_i\,[0]$ in $\mathbf{ZD}_N$. Then $F(\overline{\alpha}) : rP_j \to rP_i/\mathrm{soc}\, P_i$ is irreducible. Therefore $rP_i/\mathrm{soc}\, P_i$ is a direct summand of $rP_j/\mathrm{soc}\, P_j$. Thus $rP_i/\mathrm{soc}\, P_i \simeq rP_j/\mathrm{soc}\, P_j$ since $rP_j/\mathrm{soc}\, P_j$ is indecomposable. $\quad\square$

2.2. Lifting rP_j to $\mathbb{Z}\mathbf{D}_N$ under certain general conditions.

Lemma 2.4. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$, and $i \to t \to j$ be arrows of Q_Λ such that $i \neq j$.*

(a) *If $P_j/\mathrm{soc}\, P_j \simeq rP_t$ then $rP_j/\mathrm{soc}\, P_j$ is indecomposable. Moreover, for any lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at $rP_j/\mathrm{soc}\, P_j$, we get that: $rP_j[-1]$ belongs to some border of $\mathbb{Z}\mathbf{D}_N$, $\tau^{-1}rP_j[-1] = rP_t[0]$ and*

$$rP_i[0] \in \mathrm{Supp}\,(\tau^{-1}rP_t[0], -) \setminus (\mathrm{Supp}\,(rP_t[0], -) \cup \mathrm{Supp}\,(rP_j[-1], -)).$$

(b) *If $P_j/\mathrm{soc}\, P_j \simeq rP_t$ and $P_t/\mathrm{soc}\, P_t \simeq rP_i$, then rP_j lifts to the bottom border of $\mathbb{Z}\mathbf{D}_N$.*

Proof.   (a) Let $P_j/\mathrm{soc}\, P_j \simeq rP_t$. By 1.12 (b), we have that $rP_j[-1]$ belongs to some border of $\mathbb{Z}\mathbf{D}_N$; and therefore, $rP_j/\mathrm{soc}\, P_j$ is indecomposable. On the other hand, $\tau^{-1}rP_j[-1] = rP_t[0]$ since $\tau^{-1}rP_j = rP_t$. Moreover, since $i \to t$ is an arrow of Q_Λ, we get, by 1.12 (a), that $rP_i[0] \in \mathrm{Supp}\,(\tau^{-1}rP_t[0], -) \setminus \mathrm{Supp}\,(rP_t[0], -)$. Furthermore $k(\mathbb{Z}\mathbf{D}_N)(rP_j[-1], rP_i[0]) = 0$ because $i \neq j$; proving (a).

(b) Let $P_j/\mathrm{soc}\, P_j \simeq rP_t$ and $P_t/\mathrm{soc}\, P_t \simeq rP_i$. From (a), we know that rP_j lifts to some border of $\mathbb{Z}\mathbf{D}_N$. Let ${}_S\Gamma_\Lambda[0]$ be a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_j; and suppose that $rP_j[0]$ is either in the middle or in the top border of $\mathbb{Z}\mathbf{D}_N$. Then, we get that $rP_j[0]$, $rP_t[0]$ and $rP_i[0]$ are consecutive vertices either in the middle or in the top border of $\mathbb{Z}\mathbf{D}_N$. So, by 1.11, we obtain that $k(\mathbb{Z}\mathbf{D}_N)(rP_j[0], rP_i[0]) \neq 0$; contradicting that $i \neq j$. Therefore $rP_j[0]$ is in the bottom border of $\mathbb{Z}\mathbf{D}_N$. $\quad\square$

Remark 2.5. *In case $i \to t \to j$ is a path in Q_Λ with $i \neq j$ and $P_j/soc P_j \simeq rP_t$, we illustrate in the picture below the shape of the set*

$$\mathrm{Supp}\,(\tau^{-1}rP_t[0], -) \setminus (\mathrm{Supp}\,(rP_t[0], -) \cup \mathrm{Supp}\,(rP_j[-1], -))$$

for $h_{\mathbb{Z}\mathbf{D}_N}(rP_j[-1]) = N - 1$. *The vertices indicated with $\square$, in the following pictures, are the elements of such a set; and one of these vertices is $rP_i[0]$.*
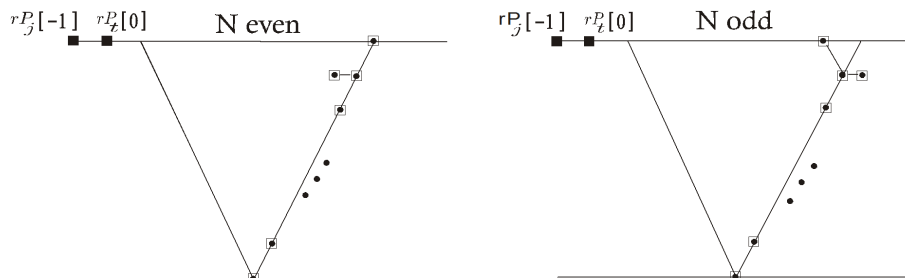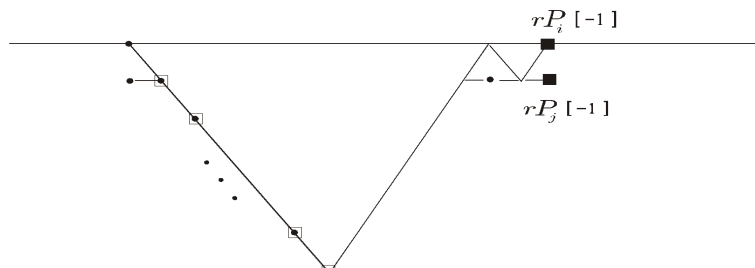

Lemma 2.6. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ with $N > 4$; and let Λ be such that either $i \leftarrow t \rightarrow j$ or $i \rightarrow t \leftarrow j$ is a subquiver of Q_Λ with $rP_i/socP_i \simeq rP_j/socP_j$. Then, for any lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at $rP_j/socP_j$, there is an arrow $rP_i[-1] \rightarrow rP_j/socP_j[0]$ in $\mathbb{Z}\mathbf{D}_N$, where $rP_i[-1]$ is either in the middle or in the top border of $\mathbb{Z}\mathbf{D}_N$.*

In the pictures below, we indicated with $\square$ the possible position of $rP_t[0]$ (respectively $rP_t[-1]$) in $\mathbb{Z}\mathbf{D}_N$.

(a) $rP_t[0]$: *In case $i \leftarrow t \rightarrow j$ is a subquiver of Q_Λ.*
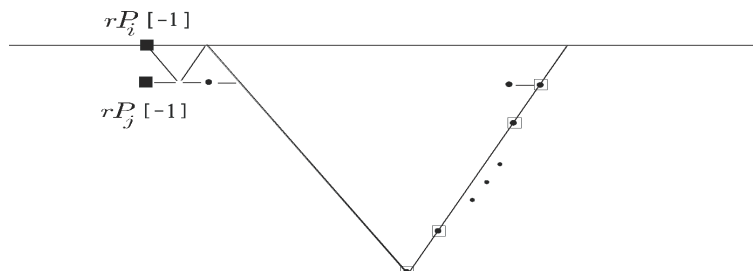
(b) $rP_t[-1]$: *In case $i \rightarrow t \leftarrow j$ is a subquiver of Q_Λ.*

Proof. Let $i \leftarrow t \rightarrow j$ be a subquiver of Q_Λ such that $rP_i/\mathrm{soc}\, P_i \simeq rP_j/\mathrm{soc}\, P_j$ (the other case is dual). By 2.3, we get an arrow $rP_i[-1] \rightarrow rP_j/\mathrm{soc}\, P_j[0]$ in $\mathbb{Z}\mathbf{D}_N$ with $rP_i[-1]$ either in the middle or in the top border of $\mathbb{Z}\mathbf{D}_N$ since $N > 4$. Then, applying 1.12 (a) to the arrows $t \rightarrow j$ and $t \rightarrow i$, we get the result. $\square$

Proposition 2.7. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ with $N > 4$, and let Λ be such that either $i \leftarrow t \rightarrow j$ or $i \rightarrow t \leftarrow j$ is a subquiver of Q_Λ.*

    (a) *The following conditions are equivalent:*

        (a1) $rP_i/\mathrm{soc}\, P_i \simeq rP_j/\mathrm{soc}\, P_j$ *and* $rP_t/\mathrm{soc}\, P_t$ *is indecomposable;*

        (a2) $rP_t[0]$ *belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$.*

    (b) *If $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ and $h_{\mathbb{Z}\mathbf{D}_N}(rP_j[0]) > N - 2$, then*
$$h_{\mathbb{Z}\mathbf{D}_N}(rP_t[0]) \leq N - 2.$$

Proof. Let $_S\Gamma_\Lambda[0]$ be a lifting of $_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_t.

(a) The fact that (a1) implies (a2) follows easily from 2.6. Suppose that $rP_t[0]$ is in the bottom border of $\mathbb{Z}\mathbf{D}_N$ and assume that $i \leftarrow t \rightarrow j$ is a subquiver of Q_Λ (the other case is dual). So, by 1.12 (a), we get that $\tau^{-1}rP_i[-1]$ and $\tau^{-1}rP_j[-1]$ are in $\mathrm{Supp}\,(-, rP_t[0])$. Then $rP_i[-1]$ and $rP_j[-1]$ belong to $\mathrm{Supp}\,(-, \tau rP_t[0])$. On the other hand, since $k(\mathbb{Z}\mathbf{D}_N)(rP_i[0], rP_j[0]) = 0 = k(\mathbb{Z}\mathbf{D}_N)(rP_j[0], rP_i[0])$, we get by 1.11 (a) that the vertices $rP_i[-1]$ and $rP_j[-1]$ correspond to the vertices of $\mathrm{Supp}\,(-, \tau rP_t[0])$ indicated with $\square$ in the picture below.
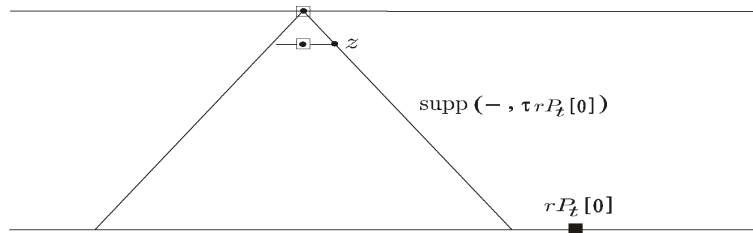
Hence $z = rP_i/\operatorname{soc} P_i[-1] = rP_j/\operatorname{soc} P_j[-1]$; and then

$$rP_i/\operatorname{soc} P_i \simeq rP_j/\operatorname{soc} P_j.$$

(b) Let $i \to t \leftarrow j$ be a subquiver of Q_Λ (for the other case the proof is dual). Suppose that $rP_t[0]$ is in the top border of $\mathbb{Z}\mathbf{D}_N$. Then, by 1.12 (a), we get that $rP_i[0]$ and $rP_j[0]$ are in $\operatorname{Supp}(\tau^{-1}rP_t[0], -)$. The shaded region in the picture below corresponds to the vertices of $\operatorname{Supp}(\tau^{-1}rP_t[0], -)$.
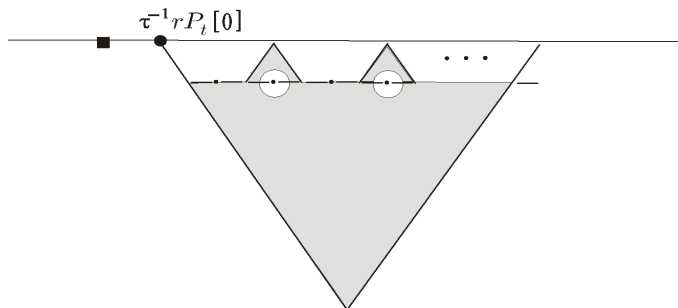
Since $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ and $h_{\mathbb{Z}\mathbf{D}_N}(rP_j[0]) > N - 2$, we get that either $k(\mathbb{Z}\mathbf{D}_N)(rP_i[0], rP_j[0]) \neq 0$ or $k(\mathbb{Z}\mathbf{D}_N)(rP_j[0], rP_i[0]) \neq 0$, which is a contradiction since $k(\mathbb{Z}\mathbf{D}_N)(rP_i[0], rP_j[0])$ and $k(\mathbb{Z}\mathbf{D}_N)(rP_j[0], rP_i[0])$ are both zero because $i \neq j$. Therefore $rP_t[0]$ is not in the top border of $\mathbb{Z}\mathbf{D}_N$. Analogously, it can be proven that $rP_t[0]$ is not in the middle border of $\mathbb{Z}\mathbf{D}_N$. $\square$

2.3. Lifting rP_j to $\mathbb{Z}\mathbf{D}_N$ for some special vertices of Q_Λ.

Proposition 2.8. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$.*

    (a) *Let $\gamma = z_1 \to z_2 \to \cdots \to z_{\ell-1} \to z_\ell$ be a path contained in a elemental cycle of Q_Λ such that $P_{z_\ell}/\operatorname{soc} P_{z_\ell} \simeq rP_{z_{\ell-1}}, \cdots$, $P_{z_2}/\operatorname{soc} P_{z_2} \simeq rP_{z_1}$. Then, for any lifting of $_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_{z_ℓ}, we have that $rP_{z_\ell}[0], rP_{z_{\ell-1}}[0], \cdots, rP_{z_1}[0]$ are consecutive vertices in one of the three borders of $\mathbb{Z}\mathbf{D}_N$; and moreover, this border is the bottom one if $\ell \geq 3$.*

    (b) *If Λ is of class I then $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ for $i = 1, 4$.*

    (c) *Let Λ be of class II and $i = 1, 2, 3$. Then, we have that $rP_i[0]$ belongs to some border of $\mathbb{Z}\mathbf{D}_N$; and moreover, if $N > 4$ and Λ is either of class II_2 or II_3, then $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$.*

Proof.   (a) Follows easily from 2.4.

(b) Since Λ is of class I, by 2.2 (b), we get that $rP_1/\operatorname{soc} P_1 \simeq rP_4/\operatorname{soc} P_4$. Then, by 2.3 (b), we obtain the result.

(c) By 2.2 (c), we have that $rP_i/\operatorname{soc} P_i$ is indecomposable; and therefore, $rP_i[0]$ is in some border of $\mathbb{Z}\mathbf{D}_N$. Suppose that $N > 4$ and Λ is either of class II_2 or II_3. So, by 2.2 (c2), we conclude that $rP_{c_t}/\operatorname{soc} P_{c_t} \not\simeq rP_3/\operatorname{soc} P_3$, $rP_{b_m}/\operatorname{soc} P_{b_m} \not\simeq rP_2/\operatorname{soc} P_2$, and $rP_{c_1}/\operatorname{soc} P_{c_1} \not\simeq rP_3/\operatorname{soc} P_3$. Therefore, applying 2.7 to the arrows $c_t \to 1 \leftarrow 3, 2 \to 3 \leftarrow b_m, c_1 \leftarrow 2 \to 3$ of Q_Λ, we get that $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ since $rP_i[0]$ belongs to some border of $\mathbb{Z}\mathbf{D}_N$ for $i = 1, 2, 3$. $\square$

Proposition 2.9. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ with Λ of class II_1 and $N > 4$. Then, the following statements hold.*

    (a) *$h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ for $i = 1, 2, 3$.*

    (b) *For any lifting of $_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_3 such that $h_{\mathbb{Z}\mathbf{D}_N}(rP_3[0]) = N - 1$, we have that rP_1, rP_2 and rP_3 lift to $\mathbb{Z}\mathbf{D}_N$ as follows*
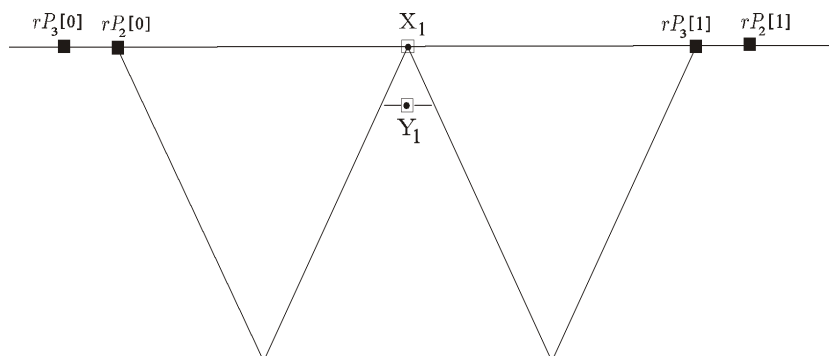
$$\text{where } rP_1[0] = \begin{cases} X_1 & \text{if } N \text{ is odd,} \\ Y_1 & \text{if } N \text{ is even.} \end{cases}$$

Proof.   By 2.2 (c1), it follows $rP_1/\mathrm{soc}\, P_1 \not\simeq rP_{a_n}/\mathrm{soc}\, P_{a_n}$ and $P_3/\mathrm{soc}\, P_3 \simeq rP_2$. Hence, applying 2.7 to the subquiver $1 \rightarrow 2 \leftarrow a_n$ of Q_Λ, we get that $rP_2[0]$ is not in the bottom border of $\mathbb{Z}\mathbf{D}_N$. On the other hand, by 2.8 (c), we know that $rP_2[0]$ is in some border of $\mathbb{Z}\mathbf{D}_N$. Hence, we conclude that $h_{\mathbb{Z}\mathbf{D}_N}(rP_2[0]) > N - 2$.

Furthermore, since $\tau^{-1}rP_3 \simeq rP_2$, we have that rP_3 and rP_2 lift consecutively to the same border of $\mathbb{Z}\mathbf{D}_N$. Thus, for a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_3 with $h_{\mathbb{Z}\mathbf{D}_N}(rP_3[0]) = N - 1$, we get the picture given in 2.9 (b). We prove now that $rP_1[0] = X_1$ if N is odd, otherwise $rP_1[0] = Y_1$.

Since $1 \rightarrow 2$ and $3 \rightarrow 1$ are arrows of Q_Λ, we conclude by 1.12 (a) that $rP_1[0] \in \mathrm{Supp}\,(\tau^{-1}rP_2[0], -)$ and $\tau^{-1}rP_1[0] \in \mathrm{Supp}\,(-, rP_3[1])$; and so,
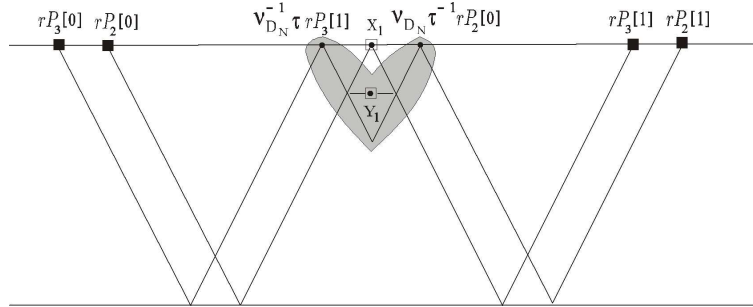
$$rP_1[0] \in \mathrm{Supp}\,(\tau^{-1}rP_2[0], -) \cap \mathrm{Supp}\,(-, \tau rP_3[1]).$$

Furthermore, by [8, Proposition 3.1] $\mathrm{Supp}\,(x, -) = \mathrm{Supp}\,(-, \nu_{\mathbf{D}_N}(x))$ for any vertex x of $\mathbb{Z}\mathbf{D}_N$. Thus

$$rP_1[0] \in \mathrm{Supp}\,(-, \nu_{\mathbf{D}_N}(\tau^{-1}rP_2[0]) \cap \mathrm{Supp}\,(\nu_{\mathbf{D}_N}^{-1}(\tau rP_3[1]), -).$$

Suppose that N is even. Then, by 1.11 (b), the shaded region of the picture below correspond to the vertices of the set

$$\mathrm{Supp}\,(\nu_{\mathbf{D}_N}^{-1}(\tau rP_3[1]), -) \cap \mathrm{Supp}\,(-, \nu_{\mathbf{D}_N}(\tau^{-1}rP_2[0]).$$

Since $rP_1[0]$ is in some border of $\mathbb{Z}\mathbf{D}_N$, we get that

$$rP_1[0] \in \{\nu_{\mathbf{D}_N}^{-1}(\tau rP_3[1]), Y_1, \nu_{\mathbf{D}_N}(\tau^{-1}rP_2[0])\}.$$

Therefore, $rP_1[0] = Y_1$ since $k(\mathbb{Z}\mathbf{D}_N)(rP_3[0], \nu_{\mathbf{D}_N}^{-1}(\tau rP_3[1])) \neq 0$ and $k(\mathbb{Z}\mathbf{D}_N)(\nu_{\mathbf{D}_N}(\tau^{-1}rP_2[0]), rP_2[1]) \neq 0$. Analogously, it can be proven that $rP_1[0] = X_1$ if N is odd. $\square$

Proposition 2.10. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ with Λ of class III. Then, the following statements hold.*

(a) $h_{\mathbb{Z}\mathbf{D}_N}(rP_1[0]) = N - 2$.

(b) *For $i = 2, 3$, we have that $rP_i[0]$ belongs to some border of $\mathbb{Z}\mathbf{D}_N$; and moreover, $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ if $N > 4$.*

(c) *For any lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_1 such that $h_{\mathbb{Z}\mathbf{D}_N}(rP_3[0]) = N - 1$, we have that rP_1, rP_2 and rP_3 lift to $\mathbb{Z}\mathbf{D}_N$ as follows*
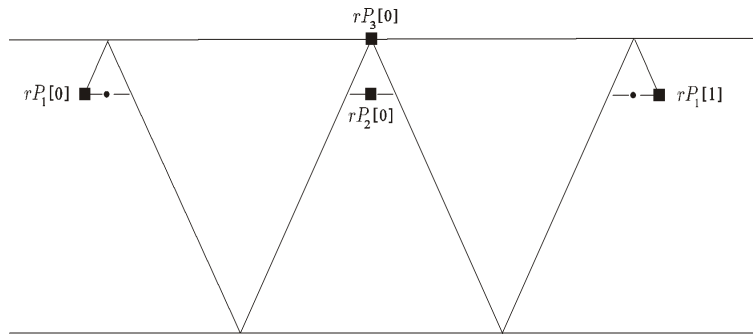
Proof. By 2.2 (d), we have that $rP_1/\mathrm{soc}\, P_1$ has three non-isomorphic indecomposable direct summands; so, we get that $h_{\mathbb{Z}\mathbf{D}_N}(rP_1[0]) = N - 2$. On the other hand, by 2.2 (d), we obtain that $rP_2/\mathrm{soc}\, P_2 \simeq rP_3/\mathrm{soc}\, P_3$. Therefore, by 2.3 (a), it follows that $rP_i[0]$ belongs to some border of $\mathbb{Z}\mathbf{D}_N$ for $i = 2, 3$; and moreover, $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ if $N > 4$.

Let $N > 4$, and consider a lifting of $_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_1 with $h_{\mathbb{Z}\mathbf{D}_N}(rP_3[0]) = N - 1$. Applying 1.12 to the arrows $2 \to 1$ and $3 \to 1$ of Q_Λ, we obtain that $rP_i[0] \in \mathrm{Supp}\,(\tau^{-1}rP_1[0], -) \setminus \mathrm{Supp}\,(rP_1[0], -)$ for $i = 2, 3$. In the picture below, we have indicated with a circle the elements of $\mathrm{Supp}\,(\tau^{-1}rP_1[0], -) \setminus \mathrm{Supp}\,(rP_1[0], -)$ which are in some border of $\mathbb{Z}\mathbf{D}_N$.
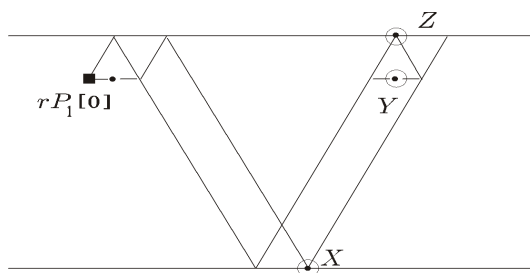
Since $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ for $i = 2, 3$, we have that $\{rP_2[0], rP_3[0]\} = \{Y, Z\}$. Hence, using that $h_{\mathbb{Z}\mathbf{D}_N}(rP_3[0]) = N - 1$, it follows that $Y = rP_2[0]$ and $Z = rP_3[0]$. $\square$
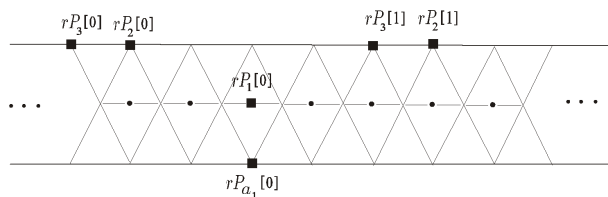
2.4. The configurations for the case $\mathbf{D}_4$.

In the following proposition, we study more closely the trivial extensions Λ of Cartan class $\mathbf{D}_4$. In this case, we have that Λ is either of class II_1 or III. From 2.8 (c) and 2.10 (b), we know that rP_i can be lifted to any border of $\mathbb{Z}\mathbf{D}_4$ for any vertex $i \in Q_\Lambda$ ($i \neq 1$ if Λ is of class III). We choose the universal covering $\pi : \mathbb{Z}\mathbf{D}_4 \to {}_S\Gamma_\Lambda$ such that $h_{\mathbb{Z}\mathbf{D}_4}(rP_t[0]) > 2$ for some fixed vertices $t = t_1, t_2$ of Q_Λ.

Proposition 2.11. *Let Λ be the trivial extension of Cartan class $\mathbf{D}_4$ given by one of the following quivers*
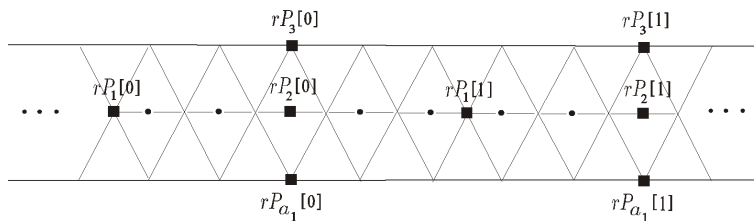
Let $\pi : \mathbb{Z}\mathbf{D}_4 \rightarrow {}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$ with $h_{\mathbb{Z}\mathbf{D}_4}(\pi^{-1}(rP_3)) = 3$ and $h_{\mathbb{Z}\mathbf{D}_4}(\pi^{-1}(rP_{a_1})) = 1$.

(a) If Λ is of class II_1 and ${}_S\Gamma_\Lambda$ is lifted to $\mathbb{Z}\mathbf{D}_4$ at rP_3, then the configuration $\widetilde{\mathcal{C}_\Lambda}$ of $\mathbb{Z}\mathbf{D}_4$ associated to Λ is

(b) If Λ is of class III and ${}_S\Gamma_\Lambda$ is lifted to $\mathbb{Z}\mathbf{D}_4$ at rP_1, then the configuration $\widetilde{\mathcal{C}_\Lambda}$ of $\mathbb{Z}\mathbf{D}_4$ associated to Λ is


Proof.   It is straightforward. $\square$

2.5. The configurations for Λ minimal of type $\mathbf{D}_N$ with $N > 4$.

In what follows, we determine the vertices of $\mathbb{Z}\mathbf{D}_N$ corresponding to the configuration $\widetilde{\mathcal{C}_\Lambda}$ if Λ is minimal and $N > 4$.

Proposition 2.12. *Let Λ be a trivial extension of Class I and Cartan class $\mathbf{D}_N$ with $N > 4$, $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$ such that $h_{\mathbb{Z}\mathbf{D}_N}(\pi^{-1}(rP_1)) = N - 1$; and take a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at $\tau(rP_1/\mathrm{soc}\, P_1)$. Then rP_1 and rP_4 lift to $\mathbb{Z}\mathbf{D}_N$ as in the picture below; and furthermore, if Λ is minimal (see the picture in 1.2), the configuration $\widetilde{\mathcal{C}_\Lambda}$ of $\mathbb{Z}\mathbf{D}_N$ associated to Λ is as follows*
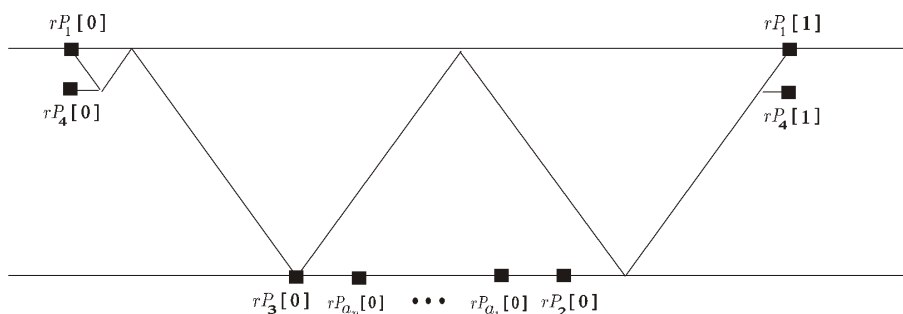

Proof. By 2.2 (b), we know that $rP_1/\mathrm{soc}\, P_1 \simeq rP_4/\mathrm{soc}\, P_4$. Then, from 2.3 (a), we get that $rP_1[0]$ and $rP_4[0]$ are immediate predecessors of $rP_1/\mathrm{soc}\, P_1[0]$ and $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ for $i = 1, 4$ since $N > 4$. Using that the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ we have chosen is such that $h_{\mathbb{Z}\mathbf{D}_N}(rP_1[0]) = N - 1$, we conclude that rP_1 and rP_4 lift to $\mathbb{Z}\mathbf{D}_N$ as in the previous picture. On the other hand, by 1.5 and since Λ is minimal of class I, it can be proven that $P_3/\mathrm{soc}\, P_3 \simeq rP_{a_n}, P_{a_n}/\mathrm{soc}\, P_{a_n} \simeq rP_{a_{n-1}}, \cdots, P_{a_2}/\mathrm{soc}\, P_{a_2} \simeq rP_{a_1}, P_{a_1}/\mathrm{soc}\, P_{a_1} \simeq rP_2$. So, applying 2.8 (a) to the path $2 \to a_1 \to a_2 \to \cdots \to a_n \to 3$ in Q_Λ, we get that the radicals $rP_3, rP_{a_n}, \cdots, rP_{a_1}, rP_2$ lift consecutively to the bottom border of $\mathbb{Z}\mathbf{D}_N$. On the other hand $rP_3[0] \in \mathrm{Supp}\,(\tau^{-1}rP_1[0], -)$ since $1 \leftarrow 3$ is an arrow in Q_Λ; and so, we get the previous picture, proving the result. $\square$

Let Λ be a trivial extension of class II. We recall (see 1.2, 1.3 and 1.9) that the class II is divided in 3 subclasses. Moreover, in case

$N > 4$, we have by 2.8 (c) and 2.9 (a) that $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) > N - 2$ for the vertices $i = 1, 2, 3$ of Q_Λ.

Lemma 2.13. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ with $N > 4$ and Λ minimal of class II. For $m > 0$, we consider the cycle $B = 3 \to 1 \to b_1 \to b_2 \to \cdots \to b_m \to 3$ of Q_Λ and the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ of ${}_S\Gamma_\Lambda$ with $h_{\mathbb{Z}\mathbf{D}_N}(\pi^{-1}(rP_3)) = N - 1$. Then, for a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_3, the radical of the projective associated to each vertex of the cycle B lifts to $\mathbb{Z}\mathbf{D}_N$ as is indicated in the following picture*
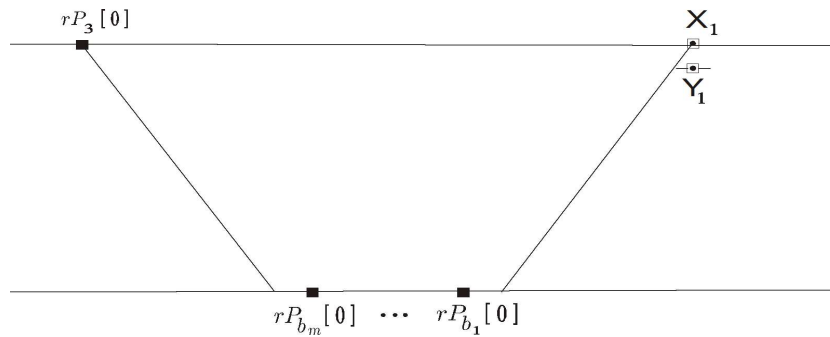
where $rP_1[0] = \begin{cases} X_1 & \text{if } N - m - 3 \text{ is even,} \\ Y_1 & \text{if } N - m - 3 \text{ is odd.} \end{cases}$

Proof.   Assume that $m > 0$, and let γ be the path $b_1 \to b_2 \to \cdots \to b_m$ in Q_Λ. Since Λ is minimal of class II, we get by 1.8 and 2.1 that γ belongs to a single elemental cycle of Q_Λ and that $b_1, b_2, \cdots, b_m$ are insertion vertices of Q_Λ. So we obtain from 2.2 (a) the following isomorphisms

$$P_{b_m}/\operatorname{soc} P_{b_m} \simeq rP_{b_{m-1}}, \cdots, P_{b_2}/\operatorname{soc} P_{b_2} \simeq rP_{b_1}.$$

Hence, by 2.8 (a), we have that $rP_{b_m}, rP_{b_{m-1}}, \cdots, rP_{b_1}$ lift consecutively to some border of $\mathbb{Z}\mathbf{D}_N$. We assert that this border is the bottom one.   Indeed, suppose that $h_{\mathbb{Z}\mathbf{D}_N}(rP_{b_m}[0]) > N - 2$. Applying 2.7 (b) to the subquiver $2 \to 3 \leftarrow b_m$ of Q_Λ, we obtain that $h_{\mathbb{Z}\mathbf{D}_N}(rP_3[0]) \leq N - 2$; giving a contradiction. Thus $rP_{b_m}[0]$ is in the bottom border of $\mathbb{Z}\mathbf{D}_N$ and $\tau^{-i}rP_{b_m}[0] = rP_{b_{m-i}}[0]$ for $i = 1, 2, \cdots, m-1$. On the other hand, applying 1.12 (a) to the arrows $b_m \to 3, 1 \to b_1$ of Q_Λ, we have that $rP_{b_m}[0] \in \operatorname{Supp}(\tau^{-1}rP_3[0], -)$

and $rP_1[0] \in \mathrm{Supp}\,(\tau^{-1}rP_{b_1}[0], -)$. So we get the previous picture and also that $rP_1[0] \in \{X_1, Y_1\}$ since $h_{\mathbb{Z}\mathbf{D}_N}(rP_1[0]) > N - 2$. Now, we prove that $rP_1[0]$ is either X_1 or Y_1 depending on the parity of $N - m - 3$.

Since $3 \to 1$ is an arrow in Q_Λ, we have from 1.2 and [8, Proposition 3.1 (a)] that

$$rP_1[0] \in \mathrm{Supp}\,(-, \tau r P_3[1]) = \mathrm{Supp}\,(\nu_{\mathbf{D}_N}^{-1}(\tau r P_3[1]), -).$$

Suppose that N is even (if N is odd the proof is likewise). Consider the following picture.
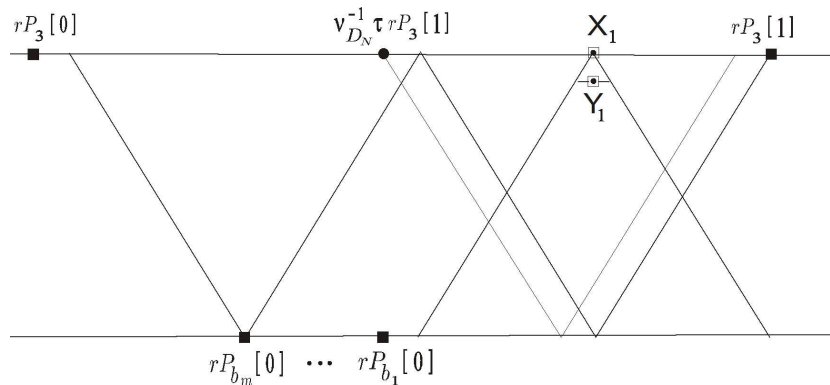
Since $rP_1[0] \in \mathrm{Supp}\,(\nu_{\mathbf{D}_N}^{-1}(\tau r P_3[1]), -) \cap \{X_1, Y_1\}$, we obtain from 1.11 (a) that

$$rP_1[0] = \begin{cases} X_1 & \text{if } m+1 \text{ is even,} \\ Y_1 & \text{si } m+1 \text{ is odd.} \end{cases}$$

Using that $N - m - 3 = N - (m + 1) - 2$ and N is even, it follows that $N - m - 3$ is even (resp. odd) if and only if $m + 1$ is even (resp. odd); proving the result. $\square$

Proposition 2.14. *Let Λ be a minimal trivial extension of Class II and Cartan class $\mathbf{D}_N$ with $N > 4$. Consider the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ of ${}_S\Gamma_\Lambda$ such that $h_{\mathbb{Z}\mathbf{D}_N}(\pi^{-1}(rP_3)) = N - 1$. Then, for a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_3, the configuration $\widetilde{\mathcal{C}_\Lambda}$ associated to Λ is as follows*
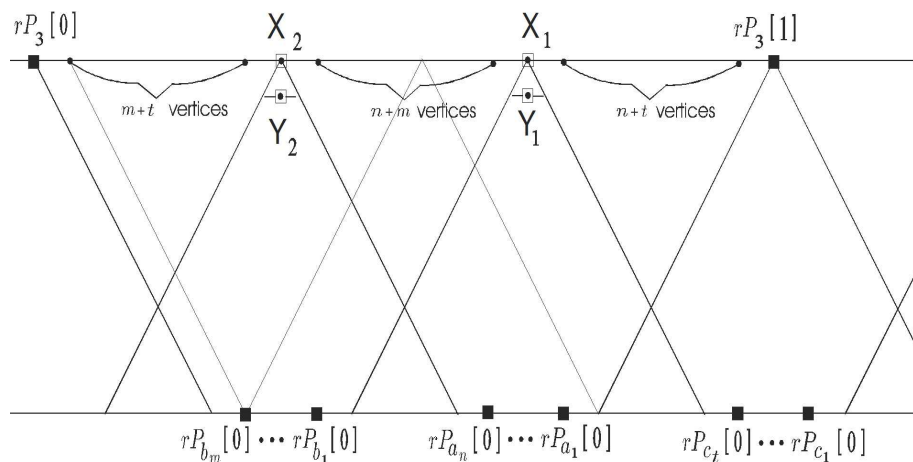
$$\text{where } rP_1[0] = \begin{cases} X_1 & \text{if } t+n \text{ is even,} \\ Y_1 & \text{if } t+n \text{ is odd.} \end{cases} \quad \text{and}$$

$$rP_2[0] = \begin{cases} X_2 & \text{if } t+m \text{ is even,} \\ Y_2 & \text{if } t+m \text{ is odd.} \end{cases}$$

Proof.   Suppose that Λ is minimal of class II_1. By 2.9 (b), we have that rP_2 and rP_3 lift to the same border of $\mathbb{Z}\mathbf{D}_N$. So, we can apply 2.13 to the cycle $A = 2 \to 3 \to a_1 \to \cdots \to a_n \to 2$ of Q_Λ; and so, from 2.9, we get that the configuration $\widetilde{\mathcal{C}_\Lambda}$ associated to Λ is as follows
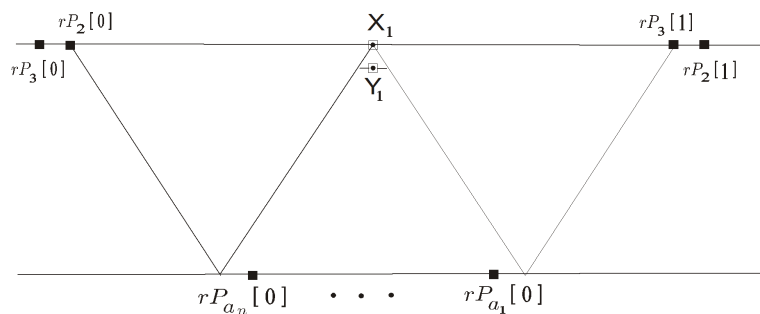
where $rP_1[0] = \begin{cases} X_1 & \text{if } N \text{ is odd,} \\ Y_1 & \text{if } N \text{ is even.} \end{cases}$

We know that $N = n+3$ since Λ is minimal of class II_1; and so, N is odd if and only if n is even, proving the result for Λ of class II_1.

Suppose that Λ is either of class II_2 or II_3. Applying 1.12 (a) to the arrows $c_t \to 1 \to b_1$, we have that $rP_1[0] \in \mathrm{Supp}\,(\tau^{-1}rP_{b_1}[0], -)$ and $rP_{c_t}[0] \in \mathrm{Supp}\,(\tau^{-1}rP_1[0], -)$; and so, by applying 2.13 to the elemental cycles $1 \to 2 \to c_1 \to c_2 \to \cdots \to c_t \to 1$ and $3 \to 1 \to b_1 \to b_2 \to \cdots \to b_m \to 3$ of Q_Λ, we obtain the following picture
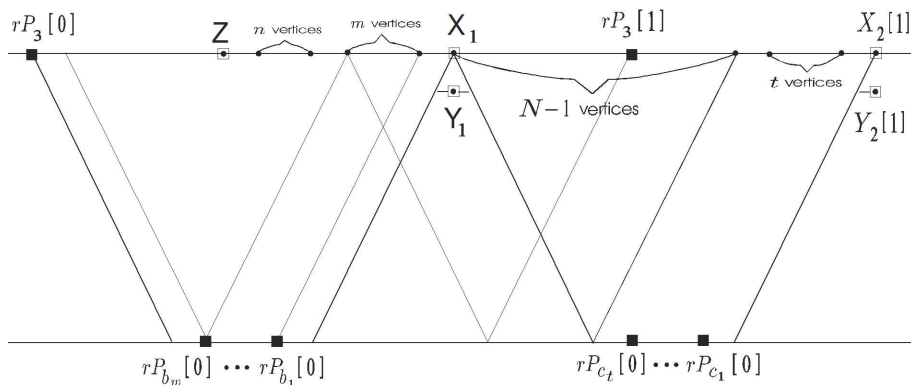

where $rP_1[0] \in \{X_1, Y_1\}$ since $h_{\mathbb{Z}\mathbf{D}_N}(rP_1[0]) > N - 2$. Moreover, we have the arrow $2 \to c_1$ in Q_Λ; and therefore, by 1.12, we obtain that $rP_2[1] \in \{X_2[1], Y_2[1]\}$ since $h_{\mathbb{Z}\mathbf{D}_N}(rP_2[1]) > N - 2$.

We assert that $Z = X_2$ for the Z in the above picture. Indeed, since Λ is minimal, we get $N = n+m+t+3$. Thus $1+n+m+N-1+t = 2N-3$ and hence $z = \tau^{1+n+m+N-1+t}X_2[1] = \tau^{2N-3}X_2[1] = X_2$.

We assert that $rP_2[0]$ and $rP_1[0]$ satisfy the stated equalities. For $rP_1[0]$, it can be obtained from 2.13 using that $N = n + m + t + 3$. For $rP_2[0]$, we apply 1.12 (a) to the arrow $2 \to 3$ in Q_Λ obtaining that $rP_2[1] \in \mathrm{Supp}\,(\tau^{-1}rP_3[1], -)$. Furthermore, from the above picture, it can be seen that $\tau^{-(m+t+1)}rP_3[1] = X_2[1]$. Then, by 1.11 (b), we conclude that $rP_2[1] = X_2[1]$ if $t + m$ is even, and $rP_2[1] = Y_2[1]$ if $t + m$ is odd. $\square$

Proposition 2.15. *Let Λ be a minimal trivial extension of Class III (see 1.2) and Cartan class $\mathbf{D}_N$ with $N > 4$. Consider the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \rightarrow {}_S\Gamma_\Lambda$ of ${}_S\Gamma_\Lambda$ with $h_{\mathbb{Z}\mathbf{D}_N}(\pi^{-1}(rP_3)) = N - 1$. Then, for a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_1, the configuration $\widetilde{\mathcal{C}_\Lambda}$ associated to Λ is as follows*
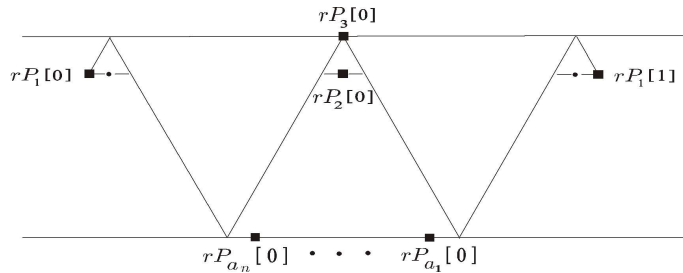
Proof. The radicals rP_1, rP_2 and rP_3 lift to $\mathbb{Z}\mathbf{D}_N$ as in 2.10. We prove that $rP_{a_n}, \cdots, rP_{a_1}$ lift consecutively to the bottom border of $\mathbb{Z}\mathbf{D}_N$ as in the preceding picture. Applying 1.12 (a) to the arrows $2 \rightarrow 1, 3 \rightarrow 1$ and $a_n \rightarrow 1$ of Q_Λ, we have $rP_i[0] \in \mathrm{Supp}\,(\tau^{-1}rP_1[0], -) \setminus \mathrm{Supp}\,(rP_1[0], -)$ for $i = 2, 3, a_n$. In the picture given in the proof of 2.10, we indicated with a circle the elements X, Y and Z of $\mathrm{Supp}\,(\tau^{-1}rP_1[0], -) \setminus \mathrm{Supp}\,(rP_1[0], -)$ which are in some border of $\mathbb{Z}\mathbf{D}_N$.
Consider the path $\gamma = a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n$ of Q_Λ. Since Λ is minimal of class III, we obtain that γ belongs to a elemental cycle of Q_Λ and also that $a_1, a_2, \cdots, a_n$ are insertion vertices (see 1.8 and 2.1) in Q_Λ. Hence, by 2.2 (a), we get that $P_{a_n}/\mathrm{soc}\, P_{a_n} \simeq rP_{a_{n-1}}, \cdots, P_{a_2}/\mathrm{soc}\, P_{a_2} \simeq rP_{a_1}$. Then, by 2.8 (a), the radicals rP_{a_n}, $rP_{a_{n-1}}, \cdots, rP_{a_1}$ lift consecutively to some border of $\mathbb{Z}\mathbf{D}_N$. Thus, it is enough to prove that $X = rP_{a_n}[0]$ (see the picture given in the proof of 2.10). By 2.10, we know that $Y = rP_3[0]$ and $Z = rP_2[0]$; therefore $X = rP_{a_n}[0]$, proving the result. $\square$

2.6. Lifting rP_j to $\mathbb{Z}\mathbf{D}_N$ for the vertices j belonging to an elimination cycle of Q_Λ.

In order to find, in the general case, the configuration $\widetilde{\mathcal{C}_\Lambda}$ associated to a given trivial extension Λ, we need firstly to know how can be lifted to $\mathbb{Z}\mathbf{D}_N$ the radicals of the projective modules associated with

the vertices of an elimination cycle of Q_Λ (see 2.1). So, the following result will be very useful throughout this paper.

Theorem 2.16. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ with $N > 4$, and $C = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ an elimination cycle of Q_Λ. Then, for a lifting $_S\Gamma_\Lambda[0]$ of $_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_{z_1}, we have that*

(a) $h_{\mathbb{Z}\mathbf{D}_N}(rP_{z_1}[0]) = 1$ *and* $\tau^{-i}rP_{z_1}[0] = rP_{z_{i+1}}[0]$ *for* $1 \leq i \leq m-2$,

(b) $\mathrm{Supp}\,(\tau^{-1}rP_{z_{m-1}}[0], -) \cap \mathrm{Supp}\,(-, \tau rP_{z_1}[1]) = \{rP_z[0]\}$, *and*

(c) $h_{\mathbb{Z}\mathbf{D}_N}(rP_z[0]) = m$.

Proof.   (a) Consider the path $\gamma := z_{m-1} \to z_{m-2} \to \cdots \to z_1$ in Q_Λ. Then γ belongs to the elemental cycle C in Q_Λ. Therefore $z_1, z_2, \cdots, z_{m-1}$ are insertion vertices of Q_Λ. Hence, by 2.2 (a), we get the isomorphisms $P_{z_1}/\mathrm{soc}\, P_{z_1} \simeq rP_{z_2}, \cdots, P_{z_{m-2}}/\mathrm{soc}\, P_{z_{m-2}} \simeq rP_{z_{m-1}}$. Thus, by 2.8 (a), it follows that $\tau^{-i}rP_{z_1}[0] = rP_{z_{i+1}}[0]$ for $1 \leq i \leq m-2$ and $rP_{z_1}[0]$ belongs to some border of $\mathbb{Z}\mathbf{D}_N$; furthermore, this border is the bottom one if $m \geq 4$. So, we have to analyze the cases $m = 2$ and $m = 3$.

Suppose that $m = 2$. Then $C = z \leftarrow z_1 \leftarrow z$; and hence by 1.12 (a) we get that $rP_z[0] \in \mathrm{Supp}\,(\tau^{-1}rP_{z_1}[0], -) \cap \mathrm{Supp}\,(-, \tau rP_{z_1}[1])$. If $h_{\mathbb{Z}\mathbf{D}_N}(rP_{z_1}[0]) > N-2$. Then, by using 1.11, it is not difficult to see that $h_{\mathbb{Z}\mathbf{D}_N}(rP_z[0]) = N-2$ and $\mathrm{Supp}\,(\tau^{-1}rP_{z_1}[0], -) \cap \mathrm{Supp}\,(-, \tau rP_{z_1}[1]) = \{rP_z[0]\}$. In the following picture, we illustrate this fact if $h_{\mathbb{Z}\mathbf{D}_N}(rP_{z_1}[0]) = N-1$ and N is even.
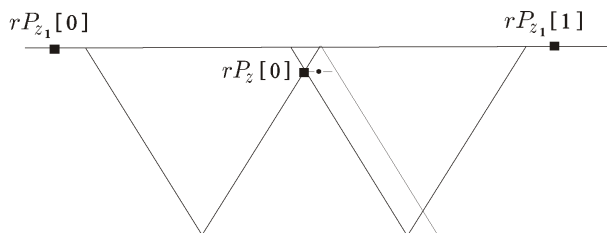
The equality $h_{\mathbb{Z}\mathbf{D}_N}(rP_z[0]) = N-2$ implies that $rP_z/\mathrm{soc}\, P_z$ has three indecomposable non-isomorphic direct summands. Using, on one hand, the combinatorial description (see [6, Corollary 4.12]) of

the number of the indecomposable non-isomorphic direct summands of $rP_j/\operatorname{soc}P_j$ for a vertex j in Q_Λ; and on the other hand, the description of Λ given in 1.8, it can be proven that Λ is of class III and $z = 1$. Moreover, Λ is minimal since C is an elimination cycle of Q_Λ. Furthermore, since $N > 4$, we get that $C = 1 \leftarrow a_n \leftarrow \cdots \leftarrow a_1 \leftarrow 1$, contradicting that C is an elimination cycle of Q_Λ (see 1.8). Hence, for $m = 2$, we have that $rP_{z_1}[0]$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$.

Assume that $m = 3$. Then $C = z \leftarrow z_1 \leftarrow z_2 \leftarrow z$; and so $P_{z_1}/\operatorname{soc}P_{z_1} \simeq rP_{z_2}$. Suppose that $h_{\mathbb{Z}\mathbf{D}_N}(rP_{z_1}[0]) > N - 2$. Since $\tau^{-1}rP_{z_1}[0] = rP_{z_2}[0]$, it can be proven by using 1.11 that there is only one vertex $x \in {}_S\Gamma_\Lambda[0]$ such that $h_{\mathbb{Z}\mathbf{D}_N}(x) > N - 2$ and $x \notin \operatorname{Supp}(rP_{z_i}[0], -) \cup \operatorname{Supp}(-, rP_{z_i}[1])$ for $i = 1, 2$. In the following picture, we illustrate this fact in case $h_{\mathbb{Z}\mathbf{D}_N}(rP_{z_1}[0]) = N - 1$, where $x = X_1$ if N is odd and $x = X_2$ if N is even.
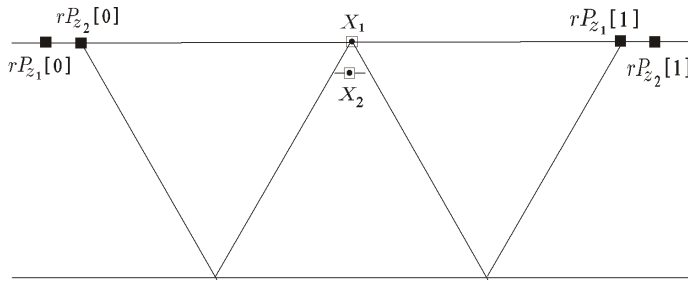
On the other hand, since C is an elimination cycle of Q_Λ, we conclude from 2.8 (b) and (c), 2.9 (a) and 2.10 (b), that there are at least two vertices $i, j \in Q_\Lambda$ such that $i, j \notin \{z_1, z_2\}$ and $h_{\mathbb{Z}\mathbf{D}_N}(rP_t[0]) > N - 2$ for $t = i, j$. Then we get two vertices $rP_i[0]$ and $rP_j[0]$ in ${}_S\Gamma_\Lambda[0]$ satisfying the same property as the vertex x, contradicting the fact that there exists just one. Therefore, for $m = 3$, we also obtain that $rP_{z_1}[0]$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$.

Finally, the proof of (b) and (c) follows easily from (a), 1.11 (a) and 1.12 (a). $\square$

Remark 2.17. *Let Λ and $C = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ be as in 2.16. Then the radicals $rP_{z_1}, rP_{z_2}, \cdots, rP_{z_{m-1}}, rP_z$ lift to $\mathbb{Z}\mathbf{D}_N$ as in the following picture*
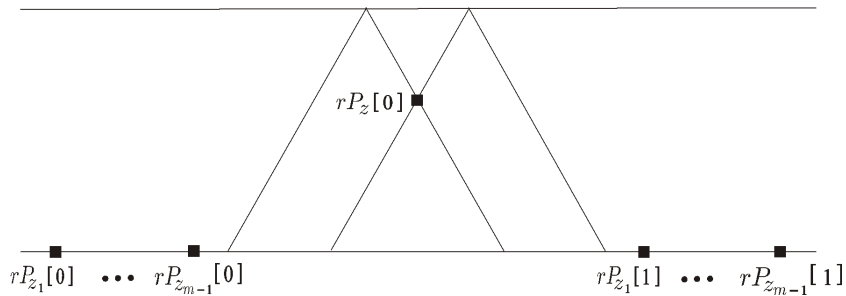
## 3. The embedding of $_S\Gamma_{\mathrm{End}_\Lambda(P)^{op}}$ in $_S\Gamma_\Lambda$

For a given artin algebra Λ, we will use freely properties of the module category $\mathrm{mod}\,\Lambda$ of finitely generated left Λ-modules, the stable category $\underline{\mathrm{mod}}\,\Lambda$ module projectives and the Auslander-Reiten translations τ and τ^{-1} as is found in [3].

Let Γ be a trivial extension of Cartan class $\mathbf{D}_N$, z an insertion vertex of Q_Γ (see 2.1) and Λ be the trivial extension of Cartan class $\mathbf{D}_{N+m-1}$ obtained from Γ by inserting the cycle $C_z = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. We will embed $_S\Gamma_\Gamma$ into $_S\Gamma_\Lambda$. This will allows us to deduce properties of $_S\Gamma_\Lambda$ from properties of $_S\Gamma_\Gamma$. Since Q_Γ has less cycles than Q_Λ, we will use this embedding to reduce our problem to the minimal case, studied in the previous section.

We recall that $\Gamma \simeq \mathrm{End}_\Lambda(P)^{op}$ where $P := \coprod_{i \in (Q_\Gamma)_0} {}_\Lambda P_i$. Consider (see [9, Section 2]) the full subcategory $\mathcal{C}_P$ of $\mathrm{mod}\,\Lambda$ whose objects are the modules X such that the projective cover $P_0(X)$ of X and the injective envelope $I_0(X)$ of X belong to $\mathrm{add}\,P$. Since Λ is a weakly-symmetric algebra, from [9, Section 2], we have that the evaluation functor $e_P = \mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$, allowing the identification of $\mathrm{mod}\,\Gamma$ with the full subcategory $\mathcal{C}_P$ of $\mathrm{mod}\,\Lambda$, induces an equivalence $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\mathrm{mod}}\,\Gamma$ of stable categories. Let $\imath : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\,\Lambda$ be the full and faithful functor obtained by the composition of the inverse equivalence of $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\mathrm{mod}}\,\Gamma$ and the inclusion $\underline{\mathcal{C}_P} \subseteq \underline{\mathrm{mod}}\,\Lambda$. In this section, we will study the behavior of the irreducible morphisms of $\underline{\mathrm{mod}}\,\Gamma$ inside $\underline{\mathrm{mod}}\,\Lambda$ using the embedding $\imath : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\,\Lambda$.

Proposition 3.1. *Let Γ be a trivial extension of Cartan class $\mathbf{D}_N$, z an insertion vertex of Q_Γ and Λ be the trivial extension obtained from Γ by inserting the cycle $C_z = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. Consider the projective Λ-module $P = \coprod_{i \in (Q_\Gamma)_0} {}_\Lambda P_i$ and the equivalence of stable categories $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\mathrm{mod}}\,\Gamma$. Then, the following statements hold.*

(a) $\mathrm{ind}\,\underline{\mathcal{C}_P}$ *decomposes in four connected components* X^-, X^+, Y^- *and* Y^+ *which are convex subquivers of* ${}_S\Gamma_\Lambda$ *as in the picture below*
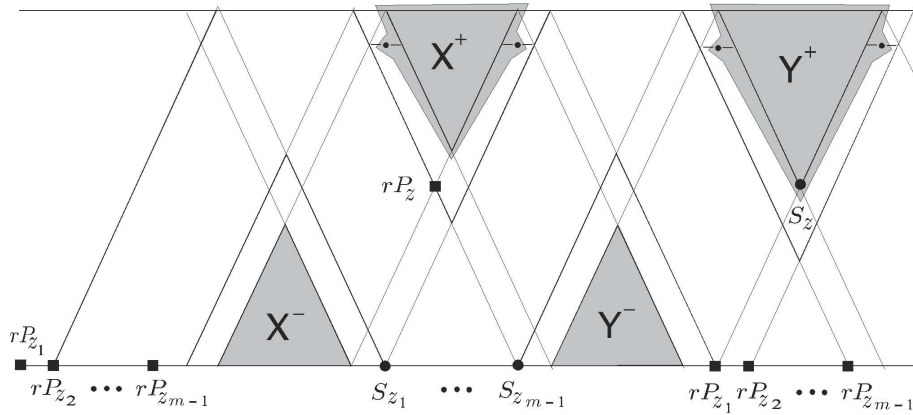
(b) *For any $X, Y \in \mathrm{ind}\,\underline{\mathcal{C}_P}$ the following conditions are equivalent.*

(b1) $f : X \to Y$ *is irreducible in* $\underline{\mathrm{mod}}\,\Lambda$.

(b2) X *and* Y *are in the same connected component of* $\mathrm{ind}\,\underline{\mathcal{C}_P}$ *in* ${}_S\Gamma_\Lambda$ *and* $\underline{e}_P(f) : \underline{e}_P(X) \to \underline{e}_P(Y)$ *is irreducible in* $\underline{\mathrm{mod}}\,\Gamma$.

(c) *If χ is one of the quivers $\mathsf{X}^\pm$, $\mathsf{Y}^\pm$, then $\underline{e}_P(\chi)$ is a convex subquiver of ${}_S\Gamma_{\mathrm{End}_\Lambda(P)^{op}}$ and $\underline{e}_P|_\chi : \chi \to \underline{e}_P(\chi)$ is an isomorphism of quivers.*

Proof. (a) Let Q be the projective Λ-module $\coprod_{i=1}^{m-1} {}_\Lambda P_{z_i}$. Then, by [9, Theorem 2.4], we know that the objects of $\underline{\mathrm{mod}}\,\Lambda$ which do not belong to $\underline{\mathcal{C}_P}$ coincides with $\mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(Q/\mathrm{soc}\,Q, -) \cup \mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(-, rQ)$.

Thus, for any lifting $_S\Gamma_\Lambda[0]$ of $_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_{z_1}, we get the equality

$$_S\Gamma_\Lambda[0]\backslash\operatorname{ind}\underline{\mathcal{C}_P}[0] = \cup_{i=1}^{m-1}\operatorname{Supp}(\tau^{-1}rP_{z_i}[0],-)\cup\cup_{i=1}^{m-1}\operatorname{Supp}(-,rP_{z_i}[0]).$$

On the other hand, by 2.17, we know that the radicals $rP_{z_1},\cdots,rP_{z_{m-1}}$ lift consecutively to the bottom border of $\mathbb{Z}\mathbf{D}_N$; furthermore, by using the shape (see in 1.11) that the supports of the functors $k(\mathbb{Z}\mathbf{D}_N)(x,-)$ and $k(\mathbb{Z}\mathbf{D}_N)(-,x)$ have in $\mathbb{Z}\mathbf{D}_N$, we obtain that the set

$$\cup_{i=1}^{m-1}\operatorname{Supp}(\tau^{-1}rP_{z_i}[0],-)\cup\cup_{i=1}^{m-1}\operatorname{Supp}(-,rP_{z_i}[0]))$$

corresponds to the shaded region of the picture below.
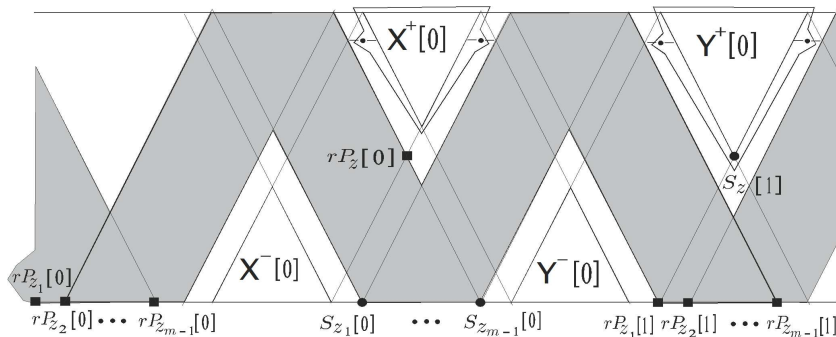
The lifting of the simple Λ-module S_i for $i=z,z_1,\cdots,z_{m-1}$ was carried out using that $S_i[0] = \nu_{\mathbb{Z}\mathbf{D}_N}(\tau^{-1}rP_i[0])$ (see [9, Proposition 3.3]); proving (a).

(b) Let $X,Y\in\operatorname{ind}\underline{\mathcal{C}_P}$.

(b1)$\Rightarrow$(b2) Assume that $f:X\to Y$ is irreducible in $\underline{\operatorname{mod}}\,\Lambda$. It follows directly, from [9, Proposition 2.9 (a)], that $\underline{e}_P(f):\underline{e}_P(X)\to\underline{e}_P(Y)$ is irreducible in $\underline{\operatorname{mod}}\,\Gamma$. On the other hand, there is an arrow $X\to Y$ in $_S\Gamma_\Lambda$ with $X\in\chi$, where χ is one of the quivers $\mathsf{X}^\pm$, $\mathsf{Y}^\pm$. Suppose that X and Y are not in the same component of $\operatorname{ind}\underline{\mathcal{C}_P}$ in $_S\Gamma_\Lambda$. Then, any path in $_S\Gamma_\Lambda$ starting at X and ending at Y has length at least two, contradicting that $X\to Y$ is an arrow of $_S\Gamma_\Lambda$; and so, we get that X and Y are in χ.

(b2)$\Rightarrow$(b1) Assume that X and Y belong to χ, where χ is one of the components $\mathsf{X}^\pm$, $\mathsf{Y}^\pm$, and also that $\underline{e}_P(f) : \underline{e}_P(X) \to \underline{e}_P(Y)$ is irreducible in $\underline{\mathrm{mod}}\,\Gamma$. Suppose that $f : X \to Y$ is not irreducible in $\underline{\mathrm{mod}}\,\Lambda$. Then, there is a path in ${}_S\Gamma_\Lambda$ induced by irreducible morphisms

$$X = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{r-1} \xrightarrow{f_r} M_r = Y$$

having non zero composition in $\underline{\mathrm{mod}}\,\Lambda$. Hence, by [9, Proposition 2.9 (c)], it follows that $M_j \notin \mathcal{C}_P$ for $j = 1, 2, \cdots, r-1$. This is a contradiction, since by (a) we know that χ is a convex subquiver of ${}_S\Gamma_\Lambda$.

(c) It follows easily from (a) and (b). $\square$

Definition 3.2. *Let Λ be a trivial extension of Dynkin Cartan class. For $X \in {}_S\Gamma_\Lambda$, we set:*

- $\underline{\alpha}_\Lambda(X)$ *is the number of arrows in ${}_S\Gamma_\Lambda$ starting at X;*
- $\vec{\mathcal{S}}_X$ *(resp. ${}_X\vec{\mathcal{S}}$) is the section of ${}_S\Gamma_\Lambda$ starting (resp. ending) at X;*
- $X^+ := \{Y \in \underline{\mathrm{ind}}\,\Lambda \ : \ \text{there exists an arrow } X \to Y \text{ in } {}_S\Gamma_\Lambda\}.$

Corollary 3.3. *With the hypothesis in 3.1, let $X \in \chi$ be such that $X \neq_\Lambda S_z$, where χ is one of the components $\mathsf{X}^\pm$, $\mathsf{Y}^\pm$ of $\mathrm{ind}\,\underline{\mathcal{C}_P}$ in ${}_S\Gamma_\Lambda$. If $X^+ \not\subseteq \chi$ then*

(a) *there exists a sectional path in ${}_S\Gamma_\Lambda$ (denoted also as $X \rightsquigarrow Y$)*

$$X \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{m-1} \xrightarrow{f_m} Y$$

such that $Y \in \mathrm{ind}\,\underline{\mathcal{C}_P}$, $Y \notin \chi$ and $M_\ell \notin \mathrm{ind}\,\underline{\mathcal{C}_P}$ for $\ell = 1, 2, \cdots m - 1$;

(b) *the morphism $\underline{e}_P(f) : \underline{e}_P(X) \to \underline{e}_P(Y)$ is irreducible in $\underline{\mathrm{mod}}\,\Gamma$, where $f := f_m \cdots f_2 f_1$.*

Proof. (a) follows easily from 3.1; and (b) from [9, Proposition 2.9(b)]. $\square$

Theorem 3.4. *With the hypothesis in 3.1, the following statements hold.*

(a) *For any X and Y in $\mathrm{ind}\,\underline{\mathcal{C}_P}$, which do not belong to the same component of $\mathrm{ind}\,\underline{\mathcal{C}_P}$ in ${}_S\overline{\Gamma_\Lambda}$, the following conditions are equivalent:*

(a1) *there exists an irreducible morphism $\underline{e}_P(X) \to \underline{e}_P(Y)$ in $\underline{\mathrm{mod}}\,\Gamma$;*

(a2) *there exists a sectional path $X \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{r-1} \xrightarrow{f_r} Y$ in ${}_S\Gamma_\Lambda$ such that $M_i \notin \mathrm{ind}\,\underline{\mathcal{C}}_P$ for $i = 1, 2, \cdots, r-1$.*

Moreover, if (a2) holds then, for $f := f_r f_{r-1} \cdots f_1$, we have that $\underline{e}_P(f) : \underline{e}_P(X) \to \underline{e}_P(Y)$ is irreducible in $\underline{\mathrm{mod}}\,\Gamma$.

(b) *For any $X \in \mathrm{ind}\,\underline{\mathcal{C}}_P$, we have that*

$$\underline{\alpha}_\Gamma(\underline{e}_P(X)) = \begin{cases} \underline{\alpha}_\Lambda(X) & \text{if } X \text{ is not isomorphic to } {}_\Lambda S_z, X_1 \text{ or } X_2, \\ \underline{\alpha}_\Lambda(X) - 1 & \text{if } X \simeq {}_\Lambda S_z, \\ \underline{\alpha}_\Lambda(X) + 1 & \text{if } X \simeq X_1 \text{ or } X \simeq X_2, \end{cases}$$

where $X_1 := \vec{\mathcal{S}}_{\tau^{-2}r_\Lambda P_{z_{m-1}}} \cap {}_{\tau_\Lambda S_{z_1}}\vec{\mathcal{S}}$ and $X_2 := \vec{\mathcal{S}}_{\tau^{-1}{}_\Lambda S_{z_{m-1}}} \cap {}_{\tau r_\Lambda P_{z_1}}\vec{\mathcal{S}}$.

(c) $\tau^{-1}(\underline{e}_P({}_\Lambda S_z)) \simeq \underline{e}_P(\tau^{-2}r_\Lambda P_{z_{m-1}})$.

Proof. By 3.1 we know that $\mathrm{ind}\,\underline{\mathcal{C}}_P$ decomposes in four connected components: $\mathsf{X}^-, \mathsf{X}^+, \mathsf{Y}^-$ and Y^+. Moreover, $\underline{e}_P(\mathsf{X}^\pm)$ and $\underline{e}_P(\mathsf{Y}^\pm)$ are convex subquivers of ${}_S\Gamma_{\mathrm{End}_\Lambda(P)^{op}}$. Through this proof, we will frequently use the picture appearing in 3.1; and also, from this picture we see that the vertex X_1 (resp. X_2) corresponds to the top of the shaded triangle X^- (resp. Y^-).

Let $X \in \mathrm{ind}\,\underline{\mathcal{C}}_P$. Then X belongs to χ, where χ is one of the quivers $\mathsf{X}^\pm, \mathsf{Y}^\pm$. Since Λ is of Cartan class $\mathbf{D}_{N+m-1}$, we get that $1 \leq \underline{\alpha}_\Lambda(X) \leq 3$. If $\mathsf{X}^+ \subseteq \chi$ then the result is obtained as a direct consequence of 3.1.

Assume that $\mathsf{X}^+ \not\subseteq \chi$. If $X \neq {}_\Lambda S_z$ then, by 3.3, there is a sectional path $X \rightsquigarrow Z$ in ${}_S\Gamma_\Lambda$ such that $Z \in \mathrm{ind}\,\underline{\mathcal{C}}_P$, $Z \notin \chi$ and $X \rightsquigarrow Z$ induces an irreducible morphism $\underline{e}_P(X) \to \underline{e}_P(Z)$ in $\underline{\mathrm{mod}}\,\Gamma$. On the other hand, if $\mathsf{X}^+ \cap \chi \neq \emptyset$ then there is an arrow $X \to Y'$ in χ and this arrow induces, by 3.1, an irreducible morphism $\underline{e}_P(X) \to \underline{e}_P(Y')$ in $\underline{\mathrm{mod}}\,\Gamma$. Hence, we have two manners of constructing irreducible morphisms in $\underline{\mathrm{mod}}\,\Gamma$: (i) by using an arrow $X \to Y'$ of χ and (ii) by using a sectional path $X \rightsquigarrow Z$ as above. Then, to prove (a), it is enough to see that the irreducible morphisms in $\underline{\mathrm{mod}}\,\Gamma$, starting at $\underline{e}_P(X)$, are obtained either by (i) or (ii).

For any $W \in \chi$ with $W^+ \not\subseteq \chi$, we denote by $\mathcal{A}_1(W)$ the set of arrows

$W \to Y'$ of χ and by $\mathcal{A}_{>1}(W)$ the set of sectional paths $W \rightsquigarrow Z$ in $_S\Gamma_\Lambda$ (as in (ii)) such that $Z \in \operatorname{ind}\underline{\mathcal{C}_P}$ and $Z \notin \chi$.

We will prove the result by considering the cardinal of the set $\mathcal{A}_1(X) \cup \mathcal{A}_{>1}(X)$. If $\operatorname{card}(\mathcal{A}_1(X) \cup \mathcal{A}_{>1}(X)) = 2$ or $\operatorname{card}(\mathcal{A}_1(X) \cup \mathcal{A}_{>1}(X)) = 3$, the proof is straightforward.

Let $\operatorname{card}(\mathcal{A}_1(X) \cup \mathcal{A}_{>1}(X)) = 1$. So $\operatorname{card}(\mathcal{A}_1(X)) = 1$ or $\operatorname{card}(\mathcal{A}_{>1}(X)) = 1$. In case $\operatorname{card}(\mathcal{A}_{>1}(X)) = 1$ the proof is also straightforward.

Suppose that $\operatorname{card}(\mathcal{A}_1(X)) = 1$. Then $X = {}_\Lambda S_z \in \mathsf{Y}^+$ since we have an arrow $X \to Y_1$ in Y^+ and there are no sectional paths $X \rightsquigarrow Y$ as in 3.3. Let $W \in \operatorname{ind}\underline{\mathcal{C}_P}$, we assert that if there exists an irreducible morphism $\underline{e}_P(X) \to \underline{e}_P(W)$ then $W \simeq Y_1$. Indeed, let Q be the projective Λ-module $\coprod_{j=1}^{m-1} {}_\Lambda P_{z_i}$ and $X \to Y_1 \coprod Y_2$ be an irreducible morphism in $\underline{\operatorname{mod}}\,\Lambda$. Consider the section $\vec{\mathcal{S}}_X$ starting at X. Then $Y_1, Y_2 \in \vec{\mathcal{S}}_X$ and therefore

$$\operatorname{Supp}\underline{\operatorname{Hom}}_\Lambda(Y_2, -) \subseteq \bigcup_{i=1}^{m-1} \operatorname{Supp}\underline{\operatorname{Hom}}_\Lambda(\tau^{-1}rP_{z_i}, -) = \operatorname{Supp}\underline{\operatorname{Hom}}_\Lambda(M, -)$$

where $M := Q/\operatorname{soc}Q$, as can be deduced from the picture below
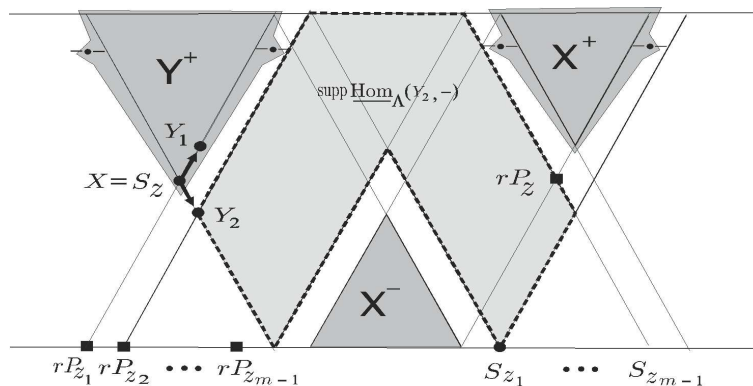
Let $\underline{e}_P(\varphi) : \underline{e}_P(X) \to \underline{e}_P(W)$ be irreducible. Then $\varphi = f_1 + f_2$ where f_i, for $i = 1, 2$, is the composition of a chain $X \to Y_i \to . \to \cdots . \to W$ of irreducible morphisms in $\underline{\operatorname{ind}}\,\Lambda$. If f_2 is non zero then $W \in \operatorname{Supp}\underline{\operatorname{Hom}}_\Lambda(Y_2, -) \subseteq \operatorname{Supp}\underline{\operatorname{Hom}}_\Lambda(Q/\operatorname{soc}Q, -)$; giving a contradiction since $W \in \operatorname{ind}\underline{\mathcal{C}_P}$ implies, from [9, Theorem 2.4], that

$\underline{\mathrm{Hom}}_\Lambda(Q/\mathrm{soc}\, Q, W) = 0$. Thus $\varphi = f_1$ and therefore $\underline{e}_P(W) \simeq \underline{e}_P(Y_1)$ since $\underline{e}_P(X) \to \underline{e}_P(Y_1)$ is irreducible. Using the fidelity of the functor $\underline{e}_P$, we get that $W \simeq Y_1$. Hence, we proved that the unique arrow in $_S\Gamma_\Gamma$ starting at $\underline{e}_P(X)$ is $\underline{e}_P(X) \to \underline{e}_P(Y_1)$. Hence $\underline{\alpha}_\Gamma(\underline{e}_P(X)) = 1 = \underline{\alpha}_\Lambda(X) - 1$.

To prove (c), we use the sectional path $h : Y_1 \rightsquigarrow \tau^{-2}r_\Lambda P_{z_{m-1}}$ (see the picture above). Since $\underline{e}_P(h) : \underline{e}_P(Y_1) \to \underline{e}_P(\tau^{-2}r_\Lambda P_{z_{m-1}})$ is irreducible and $\underline{\alpha}_\Gamma(\underline{e}_P(X)) = 1$, we conclude that $\tau^{-1}(\underline{e}_P(_\Lambda S_z)) \simeq \underline{e}_P(\tau^{-2}r_\Lambda P_{z_{m-1}})$; proving the result. □

Theorem 3.5. *Let Γ be a trivial extension of Cartan class $\mathbf{D}_N$ with $N > 4$. If z is an insertion vertex of Q_Γ, then $r_\Gamma P_z[0]$ and $_\Gamma S_z[0]$ belong to the bottom border of $\mathbb{Z}\mathbf{D}_N$.*

Proof. Let Λ be the trivial extension obtained from Γ by inserting the cycle $C_z = z \leftarrow z_1 \leftarrow z$ at z. Consider the projective Λ-module $P = \coprod_{i \in (Q_\Gamma)_0} {}_\Lambda P_i$ and the equivalence $\underline{e}_P : \underline{\mathcal{C}}_P \to \underline{\mathrm{mod}}\,\Gamma$ of stable categories. Since, by [9, Proposition 3.3], $r_\Gamma P_z[0] = \tau\nu_{\mathbf{D}_N}^{-1}(_\Gamma S_z[0])$ and the Nakayama permutation $\nu_{\mathbf{D}_N}$ sends the bottom border of $\mathbb{Z}\mathbf{D}_N$ to itself (see 1.11), it is enough to prove that $_\Gamma S_z[0]$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$. Indeed, by 3.1, we know that $\underline{\mathrm{ind}}\,\mathcal{C}_P$ splits inside $_S\Gamma_\Lambda$ as four connected components. In the picture below, we have drawn in $_S\Gamma_\Lambda$ three of these components and the position of the Λ-modules rP_t and S_t for $t = z, z_1$.
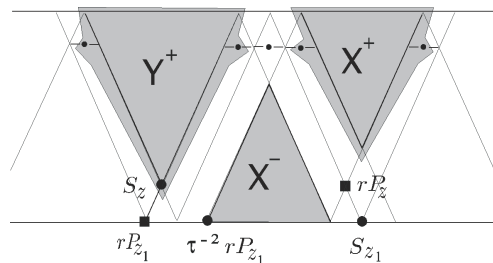
Moreover, by 3.4 (c), we know that $\tau^{-1}(\underline{e}_P(_\Lambda S_z)) \simeq \underline{e}_P(\tau^{-2}r_\Lambda P_{z_1})$; and since $\tau^{-2}r_\Lambda P_{z_1}[0]$ is in the bottom border of $\mathbb{Z}\mathbf{D}_{N+1}$ and $N > 4$, we get that the vertex $\underline{e}_P(\tau^{-2}r_\Lambda P_{z_1})[0]$ is so. Hence $\underline{e}_P(_\Lambda S_z)[0]$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$. On the other hand, by [9,

Theorem 2.8], we have that $\underline{e}_P(\,_\Lambda S_z) \simeq\,_\Gamma S_z$; and therefore $_\Gamma S_z[0]$ is in the bottom border of $\mathbb{Z}\mathbf{D}_N$. $\square$

Corollary 3.6. *Let Γ be a trivial extension of Cartan class $\mathbf{D}_N$, z an insertion vertex of Q_Γ and Λ the trivial extension obtained from Γ by inserting the cycle $C_z = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. Then, for the projective Λ-module $P := \coprod_{i \in (Q_\Gamma)_0}\,_\Lambda P_i$ and the equivalence $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\mathrm{mod}}\,\Gamma$ of stable categories, we have that: (a) $\tau\,_\Lambda S_{z_1} \in \underline{\mathcal{C}_P}$, and $\overline{(b)}$ $\underline{e}_P(\tau\,_\Lambda S_{z_1}) \simeq r_\Gamma P_z$ if $N > 4$.*
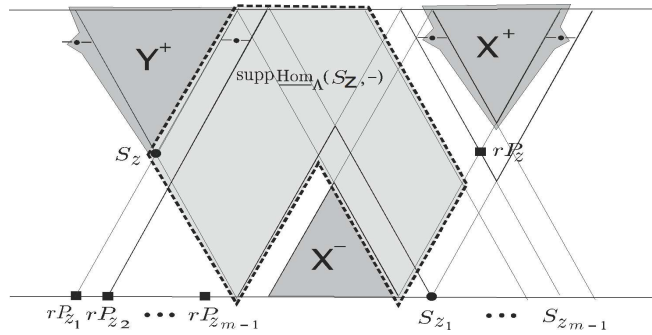
Proof.

    (a) Follows easily from 3.1 (a).

    (b) By (a), we know that $\tau\,_\Lambda S_{z_1} \in \underline{\mathcal{C}_P}$. We assert that

$$(*) \qquad \underline{\mathrm{Hom}}_\Gamma(\,_\Gamma S_z, \underline{e}_P(\tau\,_\Lambda S_{z_1})) \neq 0.$$

Indeed, by 3.1 (a), we get the following picture

Therefore we have that $\underline{e}_P$ induces an isomorphism

$$(**) \qquad \mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(\,_\Lambda S_z, -) \cap (\mathsf{Y}^+ \cup \mathsf{X}^-) \xrightarrow{\sim} \mathrm{Supp}\,\underline{\mathrm{Hom}}_\Gamma(\,_\Gamma S_z, -),$$

since by [9, Theorem 2.8] it follows that $\underline{e}_P(\,_\Lambda S_z) \simeq\,_\Gamma S_z$. Using that $\underline{\mathrm{Hom}}_\Lambda(\,_\Lambda S_z, \tau\,_\Lambda S_{z_1}) \neq 0$ (see the picture above) and $(**)$, we conclude that $\underline{\mathrm{Hom}}_\Gamma(\,_\Gamma S_z, \underline{e}_P(\tau\,_\Lambda S_{z_1})) \neq 0$; proving $(*)$.
Consider a lifting $_S\Gamma_\Gamma[0]$ of $_S\Gamma_\Gamma$ to $\mathbb{Z}\mathbf{D}_N$ at $_\Gamma S_z$. Then, by $(*)$, we get that

$$(***) \qquad \underline{e}_P(\tau\,_\Lambda S_{z_1})[0] \in \mathrm{Supp}\,(\,_\Gamma S_z[0], -).$$

So, since $\tau_\Lambda S_{z_1}$ is in the bottom border of ${}_S\Gamma_\Lambda$ and $N > 4$, the isomorphism of quivers $\underline{e}_P|_{\mathsf{X}^-} : \mathsf{X}^- \to \underline{e}_P(\mathsf{X}^-)$ gives us that $\underline{e}_P(\tau_\Lambda S_{z_1})[0]$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$. Furthermore, since z is an insertion vertex of Q_Γ, we obtain by 3.5 that ${}_\Gamma S_z[0]$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$. Then, from $(\ast\ast\ast)$, we have that $\nu_{D_N}({}_\Gamma S_z[0]) = \underline{e}_P(\tau_\Lambda S_{z_1})[0]$. Hence $r_\Gamma P_z[0] = \underline{e}_P(\tau_\Lambda S_{z_1})[0]$ since, by [9, Proposition 3.3.], we know that $\nu_{D_N}({}_\Gamma S_z[0]) = r_\Gamma P_z[0]$; proving that $\underline{e}_P(\tau_\Lambda S_{z_1}) \simeq r_\Gamma P_z$. $\square$

4. **The embedding of** $k(\mathbb{Z}\mathbf{D}_N)$ **in** $k(\mathbb{Z}\mathbf{D}_{N+m-1})$.

Throughout this section Γ is a trivial extension of Cartan class $\mathbf{D}_N$, z is an insertion vertex of Q_Γ, and Λ is the trivial extension of Cartan class $\mathbf{D}_{N+m-1}$ obtained from Γ by inserting the cycle $C_z = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. In the previous section, we described an embedding of ${}_S\Gamma_\Gamma$ in ${}_S\Gamma_\Lambda$. In this section, we will study the corresponding embedding between their coverings $k(\mathbb{Z}\mathbf{D}_N)$ and $k(\mathbb{Z}\mathbf{D}_{N+m-1})$.

Recall that $\Gamma \simeq \mathrm{End}_\Lambda(P)^{op}$ where P is the projective Λ-module $\coprod_{i\in(Q_\Gamma)_0} {}_\Lambda P_i$. In Section 3, we studied the embedding $\imath : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\Lambda$ and saw that $\mathrm{ind}\underline{\mathcal{C}_P}$ splits inside ${}_S\Gamma_\Lambda$ in four convex components $\mathsf{X}^\pm$, $\mathsf{Y}^\pm$ (see 3.1).

The partition $\{\mathsf{X}^-, \mathsf{X}^+, \mathsf{Y}^-, \mathsf{Y}^+\}$ of $\mathrm{ind}\underline{\mathcal{C}_P}$ induces, through the equivalence $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\mathrm{mod}}\,\Gamma$, the partition in ${}_S\Gamma_\Gamma$ indicated in the following picture where $x := \underline{e}_P(\tau_\Lambda S_{z_1}) \simeq r_\Gamma P_z$ (see 3.6).
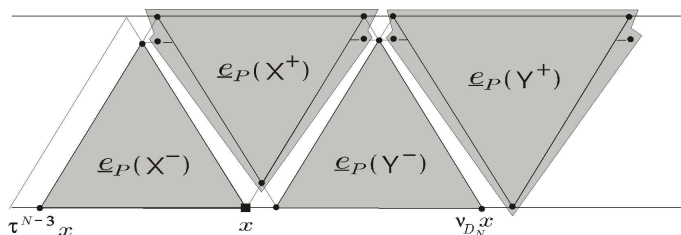
So, the partition $\{\underline{e}_P(\mathsf{X}^-), \underline{e}_P(\mathsf{X}^+), \underline{e}_P(\mathsf{Y}^-), \underline{e}_P(\mathsf{Y}^+)\}$ of ${}_S\Gamma_\Gamma$ depends on the insertion vertex z of Q_Γ, this is, of $r_\Gamma P_z$.

On the other hand, the embedding $\imath : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\,\Lambda$ induces a map

denoted by the same symbol $\imath :\ {}_S\Gamma_\Gamma \to {}_S\Gamma_\Lambda$. Indeed, let $\alpha : M \to N$ be an arrow of ${}_S\Gamma_\Gamma$. Then, by 3.1 and 3.4, there exists a unique sectional path in ${}_S\Gamma_\Lambda$ starting at $\imath(M)$ and ending at $\imath(N)$. Hence we can define $\imath(\alpha)$ to be this sectional path. It is not difficult to see that, if ρ is a mesh relation in ${}_S\Gamma_\Gamma$ then $\imath(\rho)$ is zero in $k({}_S\Gamma_\Lambda)$. Therefore the map $\imath$ induces a full and faithful functor, denoted also by $\imath : k({}_S\Gamma_\Gamma) \to k({}_S\Gamma_\Lambda)$. Let $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Gamma$ and $\pi' : \mathbb{Z}\mathbf{D}_{N+m-1} \to {}_S\Gamma_\Lambda$ be the universal coverings of ${}_S\Gamma_\Gamma$ and ${}_S\Gamma_\Lambda$ respectively. In this section, we will define a functor $\Phi : k(\mathbb{Z}\mathbf{D}_N) \to k(\mathbb{Z}\mathbf{D}_{N+m-1})$ making the following diagram commutative

$$
\begin{array}{ccc}
k(\mathbb{Z}\mathbf{D}_N) & \xrightarrow{\ \Phi\ } & k(\mathbb{Z}\mathbf{D}_{N+m-1}) \\
F\downarrow & & \downarrow F' \\
\underline{\mathrm{ind}}\,\Gamma & \xrightarrow{\ \imath\ } & \underline{\mathrm{ind}}\,\Lambda,
\end{array}
$$

where F and F' are the well behaved functors induced by the coverings π and π' respectively. In order to describe such a functor Φ, we introduce some notation.

4.1. The $\mathcal{X}, \mathcal{Y}$ partition in $\mathbb{Z}\mathbf{D}_N$.

Let X be a subset of vertices in $\mathbb{Z}\mathbf{D}_N$. We denote by $\mathrm{Conv}\,(X)$ the convex closure of X in $\mathbb{Z}\mathbf{D}_N$, which is the smallest convex subquiver of $\mathbb{Z}\mathbf{D}_N$ containing the set X. Furthermore, given a vertex $x \in \mathbb{Z}\mathbf{D}$, the section of $\mathbb{Z}\mathbf{D}_N$ starting at x will be denoted by $\vec{\mathcal{S}}_x$.

Definition 4.1. *For a given vertex x of the bottom border of $\mathbb{Z}\mathbf{D}_N$, we define the following vertices and the full convex subquivers of $\mathbb{Z}\mathbf{D}_N$.*

(a) *$\beta_S(x)$ (resp. $\beta_M(x)$) is the vertex of the section $\vec{\mathcal{S}}_x$ which is in the top (resp. middle) border of $\mathbb{Z}\mathbf{D}_N$.*

(b) *$\mathcal{X}_x^- := \mathrm{Conv}\,(\{\tau^{-1}\nu_{\mathbf{D}_N}^{-1}(x), x\})$.*

(c) *$\mathcal{X}_x^+ := \mathrm{Conv}\,(\{\tau^{-1}\nu_{\mathbf{D}_N}^{-1}(x), \beta_S(x), \beta_M(x)\}) \setminus \mathcal{X}_x^-$.*

(d) *$\mathcal{Y}_x^- := \mathrm{Conv}\,(\{\tau^{-1}x, \nu_{\mathbf{D}_N}(x)\})$.*

(e) *$\mathcal{Y}_x^+ := \mathrm{Conv}\,(\{\tau^{-1}x, \beta_S(\tau^{-1}\nu_{\mathbf{D}_N}(x)), \beta_M(\tau^{-1}\nu_{\mathbf{D}_N}(x))\}) \setminus \mathcal{Y}_x^-$.*

In the following picture, the shape of these subquivers of $\mathbb{Z}\mathbf{D}_N$ can be seen. We also remark that $\tau^{-1}\nu_{\mathbf{D}_N}^{-1}(x) = \tau^{N-3}(x)$.
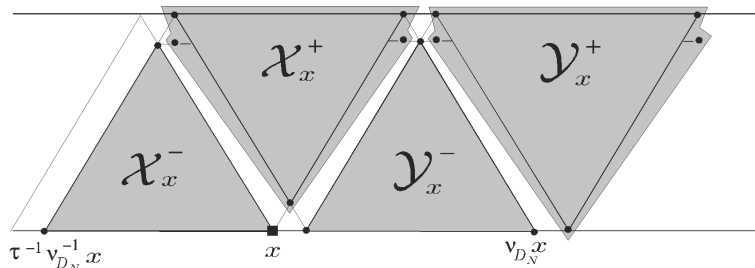
Following [4], we recall that $m_{\mathbb{Z}\mathbf{D}_N} = 2N - 3$, where $m_{\mathbb{Z}\mathbf{D}_N}$ is the Loewy Length in the mesh category $k(\mathbb{Z}\mathbf{D}_N)$. For any vertex u of $\mathbb{Z}\mathbf{D}_N$ and any integer i, we set $u[i] := \tau^{-i(2N-3)}(u)$. Moreover, if $\mathcal{Z}$ is a subquiver of $\mathbb{Z}\mathbf{D}_N$ and i is any integer, we will consider the i-shifted subquiver $\mathcal{Z}[i] := \tau^{-i(2N-3)}(\mathcal{Z})$ of $\mathbb{Z}\mathbf{D}_N$.

Definition 4.2. *For a given vertex x of the bottom border of $\mathbb{Z}\mathbf{D}_N$ and any integer i, we set:* $C_x^+[i] := \mathcal{X}_x^+[i] \cup \mathcal{Y}_x^+[i]$, $C_x^-[i] := \mathcal{X}_x^-[i] \cup \mathcal{Y}_x^-[i]$ *and* $C_x^{\pm} := \cup_{i \in \mathbb{Z}} C_x^{\pm}[i]$. *Thus,* $\{C_x^-[i], C_x^+[i] : i \in \mathbb{Z}\}$ *is a partition of* $\mathbb{Z}\mathbf{D}_N$.

Remark 4.3. *(1) Since z is an insertion vertex of Q_Γ, we get by 3.5 that if $N > 4$ then the fiber $\pi^{-1}(r_\Gamma P_z)$ is contained in the bottom border of $\mathbb{Z}\mathbf{D}_N$, where $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Gamma$ is the universal covering of ${}_S\Gamma_\Gamma$. In case $N = 4$, we choose π in such a way that $\pi^{-1}(r_\Gamma P_z)$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_4$.*

(2) The Nakayama permutation $\nu_{\mathbf{D}_N} : (\mathbb{Z}\mathbf{D}_N)_0 \to (\mathbb{Z}\mathbf{D}_N)_0$ commutes with the translation τ of $\mathbb{Z}\mathbf{D}_N$. Thus, $\nu_{\mathbf{D}_N}$ induces a permutation on $({}_S\Gamma_\Gamma)_0$ which is also denoted by $\nu_{\mathbf{D}_N}$.

Lemma 4.4. *Let z be an insertion vertex of Q_Γ, $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Gamma$ be the universal covering of ${}_S\Gamma_\Gamma$ and $\{\mathsf{X}^-, \mathsf{X}^+, \mathsf{Y}^-, \mathsf{Y}^+\}$ be the partition of $\mathrm{ind}\,\underline{\mathcal{C}_P}$ in ${}_S\Gamma_\Lambda$ (see 3.1).*

(a) *If $x \in \pi^{-1}(r_\Gamma P_z)$, then $\imath\pi|_{\mathcal{X}_x^{\pm}} : \mathcal{X}_x^{\pm} \to \mathsf{X}^{\pm}$ and $\imath\pi|_{\mathcal{Y}_x^{\pm}} : \mathcal{Y}_x^{\pm} \to \mathsf{Y}^{\pm}$ are quiver isomorphisms.*

(b) *Let $_S\Gamma_\Gamma[0]$ be a lifting of $_S\Gamma_\Gamma$ to $\mathbb{Z}\mathbf{D}_N$ at $\tau^{-1}\nu_{\mathbf{D}_N}^{-1}(r_\Gamma P_z)$. If $x = r_\Gamma P_z[0]$ then $_S\Gamma_\Gamma[0] = C_x^+[0] \cup C_x^-[0]$.*

Proof.   It follows easily from definitions and 3.1.   □

4.2. **Definition of the functor $\Phi : k(\mathbb{Z}\mathbf{D}_N) \to k(\mathbb{Z}\mathbf{D}_{N+m-1})$.**

In what follows, we define a functor $\Phi : k(\mathbb{Z}\mathbf{D}_N) \to k(\mathbb{Z}\mathbf{D}_{N+m-1})$ which is a "lifting" of the embedding $\imath : \underline{\mathrm{ind}}\,\Gamma \to \underline{\mathrm{ind}}\,\Lambda$. In order to get some feeling about the correct definition of Φ, we start by looking at the following situation. Consider the partition $\{\underline{e}_P(\mathsf{X}^-), \underline{e}_P(\mathsf{X}^+), \underline{e}_P(\mathsf{Y}^-), \underline{e}_P(\mathsf{Y}^+)\}$ of $_S\Gamma_\Gamma$. Let U and V in $\underline{\mathrm{ind}}\,\Gamma$ as in the following picture.
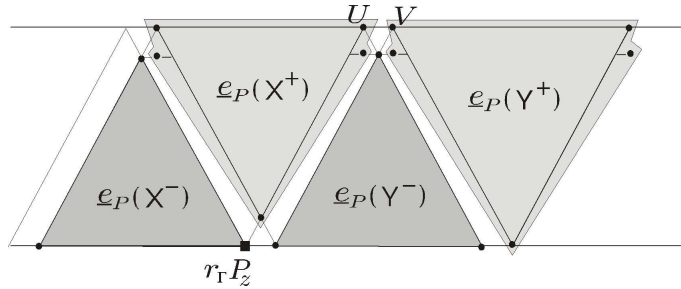
Then $\imath(U)$ (resp. $\imath(V)$) is either in the top or in the middle border of $_S\Gamma_\Lambda$. By fixing $\imath(U)$ in the top border of $_S\Gamma_\Lambda$, we assert that $\imath(U)$ and $\imath(V)$ belong to the same border if and only if $m - 1$ is even. Indeed, $\underline{\mathrm{Hom}}_\Gamma(U, V) = 0$ since $U = \tau V$; and so $\underline{\mathrm{Hom}}_\Lambda(\imath(U), \imath(V)) = 0$. Thus, by using the shape of $\mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(\imath(U), -)$ in $_S\Gamma_\Lambda$ (see 1.11 and [8, Proposition 3.3]), we get the assertion. Then, motivated by this property of the functor $\imath$, we introduce the following definitions. To do that, we will make use of the automorphism $\sigma \in \mathrm{Aut}(\mathbb{Z}\mathbf{D}_N)$ which is defined as follows: $\sigma(p, N-1) := (p, N)$, $\sigma(p, N) := (p, N-1)$ and $\sigma(p, q) := (p, q)$ in case $1 \leq q \leq N-2$. That is, σ sends the top border of $\mathbb{Z}\mathbf{D}_N$ to the middle one and fix the other vertices of $\mathbb{Z}\mathbf{D}_N$.

Let x be a vertex of the bottom border of $\mathbb{Z}\mathbf{D}_N$ and y be a vertex of the top border of $\mathbb{Z}\mathbf{D}_N$. By using the quivers C_x^- and C_x^+ (see 4.2), we will define a k-linear fully faithful functor $\Phi = \Phi_{m-1,x,y}$:

$k(\mathbb{Z}\mathbf{D}_N) \to k(\mathbb{Z}\mathbf{D}_{N+m-1})$. Such a functor depends on the automorphism $\sigma \in \mathrm{Aut}(\mathbb{Z}\mathbf{D}_N)$ and on two functions $\Phi_{m-1,x,\mathcal{X}^+}$, $\Phi_{m-1,x,\mathcal{Y}^+}$: $(\mathbb{Z}\mathbf{D}_N)_0 \to (\mathbb{Z}\mathbf{D}_{N+m-1})_0$. So, we start by defining these auxiliary functions as follows.

Definition 4.5. *The functions* $\Phi_{m-1,x,\mathcal{X}^+}$ *and* $\Phi_{m-1,x,\mathcal{Y}^+}$.

(a) *For* $(p, q) \in C_x^-[0] \cup C_x^+[0]$, *we set*

$$\Phi_{m-1,x,\mathcal{X}^+}(p,q) := \begin{cases} (p,q) & \text{if } (p,q) \in \mathcal{X}_x^-, \\ (p, q+m-1) & \text{if } (p,q) \in \mathcal{X}_x^+, \\ (p+m-1, q) & \text{if } (p,q) \in \mathcal{Y}_x^-, \\ \sigma^{m-1}(p+m-1, q+m-1) & \text{if } (p,q) \in \mathcal{Y}_x^+; \end{cases}$$

$$\Phi_{m-1,x,\mathcal{Y}^+}(p,q) := \begin{cases} (p,q) & \text{if } (p,q) \in \mathcal{X}_x^-, \\ \sigma^{m-1}(p, q+m-1) & \text{if } (p,q) \in \mathcal{X}_x^+, \\ (p+m-1, q) & \text{if } (p,q) \in \mathcal{Y}_x^-, \\ (p+m-1, q+m-1) & \text{if } (p,q) \in \mathcal{Y}_x^+. \end{cases}$$

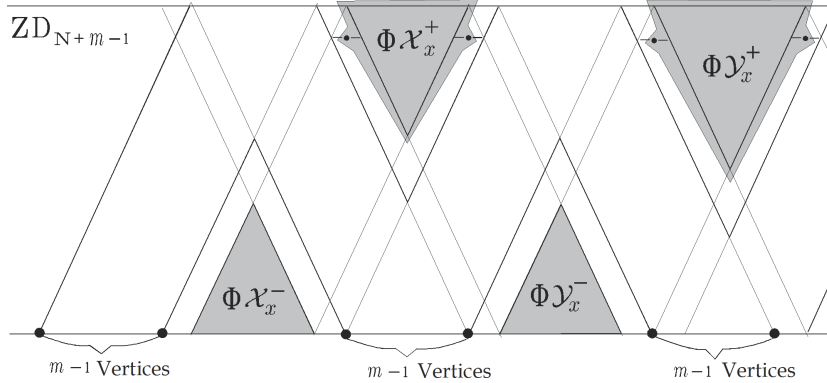(b) *For* $t \in C_x^\pm[i]$ *with* $i \neq 0$, *we set*

$$\Phi_{m-1,x,\mathcal{X}^+}(t) := \Phi_{m-1,x,\mathcal{X}^+}(t[-i])[i],$$
$$\Phi_{m-1,x,\mathcal{Y}^+}(t) := \Phi_{m-1,x,\mathcal{Y}^+}(t[-i])[i].$$

Definition 4.6. *The function* $\Phi = \Phi_{m-1,x,y}$ *on the vertices of* $\mathbb{Z}\mathbf{D}_N$.

$$\Phi(t) := \begin{cases} \Phi_{m-1,x,\mathcal{X}^+}(t) & \text{if } y \in \cup_{i \in \mathbf{Z}} \mathcal{X}_x^+[i], \\ \Phi_{m-1,x,\mathcal{Y}^+}(t) & \text{if } y \in \cup_{i \in \mathbf{Z}} \mathcal{Y}_x^+[i]. \end{cases}$$

In the following picture, we illustrate the shape of the set $\Phi((\mathbb{Z}\mathbf{D}_N)_0)$

Definition 4.7. *The function $\Phi = \Phi_{m-1,x,y}$ on the arrows of $\mathbb{Z}\mathbf{D}_N$. Let $w \xrightarrow{\alpha} z$ be an arrow of $\mathbb{Z}\mathbf{D}_N$. By 4.6, it is not difficult to see that there exists a unique sectional path γ in $\mathbb{Z}\mathbf{D}_{N+m-1}$ starting at $\Phi(w)$ and ending at $\Phi(z)$; and so we set $\Phi(\alpha) := \gamma$.*

Remark 4.8. *if α belongs to one of the components $\mathcal{X}^{\pm}[i], \mathcal{Y}^{\pm}[i]$ for some integer i, then $\Phi(\alpha)$ is an arrow in $\mathbb{Z}\mathbf{D}_{N+m-1}$. Moreover, it is not difficult to see that if ρ is a mesh relation in $\mathbb{Z}\mathbf{D}_N$ then $\Phi(\rho)$ is zero in $k(\mathbb{Z}\mathbf{D}_{N+m-1})$. Thus, we get a k-linear fully faithful functor, denoted also by Φ*

$$\Phi = \Phi_{m-1,x,y} : k(\mathbb{Z}\mathbf{D}_N) \to k(\mathbb{Z}\mathbf{D}_{N+m-1}).$$

Lemma 4.9. *Let z be an insertion vertex of Q_Γ, $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Gamma$ be the universal covering of ${}_S\Gamma_\Gamma$ (as fixed in 4.3), $\pi' : \mathbb{Z}\mathbf{D}_{N+m-1} \to {}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$ and y be a vertex of the top border of $\mathbb{Z}\mathbf{D}_N$. Let ${}_S\Gamma_\Gamma[0]$ be a lifting of ${}_S\Gamma_\Gamma$ to $\mathbb{Z}\mathbf{D}_N$ at $\tau^{-1}\nu_{\mathbf{D}_N}^{-1}(r_\Gamma P_z)$ and take $x := r_\Gamma P_z[0]$ as the fixed vertex in the bottom border of $\mathbb{Z}\mathbf{D}_N$.
We fix a lifting ${}_S\Gamma_\Lambda[0]$ of ${}_S\Gamma_\Lambda$ to $\mathbf{Z}\mathbf{D}_{N+m-1}$ at $\tau^{-1}r_\Lambda P_{z_1}$ in such a way that $\tau^{-1}r_\Lambda P_{z_1}[0] = \tau^{-1}\nu_{\mathbf{D}_{N+m-1}}^{-1}(\Phi(r_\Gamma P_z[0]))$. Then, the functor*

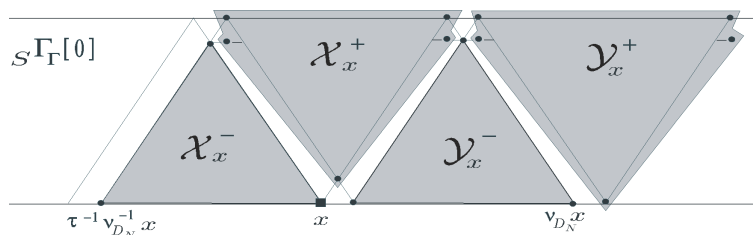$$\Phi = \Phi_{m-1,r_\Gamma P_z[0],y} : k(\mathbb{Z}\mathbf{D}_N) \to k(\mathbb{Z}\mathbf{D}_{N+m-1})$$

has the following properties:

(a) *the maps $\pi'\Phi|_{\mathcal{X}^{\pm}_{r_\Gamma P_z[0]}} : \mathcal{X}^{\pm}_{r_\Gamma P_z[0]} \to \mathsf{X}^{\pm}$ and*

   $\pi'\Phi|_{\mathcal{Y}^{\pm}_{r_\Gamma P_z[0]}} : \mathcal{Y}^{\pm}_{r_\Gamma P_z[0]} \to \mathsf{Y}^{\pm}$ *are quiver isomorphisms,*

(b) ${}_S\Gamma_\Lambda[0] = C^+_{\Phi(r_\Gamma P_z[0])}[0] \cup C^-_{\Phi(r_\Gamma P_z[0])}[0]$,

(c) $\Phi(C^{\pm}_{r_\Gamma P_z[0]}[0]) \subseteq C^{\pm}_{\Phi(r_\Gamma P_z[0])}[0]$.
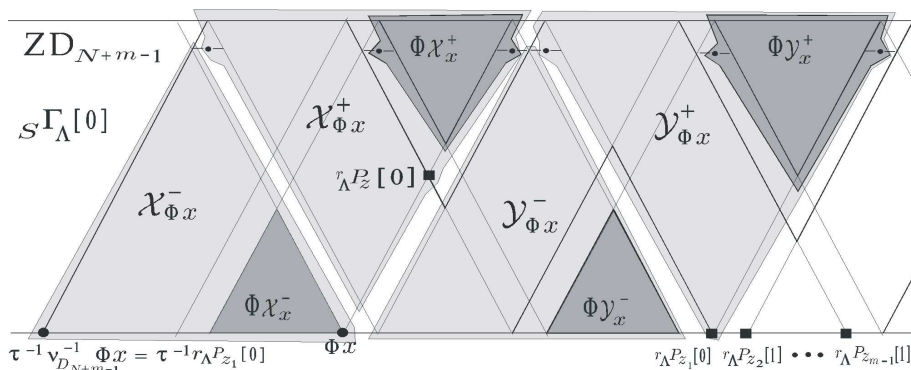
Proof. It follows directly from the definitions and 4.4. $\square$

Remark 4.10. *In the following two pictures, we illustrate 4.9.*

(a) *The partition of ${}_S\Gamma_\Gamma[0]$ induced by the vertex $x := r_\Gamma P_z[0]$.*

(b) *The partition of $_S\Gamma_\Lambda[0]$ which is compatible with the functor Φ.*

Theorem 4.11. *Let Γ be a trivial extension of Cartan class $\mathbf{D}_N$, z be an insertion vertex of Q_Γ and Λ be the trivial extension obtained from Γ by inserting the minimal cycle $C_z = z \leftarrow z_1 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. Fix $i_0 \in \mathbb{N}$ such that $i_0 := 1$ if Γ is of Class I, and $i_0 := 3$ if Γ is either of Class II or III (see 1.9). Choose the universal coverings $\pi : \mathbb{Z}\mathbf{D}_N \rightarrow {}_S\Gamma_\Gamma$ and $\pi' : \mathbb{Z}\mathbf{D}_{N+m-1} \rightarrow {}_S\Gamma_\Lambda$ as in 4.9 and such that $h_{\mathbb{Z}\mathbf{D}_N}(\pi^{-1}(r_\Gamma P_{i_0})) = N-1$. Consider also the liftings and the full and faithful functor*

$$\Phi_{z,i_0} := \Phi_{m-1, r_\Gamma P_z[0], y} : k(\mathbb{Z}\mathbf{D}_N) \rightarrow k(\mathbb{Z}\mathbf{D}_{N+m-1}),$$

fixed in 4.9, where $y := r_\Gamma P_{i_0}[0]$.
If $F' : k(\mathbb{Z}\mathbf{D}_{N+m-1}) \rightarrow \underline{\mathrm{ind}}\,\Lambda$ is a well behaved functor induced by the universal covering $\pi' : \mathbb{Z}\mathbf{D}_{N+m-1} \rightarrow {}_S\Gamma_\Lambda$, then

(a) *there is a well behaved functor $F : k(\mathbb{Z}\mathbf{D}_N) \to \underline{\mathrm{ind}}\,\Gamma$, induced by the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Gamma$, making the following diagram*

$$
\begin{array}{ccc}
k(\mathbb{Z}\mathbf{D}_N) & \xrightarrow{\ \Phi_{z,i_0}\ } & k(\mathbb{Z}\mathbf{D}_{N+m-1}) \\
F \downarrow & & \downarrow F' \\
\underline{\mathrm{ind}}\,\Gamma & \xrightarrow{\ \imath\ } & \underline{\mathrm{ind}}\,\Lambda
\end{array}
$$

commutative, where $\imath$ is the fully faithful functor defined in Section 3;

(b) *for any vertex $t \in (Q_\Gamma)_0$, the functor Φ_{z,i_0} satisfies:*

(b1) $\Phi_{z,i_0}(r_\Gamma P_t[0]) = \begin{cases} r_\Lambda P_t[0] & \text{if } t \neq z, \\ \tau(\,{}_\Lambda \check{S}_{z_1}[0]) & \text{if } t = z; \end{cases}$

(b2) $\Phi_{z,i_0}(\,{}_\Gamma S_t[0]) = {}_\Lambda S_t[0].$

Proof.   For the sake of simplicity, we set $\Phi := \Phi_{z,i_0}$.

(a) Firstly, we prove that $\pi'\Phi = \imath\pi$. To do that, we consider the following functions:

$$
\begin{array}{ll}
F_1 = \imath\pi|_{\mathcal{X}^-_{r_\Gamma P_z[0]}} & F_2 = \imath\pi|_{\mathcal{Y}^-_{r_\Gamma P_z[0]}}, \\
G_1 = \pi'\Phi|_{\mathcal{X}^-_{r_\Gamma P_z[0]}} & G_2 = \pi'\Phi|_{\mathcal{Y}^-_{r_\Gamma P_z[0]}}
\end{array}
$$

$$
\begin{array}{ll}
F_3 = \imath\pi|_{\mathcal{X}^+_{r_\Gamma P_z[0]}} & F_4 = \imath\pi|_{\mathcal{Y}^+_{r_\Gamma P_z[0]}}, \\
G_3 = \pi'\Phi|_{\mathcal{X}^+_{r_\Gamma P_z[0]}} & G_4 = \pi'\Phi|_{\mathcal{Y}^+_{r_\Gamma P_z[0]}}.
\end{array}
$$

By 4.4 and 4.9, we get that F_i and G_i are quiver isomorphisms for any i. Hence

$$
\begin{array}{ll}
G_1^{-1}F_1 \in \mathrm{Aut}(\mathcal{X}^-_{r_\Gamma P_z[0]}) = \{1\}, & G_2^{-1}F_2 \in \mathrm{Aut}(\mathcal{Y}^-_{r_\Gamma P_z[0]}) = \{1\}, \\
G_3^{-1}F_3 \in \mathrm{Aut}(\mathcal{X}^+_{r_\Gamma P_z[0]}) = \{1,\sigma\}, & G_4^{-1}F_4 \in \mathrm{Aut}(\mathcal{Y}^+_{r_\Gamma P_z[0]}) = \{1,\sigma\}.
\end{array}
$$

Then $F_1 = G_1$, $F_2 = G_2$; and to obtain $\pi'\Phi = \imath\pi$, we have to prove that $F_i = G_i$ for $i = 3, 4$. By definition we have

$$
\Phi = \begin{cases} \Phi_{m-1,r_\Gamma P_z[0],\mathcal{X}^+} & \text{if } r_\Gamma P_{i_0}[0] \in \mathcal{X}^+_{r_\Gamma P_z[0]}, \\ \Phi_{m-1,r_\Gamma P_z[0],\mathcal{Y}^+} & \text{if } r_\Gamma P_{i_0}[0] \in \mathcal{Y}^+_{r_\Gamma P_z[0]}. \end{cases}
$$

Assume that $r_\Gamma P_{i_0}[0] \in \mathcal{X}^+_{r_\Gamma P_z[0]}$ (for the other case, the proof is very similar). Then $\Phi = \Phi_{m-1, r_\Gamma P_z[0], \mathcal{X}^+}$.

•) $F_3 = G_3$.

Indeed, since $r_\Gamma P_{i_0}[0]$ is in the top border of $\mathbb{Z}\mathbf{D}_N$ and $r_\Gamma P_{i_0}[0] \in \mathcal{X}^+_{r_\Gamma P_z[0]}$, we get that $\Phi(r_\Gamma P_{i_0}[0]) \in \mathsf{X}^+[0]$ with $\Phi(r_\Gamma P_{i_0}[0])$ in the top border of $\mathbb{Z}\mathbf{D}_{N+m-1}$. Furthermore, by [9, Theorem 2.8], we know that $\imath(r_\Gamma P_{i_0}) \simeq r_\Lambda P_{i_0}$; and hence $r_\Lambda P_{i_0}[0] \in \mathsf{X}^+[0]$ since $r_\Gamma P_{i_0}[0] \in \mathcal{X}^+_{r_\Gamma P_z[0]}$. So, using that $r_\Lambda P_{i_0}[0]$ belongs to $\mathsf{X}^+[0]$ and to the top border of $\mathbb{Z}\mathbf{D}_{N+m-1}$, we obtain that $\Phi(r_\Gamma P_{i_0}[0]) = r_\Lambda P_{i_0}[0]$ since $\imath(r_\Gamma P_{i_0}) \simeq r_\Lambda P_{i_0}$ and $\underline{e}_P|_{\mathsf{X}^+} : \mathsf{X}^+ \to \pi \mathcal{X}^+_{r_\Gamma P_z[0]}$ is an isomorphism of quivers. Therefore $\pi'\Phi(r_\Gamma P_{i_0}[0]) = r_\Lambda P_{i_0}$; and hence $G_3^{-1}F_3(r_\Gamma P_{i_0}[0]) = r_\Gamma P_{i_0}[0]$. Then $G_3^{-1}F_3 = 1$ since the automorphism σ sends the top border of $\mathbb{Z}\mathbf{D}_N$ to the middle border of $\mathbb{Z}\mathbf{D}_N$.

•) $F_4 = G_4$.

Indeed, let $x = r_\Gamma P_z[0]$. Then $\beta_S(x) \in \mathcal{X}^+_x$ and $\beta_M(\tau^{-1}x) \in \mathcal{Y}^+_x$.
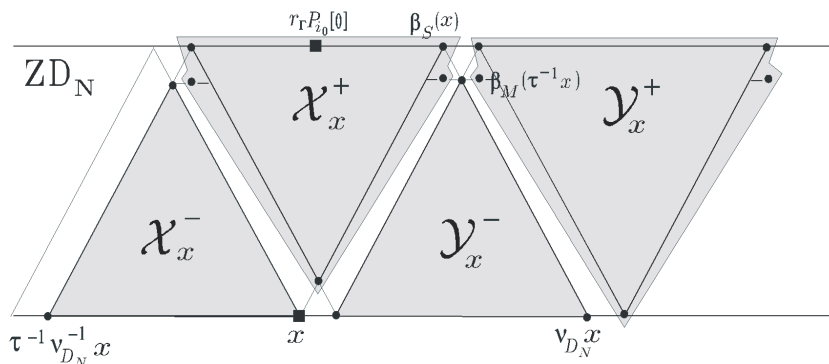
Let $F_\pi : k(\mathbb{Z}\mathbf{D}_N) \to \underline{\mathrm{ind}}\,\Gamma$ be a well-behaved functor induced by the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Gamma$. From [8, 3.2 and 3.3], we have that

$$F_\pi : k(\mathbb{Z}\mathbf{D}_N)(\beta_S(x), \beta_M(\tau^{-1}x)) \to \underline{\mathrm{Hom}}_\Gamma(\pi\beta_S(x), \pi\beta_M(\tau^{-1}x))$$

is a k-linear isomorphism. On the other hand, we know that

$$\underline{\mathrm{Hom}}_\Gamma(\pi\beta_S(x), \pi\beta_M(\tau^{-1}x)) \simeq \underline{\mathrm{Hom}}_\Lambda(\imath\pi\beta_S(x), \imath\pi\beta_M(\tau^{-1}x));$$

and by these isomorphisms, we get $\underline{\mathrm{Hom}}_\Lambda(\imath\pi\beta_S(x), \imath\pi\beta_M(\tau^{-1}x)) \neq 0$ since, by the preceding picture, we know that $k(\mathbb{Z}\mathbf{D}_N)(\beta_S(x), \beta_M(\tau^{-1}x)) \neq 0$. Using that $F_3 = G_3$, we have that $\imath\pi(\beta_S(x)) = \pi'\Phi(\beta_S(x))$. Thus

$$(**) \quad \imath\pi(\beta_M(\tau^{-1}x)) \in \mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(\pi'\Phi(\beta_S(x)), -).$$

Furthermore, by [8, Proposition 3.3], it follows that the universal covering $\pi' : \mathbb{Z}\mathbf{D}_{N+m-1} \to {}_S\Gamma_\Lambda$ induces a bijection

$$\mathrm{Supp}\,k(\mathbb{Z}\mathbf{D}_{N+m-1})(\Phi(\beta_S(x)), -) \overset{\sim}{\to} \mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(\pi'\Phi(\beta_S(x)), -).$$

Moreover, by 1.11, we know the shape of $\mathrm{Supp}\,k(\mathbb{Z}\mathbf{D}_{N+m-1})(\Phi(\beta_S(x)), -)$ in $\mathbb{Z}\mathbf{D}_{N+m-1}$. Then, by $(**)$, we conclude that

$$\imath\pi(\beta_M(\tau^{-1}x)) = \pi'\sigma^{m-1}\beta_M(\Phi(\tau^{-1}x)).$$

By the definition of Φ, we get that $\Phi(\beta_M(\tau^{-1}x)) = \sigma^{m-1}\beta_M(\Phi(\tau^{-1}x))$. Then $\imath\pi(\beta_M(\tau^{-1}x)) = \pi'\Phi(\beta_M(\tau^{-1}x))$ and hence $G_4^{-1}F_4(\beta_M(\tau^{-1}x)) = \beta_M(\tau^{-1}x)$. Thus $G_4^{-1}F_4 = 1$ since the automorphism σ sends the middle border of $\mathbb{Z}\mathbf{D}_N$ to the top one; proving that $\imath\pi = \pi'\Phi$.

Let $F' : k(\mathbb{Z}\mathbf{D}_{N+m-1}) \to \underline{\mathrm{ind}}\,\Lambda$ be a well behaved functor induced by the universal covering $\pi' : \mathbb{Z}\mathbf{D}_{N+m-1} \to {}_S\Gamma_\Lambda$. Since $\imath\pi = \pi'\Phi$. It is clear how to define F on the vertices of $\mathbb{Z}\mathbf{D}_N$ (just take $F := \pi$ on $(\mathbb{Z}\mathbf{D}_N)_0$). Now, we define F on the arrows of $\mathbb{Z}\mathbf{D}_N$. For an arrow $x \overset{\alpha}{\to} y$ of $\mathbb{Z}\mathbf{D}_N$, we set $F(\alpha) := \underline{e}_P\,F'\Phi(\alpha)$, where $\underline{e}_P = \underline{\mathrm{Hom}}_\Lambda(P, -) : \underline{\mathcal{C}}_P \overset{\sim}{\to} \underline{\mathrm{mod}}\,\Gamma$ is the equivalence of stable categories given in Section 3. So, we have a functor $F : k(\mathbb{Z}\mathbf{D}_N) \to \underline{\mathrm{ind}}\,\Gamma$. To see that F is well-behaved and makes the diagram of (a) commute, it is enough to prove that $\underline{e}_P\,F'\Phi(\alpha)$ is irreducible in $\underline{\mathrm{mod}}\,\Gamma$ for any arrow $x \overset{\alpha}{\to} y$ in $\mathbb{Z}\mathbf{D}_N$. Consider the partition $\{\mathcal{X}^+_{r_\Gamma P_z\,[0]}[j], \mathcal{X}^-_{r_\Gamma P_z\,[0]}[j], \mathcal{Y}^+_{r_\Gamma P_z\,[0]}[j], \mathcal{Y}^-_{r_\Gamma P_z\,[0]}[j] : j \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_N$, we have that $\Phi(x)$ and $\Phi(y)$ are either in the same or in different components of this partition.

If $\Phi(x)$ and $\Phi(y)$ are in the same component of the above partition. Then, by the definition of Φ, we get that $\Phi(\alpha)$ is an arrow of $\mathbb{Z}\mathbf{D}_{N+m-1}$ and therefore $F'\Phi(\alpha)$ is irreducible in $\underline{\mathrm{mod}}\,\Lambda$. Hence, from 3.1 (b), we obtain that $\underline{e}_P\,F'\Phi(\alpha)$ is irreducible in $\underline{\mathrm{mod}}\,\Gamma$.

Suppose that $\Phi(x)$ and $\Phi(y)$ are not in the same component of the above partition. Then, by the definition of Φ, we obtain that $\pi'(\Phi(x))$

and $\pi'(\Phi(y))$ belong to different components of ind $\underline{\mathcal{C}}_P$ (see 3.1 (a)). Moreover $\pi'(\Phi(\alpha))$ is the following sectional path in $\overline{{}_S\Gamma_\Lambda}$

$$\pi'(\Phi(x)) \to M_1 \to \cdots \to M_{r-1} \to \pi'(\Phi(y))$$

such that $M_i \notin \underline{\mathcal{C}}_P$ for any $i = 1, 2, \cdots, r - 1$. Therefore, by 3.4 (a), we have that $\underline{e}_P \overline{F'}\Phi(\alpha)$ is irreducible in $\underline{\mod}\,\Gamma$.

(b) Let $t \in (Q_\Gamma)_0$. We only prove (b1) since the prove of (b2) is very similar.

If $t = z$ then $\Phi(r_\Gamma P_t[0]) = \tau({}_\Lambda S_{z_1}[0])$. Assume that $t \neq z$. Then, by [9, Theorem 2.8], we have that $r_\Lambda P_t \in \underline{\mathcal{C}}_P$ and $\underline{e}_P(r_\Lambda P_t) \simeq r_\Gamma P_t$. Hence $\imath(r_\Gamma P_t) \simeq r_\Lambda P_t$ since $\imath\,\underline{e}_P \simeq 1_{\underline{\mathcal{C}}_P}$. On the other hand, by (a), we have that $\pi'\Phi(r_\Gamma P_t[0]) = \imath\pi(r_\Gamma P_t[0]) = \imath(r_\Gamma P_t)$, and then, $\pi'\Phi(r_\Gamma P_t[0]) = r_\Lambda P_t$ since $\imath(r_\Gamma P_t) \simeq r_\Lambda P_t$. Thus $\Phi(r_\Gamma P_t[0]) = r_\Lambda P_t[0]$; proving the result. $\square$

Corollary 4.12. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ with $N > 4$. For a given vertex $z \in Q_\Lambda$, the following conditions are equivalent.*

(a) *z is an insertion vertex of Q_Λ.*

(b) *$rP_z[0]$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$.*

(c) *$S_z[0]$ belongs to the bottom border of $\mathbb{Z}\mathbf{D}_N$.*

In particular, the number of vertices $z \in Q_\Lambda$ such that rP_z lifts to the bottom border of $\mathbb{Z}\mathbf{D}_N$ is larger than 1 and coincides with the number of insertion vertices of Q_Λ.

Proof. (b)$\Leftrightarrow$(c) It follows from the equality $S_z[0] = \nu_{\mathbf{D}_N}(\tau^{-1}rP_z[0])$ (see in [9, Proposition 3.3].

(a)$\Leftrightarrow$(b) If Λ is minimal then the equivalence follows from 2.12, 2.14 and 2.15. In case Λ is not minimal, we proceed by induction over the number of minimal oriented cycles of Q_Λ; and so by using 4.11 we get the result. $\square$

5. Some generalities about trivial extensions of class II.

In this section, we extend 2.14 to the general case, that is, to a trivial extension Λ of Class II (see 1.2 and 1.9). To do that, we firstly consider the case when Λ is minimal of Class II (see 2.14). Then, by

using 4.11, we see how the radicals of the projective modules, associated with the vertices $1, 2$ and 3 of Q_Λ, can be lifted to the universal covering. Furthermore, we also determine some special regions, in the bottom border of $\mathbb{Z}\mathbf{D}_N$, where the radicals of the projective modules associated with the insertion vertices of Q_Λ can be lifted. So, throughout this section, Λ will be a trivial extension of Class II.

We recall that the ordinary quiver Q_Λ of Λ is described (see 1.2 and 1.9) by drawing the quiver $Q_{\Lambda_{min}}$ of Λ_{min} and small squares on some vertices of $Q_{\Lambda_{min}}$. The small square on a vertex $x_j \in Q_{\Lambda_{min}}$, represents a quiver Q_{x_j} which is either $\{x_j\}$ or the ordinary quiver of a trivial extension Γ_{x_j} of Cartan class $\mathbf{A}_{n_j}$ with $n_j \geq 2$.

Definition 5.1. *For a trivial extension Λ of Class II, it will be considered the following subquivers of Q_Λ.*

$$Q_{\Lambda_a}: \quad \overset{a_1}{\blacksquare} \longrightarrow \overset{a_2}{\blacksquare} \longrightarrow \cdots \longrightarrow \overset{a_n}{\blacksquare} \; ,$$

$$Q_{\Lambda_b}: \quad \overset{b_1}{\blacksquare} \longrightarrow \overset{b_2}{\blacksquare} \longrightarrow \cdots \longrightarrow \overset{b_m}{\blacksquare} \; ,$$

$$Q_{\Lambda_c}: \quad \overset{c_1}{\blacksquare} \longrightarrow \overset{c_2}{\blacksquare} \longrightarrow \cdots \longrightarrow \overset{c_t}{\blacksquare} \; .$$

In case one of the numbers n, m or t is zero then the corresponding quiver Q_{Λ_a}, Q_{Λ_b} or Q_{Λ_c} is the empty set. We set $w(\Lambda) := |(Q_{\Lambda_w})_0|$ for $w \in \{a, b, c\}$ where $|X|$ stands for the cardinality of a given set X.

Definition 5.2. *For a trivial extension Λ of Class II, we fix the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ such that $h_{\mathbb{Z}\mathbf{D}_N}(\pi^{-1}(rP_3)) = N - 1$; and moreover, in case $N = 4$, we add the condition $h_{\mathbb{Z}\mathbf{D}_4}(\pi^{-1}(rP_{a_1})) = 1$ (see 2.11 (a)). Consider a lifting (see [8, 3.5]) of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ at rP_3. For each $w \in \{a, b, c\}$, we define the set $\mathrm{Ins}(Q_{\Lambda_w})$ as the subset of the bottom border of $\mathbb{Z}\mathbf{D}_N$ which is illustrated in the following picture.*
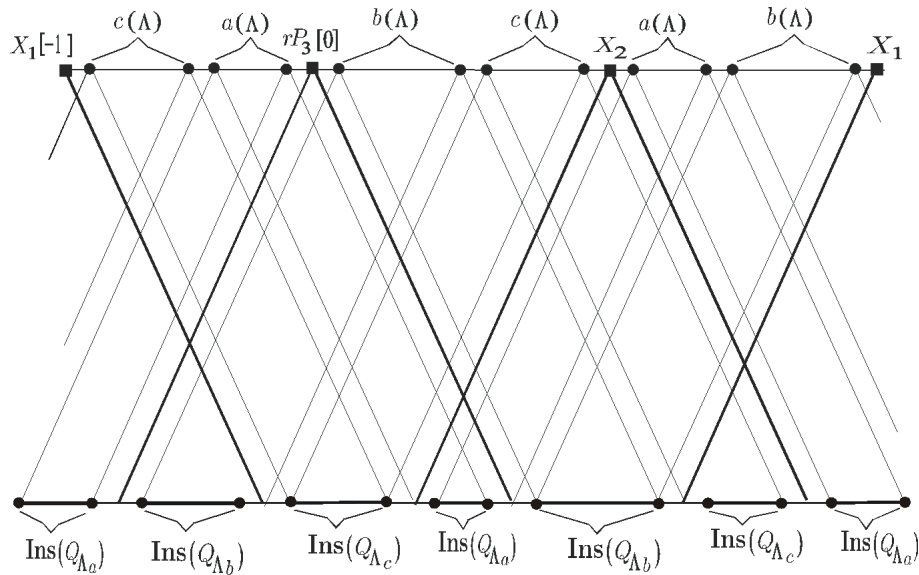
In what follows, we prove that the radicals of the projective modules associated to the insertion vertices of Q_Λ belonging to Q_{Λ_w}, for $w \in \{a, b, c\}$, lift to $\mathrm{Ins}(Q_{\Lambda_w})$.

Proposition 5.3. *Let Λ be a trivial extension of Class II and Cartan class $\mathbf{D}_N$. Consider a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{D}_N$ as in 5.2.*

(a) *Let x be an insertion vertex of Q_Λ and $w \in \{a, b, c\}$. If $x \in Q_{\Lambda_w}$ then $\pi^{-1}(r_\Lambda P_x) \subseteq \mathrm{Ins}(Q_{\Lambda_w})$.*

(b) *The radicals rP_1, rP_2 and rP_3 lift to $\mathbb{Z}\mathbf{D}_N$ as is indicated in the following picture, where*

$$rP_2[0] = \begin{cases} X_2 & \text{if } c(\Lambda) + b(\Lambda) \text{ is even,} \\ Y_2 & \text{if } c(\Lambda) + b(\Lambda) \text{ is odd.} \end{cases}$$

$$rP_1[0] = \begin{cases} X_1 & \text{if } c(\Lambda) + a(\Lambda) \text{ is even,} \\ Y_1 & \text{if } c(\Lambda) + a(\Lambda) \text{ is odd.} \end{cases}$$
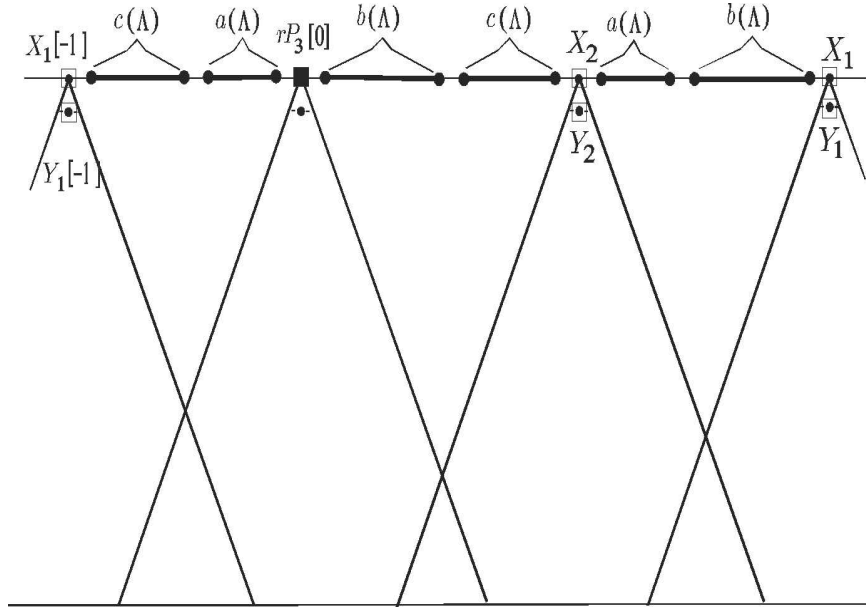
Proof.   We will proceed by induction on the number of elemental cycles of Q_Λ (see 1.4). If there are not elimination cycles in Q_Λ (see 2.1), then Λ is minimal and so $a(\Lambda) = n$, $b(\Lambda) = m$ and $c(\Lambda) = t$. Hence, in this case, the result follows from 2.11 (a) and 2.14.

Suppose there is at least one elimination cycle C in the ordinary quiver Q_Λ. Since $C \subseteq \cup_{w \in \{a,b,c\}} Q_{\Lambda_w}$, we may assume that C is contained in Q_{Λ_c} (for the other cases, the proof is quite similar). Let $C = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$. Consider the trivial extension Γ of Cartan class $\mathbf{D}_{N-m+1}$ obtained from Λ by eliminating the cycle C. So, by 4.11, we have the full and faithful functor

$$\Phi := \Phi_{z,3} : k(\mathbb{Z}\mathbf{D}_{N-m+1}) \to k(\mathbb{Z}\mathbf{D}_N).$$

By induction, we have that the result is true for Γ. Furthermore, since $z \in Q_{\Gamma_c}$ is an insertion vertex of Q_Γ, it follows that $r_\Gamma P_z[0] \in \mathrm{Ins}(Q_{\Gamma_c})$. Now, consider the partition $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i] : i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$ (see 4.2). In the following picture, we illustrate this partition where the shaded regions correspond to $C^-_{r_\Gamma P_z[0]}$.
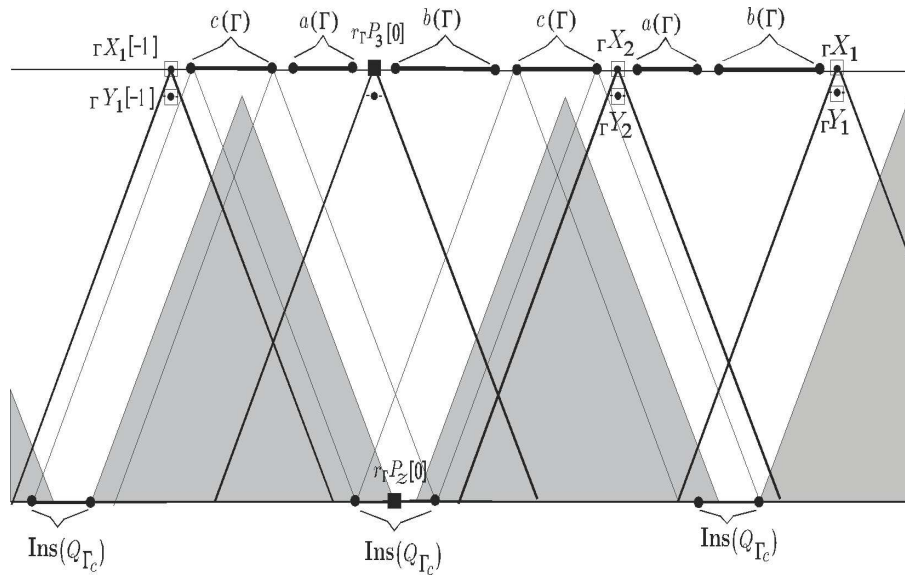
Φ changes the partition $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i] \ : \ i \in \mathbf{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$ as is indicated in the following picture (see 4.9 and 4.10).
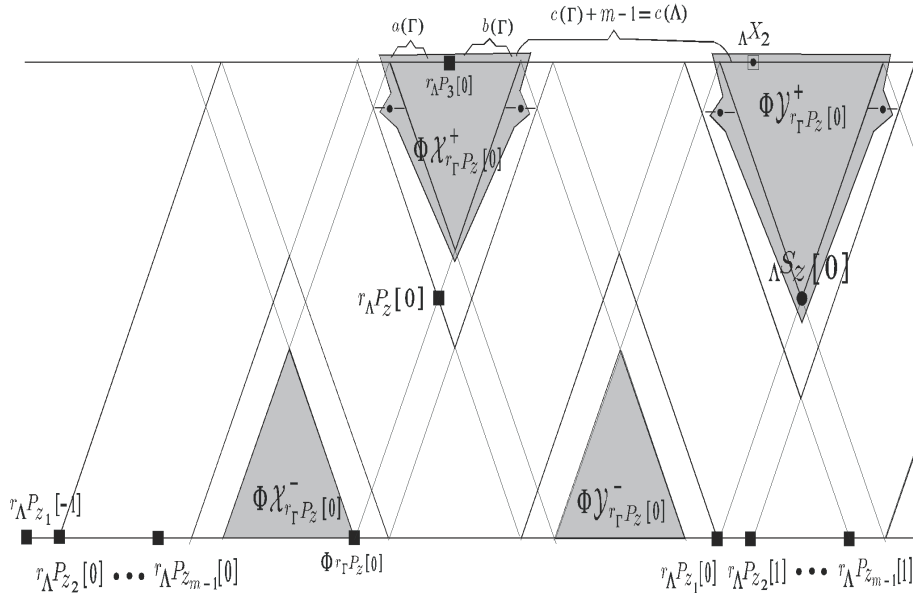
Moreover, by 4.11 (b), we have that $\Phi(r_\Gamma P_t[0]) = r_\Lambda P_t[0]$ for any $t \in (Q_\Gamma)_0$ with $t \neq z$. Now, we are ready to prove items (a) and (b).

(a) Let x be an insertion vertex in Q_Λ. To prove (a), it is enough tho consider two cases: (a1) $x \in Q_{\Lambda_a}$ and (a2) $x \in Q_{\Lambda_c}$ (the case $x \in Q_{\Lambda_b}$ is quite similar to the first one).

(a1) Let $x \in Q_{\Lambda_a}$. Since $Q_{\Lambda_a} = Q_{\Gamma_a}$, we get by induction that $r_\Gamma P_x[0] \in \mathrm{Ins}(Q_{\Gamma_a})$. Moreover, by the two figures above, it follows that $\Phi(\mathrm{Ins}(Q_{\Gamma_a})) = \mathrm{Ins}(Q_{\Lambda_a})$. Hence $r_\Lambda P_x[0] = \Phi(r_\Gamma P_x[0]) \in \Phi(\mathrm{Ins}(Q_{\Gamma_a})) = \mathrm{Ins}(Q_{\Lambda_a})$; proving (a) in this case.

(a2) Let $x \in Q_{\Lambda_c}$. Then, either $x \in (Q_{\Gamma_c})_0 \backslash \{z\}$ or $x \in \{z_1, z_2, \cdots, z_{m-1}\}$. If $x \in \{z_1, z_2, \cdots, z_{m-1}\}$, by the two figures above, we get that $r_\Lambda P_{z_i}[0] \in \mathrm{Ins}(Q_{\Lambda_c})$ for any $i = 1, 2, \cdots, m-1$. Thus $r_\Lambda P_x[0] \in \mathrm{Ins}(Q_{\Lambda_c})$.
In case $x \in (Q_{\Gamma_c})_0 \setminus \{z\}$, by induction, we obtain that $r_\Gamma P_x[0] \in \mathrm{Ins}(Q_{\Gamma_c})$. On the other hand, using the two figures above, we get that $\Phi(\mathrm{Ins}(Q_{\Gamma_c})) \subseteq \mathrm{Ins}(Q_{\Lambda_c})$. Hence $r_\Lambda P_x[0] = \Phi(r_\Gamma P_x[0]) \in \Phi(\mathrm{Ins}(Q_{\Gamma_c})) \subseteq \mathrm{Ins}(Q_{\Lambda_c})$; proving (a).

(b) Since by assumption $C \subseteq Q_{\Lambda_c}$, we get that $Q_{\Lambda_a} = Q_{\Gamma_a}$ and $Q_{\Lambda_b} = Q_{\Gamma_b}$. Therefore $a(\Lambda) = a(\Gamma)$ and $b(\Lambda) = b(\Gamma)$. On the other hand $c(\Lambda) = c(\Gamma) + m - 1$. We only check the first equality in (b) since the second one can be checked in a very similar way.

We may assume that $c(\Gamma) + b(\Gamma)$ is even (the other case is similar). Then, by induction, we have that $r_\Gamma P_2[0] = {}_\Gamma X_2$. Furthermore, by 4.11 (b), we obtain that $r_\Lambda P_2[0] = \Phi(r_\Gamma P_2[0]) = \Phi({}_\Gamma X_2)$. Now, since $r_\Gamma P_3[0] \in \mathcal{X}^+_{r_\Gamma P_z[0]}$, we have that $\Phi = \Phi_{m-1, r_\Gamma P_z[0], \mathcal{X}^+}$ (see in 4.6). Moreover, we have that ${}_\Gamma X_2 \in \mathcal{Y}^+_{r_\Gamma P_z[0]}$ (see the picture above); and hence, by the definition of Φ, it follows that

$$(*) \quad r_\Lambda P_2[0] = \Phi({}_\Gamma X_2) = \sigma^{m-1}({}_\Lambda X_2) = \begin{cases} {}_\Lambda X_2 & \text{if } m-1 \text{ is even,} \\ {}_\Lambda Y_2 & \text{if } m-1 \text{ is odd.} \end{cases}$$

Since $c(\Lambda) = c(\Gamma) + m - 1$, it follows that $c(\Lambda) + b(\Lambda) = c(\Gamma) + b(\Gamma) + m - 1$. As a consequence, we have that $c(\Lambda) + b(\Lambda)$ and $m - 1$ have the same parity since $c(\Gamma) + b(\Gamma)$ is even. Hence, from $(*)$, we get the first equality in (b). $\quad \square$

6. Construction of the configuration associated to a trivial extension of Cartan class $\mathbf{D}_N$.

Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. The ordinary quiver Q_Λ, of a trivial extension Λ is described in 1.2 and 1.9. We recall that the quiver Q_Λ can be described by drawing only the minimal quiver $Q_{\Lambda_{min}}$ of Λ_{min} (see 1.8).

Consider the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ of ${}_S\Gamma_\Lambda$. In this section, we give an algorithm to determine the configuration $\widetilde{\mathcal{C}_\Lambda}$ of $\mathbb{Z}\mathbf{D}_N$ associated to Λ. We recall that $\widetilde{\mathcal{C}_\Lambda} := \pi^{-1}(\mathcal{C}_\Lambda)$, where $\mathcal{C}_\Lambda$ is the set of vertices of ${}_S\Gamma_\Lambda$ representing the radicals of the indecomposable projective Λ-modules (see [8, 2.7]). We define the subset $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)$ of $(\mathbb{Z}\mathbf{D}_N)_0$ and prove that $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)[\mathbb{Z}] := \cup_{i \in \mathbb{Z}} r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)[i]$ is the desired configuration.

We start by defining some notations needed to state the algorithm: the sections $\mathcal{S}_x^+$ and $\mathcal{S}_x^-$, the initial vertices, and the functions border and height defined on the set of vertices of Q_Λ. The algorithm is stated

in the Theorem 6.8 and illustrated in the Example 6.3. The remainder
of the section is devoted to the proof of the Theorem 6.8.

In what follows, we introduce two sections in $\mathbb{Z}\mathbf{D}_N$ which are as-
sociated with a given vertex in $\mathbb{Z}\mathbf{D}_N$. These sections will play an im-
portant role in the algorithm to compute the configuration associated
to the trivial extension Λ.

Definition 6.1. *Associated with a given vertex* (p,q) *in* $\mathbb{Z}\mathbf{D}_N$, *we
have the following two sections in* $\mathbb{Z}\mathbf{D}_N$:

$$\mathcal{S}^+_{(p,q)} := \{(p+N-1, i) \ : \ 1 \le i \le N-1\};$$

$$\mathcal{S}^-_{(p,q)} := \begin{cases} \{(p+q, i) \ : \ 1 \le i \le N-1\} & \text{if } q \ne N, \\ \\ \mathcal{S}^+_{(p,N)} & \text{if } q = N. \end{cases}$$

In the following picture, we illustrate the shape of the two parallel
sections $\mathcal{S}^+_x$ and $\mathcal{S}^-_x$ associated with the vertex $x = (p,q)$.
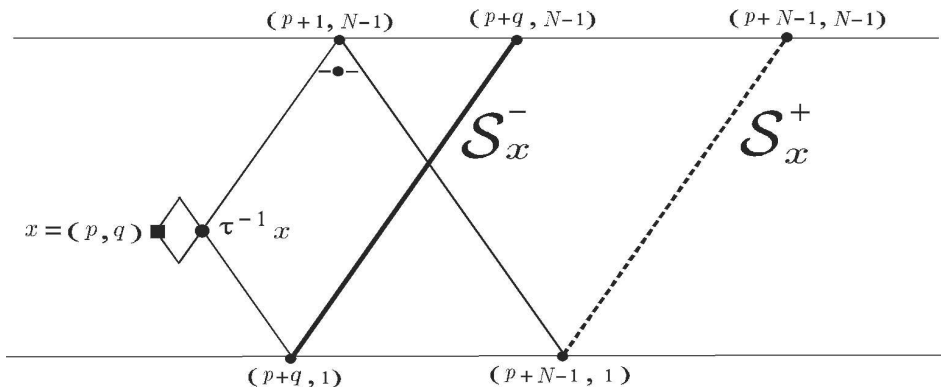
Definition 6.2. *Let* Λ *be a trivial extension of Cartan class* $\mathbf{D}_N$. *The*
initial vertices $\mathrm{Ini}\,(\Lambda)$ *of* Q_Λ *are those in the set*

$$\mathrm{Ini}\,(\Lambda) := \begin{cases} \{1,4\} & \text{if } \Lambda \text{ is of Class I,} \\ \\ \{1,2,3\} & \text{otherwise.} \end{cases}$$

The elements of the set $(Q_\Lambda)_0 \setminus \mathrm{Ini}\,(\Lambda)$ *are called the* **non-initial
vertices** *of* Q_Λ.

Definition 6.3. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. Any non-initial vertex i of Q_Λ induces two full connected subquivers of Q_Λ : namely, $Q_\Lambda^{-,i}$ and $Q_\Lambda^{+,i}$ which are defined as follows.*

(a) $Q_\Lambda^{-,i} \cap Q_\Lambda^{+,i} = \{i\}$ *and* $Q_\Lambda^{-,i} \cup Q_\Lambda^{+,i} = Q_\Lambda$;

(b) $Q_\Lambda^{-,i} = Q_{\Lambda'}$ *where Λ' is a trivial extension of Cartan class* $\mathbf{D}_{N'}$;

(c) *if i is an insertion vertex of Q_Λ, we set $Q_\Lambda^{+,i} := \{i\}$; and otherwise, $Q_\Lambda^{+,i} = Q_{\Lambda''}$ where Λ'' is a trivial extension of Cartan class $\mathbf{A}_{n''}$ with $n'' > 1$.*

The quivers $Q_\Lambda^{-,i}$ and $Q_\Lambda^{+,i}$ are said to be the splitting of Q_Λ at the vertex i.

Example 6.1. *Let Λ be the trivial extension of Class I and Cartan class D_6 given by the quiver:*
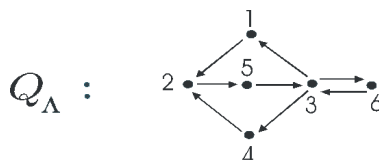
The splitting of Q_Λ at the vertex $i = 3$ is
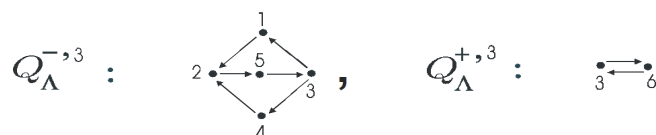
Definition 6.4. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. The* **border map** $\partial_\Lambda : (Q_\Lambda)_0 \to \{-, *, +\}$ *is defined as follows:*

(a) $\partial_\Lambda(i) := *$ *for any initial vertex i of Q_Λ;*

(b) *for any non-initial vertex $j \in Q_{\Lambda_{min}}$, we set $\partial_\Lambda(j) := +$ if there is an arrow $j \to v$ with $v \in \mathrm{Ini}\,(\Lambda)$, otherwise $\partial_\Lambda(j) := -$;*

(c) *let C and C' be two elemental cycles of Q_Λ (see 1.4) meeting only at the non-initial vertex x. Suppose we already have defined ∂_Λ on the vertices of the cycle C; so, for any vertex z of C', different from x, we set $\partial_\Lambda(z) := +$ if there is an arrow $z \to x$, otherwise $\partial_\Lambda(z) := -$.*

Definition 6.5. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. The* **height function** *$h_\Lambda : (Q_\Lambda)_0 \to \mathbb{N}$ is defined as follows.*

(a) *For the initial vertices of Q_Λ :*

(a1) *if Λ is of Class I, we set $h_\Lambda(1) := N-1$ and $h_\Lambda(4) := N$;*

(a2) *if Λ is of Class II, we set $h_\Lambda(3) := N-1$ and*

$$h_\Lambda(2) := \begin{cases} N-1 & \text{if } c(\Lambda) + b(\Lambda) \text{ is even,} \\ N & \text{if } c(\Lambda) + b(\Lambda) \text{ is odd;} \end{cases}$$

$$h_\Lambda(1) := \begin{cases} N-1 & \text{if } c(\Lambda) + a(\Lambda) \text{ is even,} \\ N & \text{if } c(\Lambda) + a(\Lambda) \text{ is odd;} \end{cases}$$

where $a(\Lambda)$, $b(\Lambda)$ and $c(\Lambda)$ are the numbers defined in 5.1

(a3) *if Λ is of Class III, we set $h_\Lambda(1) := N-2$, $h_\Lambda(3) := N-1$ and $h_\Lambda(2) := N$.*

(b) *$h_\Lambda(i) := |(Q_\Lambda^{+,i})_0|$ for any non-initial vertex i of Q_Λ.*

Remark 6.6. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. Consider the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ and a lifting of ${}_S\Gamma_\Lambda[0]$ to $\mathbb{Z}\mathbf{D}_N$ as in 2.10 (c), 2.11, 2.12 and 5.3. Then, for any initial vertex i of Q_Λ, we have that $h_\Lambda(i) = h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0])$.*

**Example 6.2.** *Let Λ be the trivial extension of Class I and Cartan class $\mathbf{D}_9$ given by the quiver*
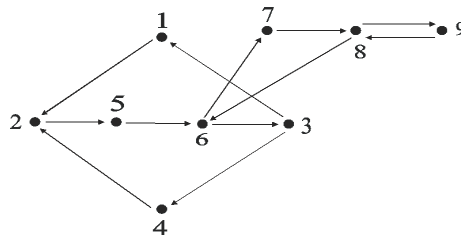
In the following table, we compute the values of the height function and the border map associated to Λ.

$i \in (Q_\Lambda)_0$	1	2	3	4	5	6	7	8	9
$\partial_\Lambda(i)$	$*$	$-$	$+$	$*$	$-$	$-$	$-$	$+$	$+$
$h_\Lambda(i)$	8	1	1	9	1	4	1	2	1

Definition 6.7. *Let Λ be a trivial extension of Cartan class $\mathcal{D}_N$ and u a vertex of the top border of $\mathbb{Z}\mathbf{D}_N$. Consider the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ and a lifting of ${}_S\Gamma_\Lambda[0]$ to $\mathbb{Z}\mathbf{D}_N$ as in 2.10 (c), 2.11, 2.12 and 5.3. We define the set $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N) := \{\widehat{rP_i} \in (\mathbb{Z}\mathbf{D}_N)_0 : i \in (Q_\Lambda)_0\}$ by the following rules.*
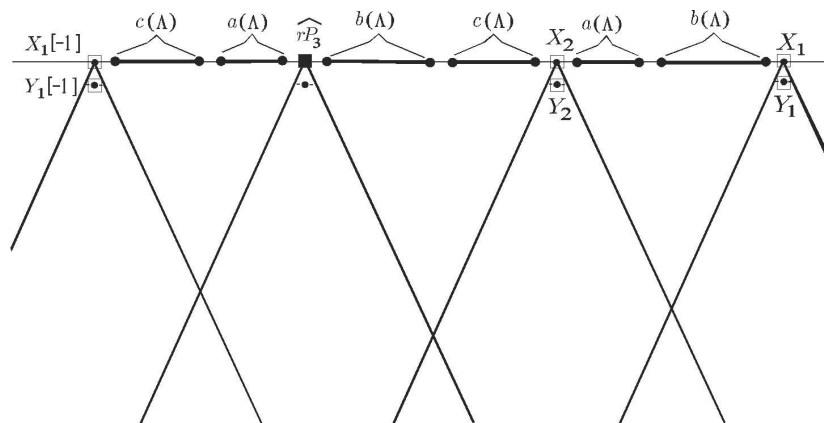
(a) *For the initial vertices of Q_Λ :*

    (a1) *if Λ is of Class I, we set $\widehat{rP_1} := u$ and define $\widehat{rP_4}$ as is indicated in the picture below*
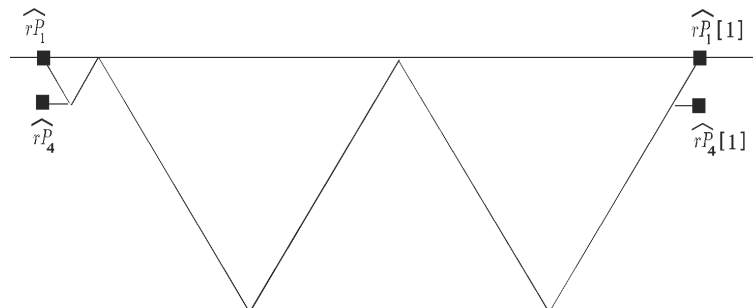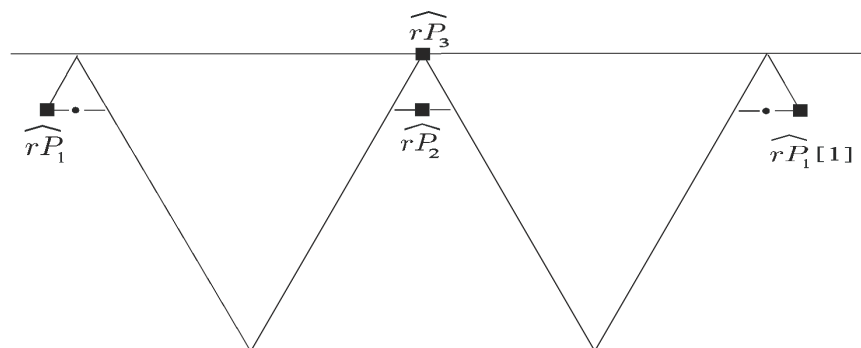

    (a2) *if Λ is of Class II, we set $\widehat{rP_3} := u$ and define $\widehat{rP_i}$, for $i = 1, 2$, as is indicated in the picture below, where*

$$\widehat{rP_2} := \begin{cases} X_2 & \text{if } c(\Lambda) + b(\Lambda) \text{ is even,} \\ Y_2 & \text{if } c(\Lambda) + b(\Lambda) \text{ is odd.} \end{cases}$$

$$\widehat{rP_1} := \begin{cases} X_1 & \text{if } c(\Lambda) + a(\Lambda) \text{ is even,} \\ Y_1 & \text{if } c(\Lambda) + a(\Lambda) \text{ is odd.} \end{cases}$$
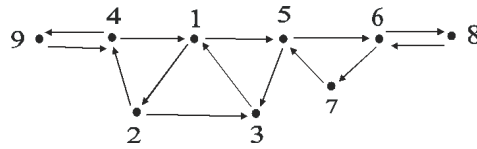
(a3) *if Λ is of Class III, we set $\widehat{rP_3} := u$ and define $\widehat{rP_i}$, for $i = 1, 2$, as is indicated in the picture below*

(b) *Let $i \to j$ be an arrow in Q_Λ with i a non-initial vertex. Suppose that we already have defined $\widehat{rP_j}$. Then, we set $\widehat{rP_i} := X_i$ for the vertex X_i in the section $\mathcal{S}_{\widehat{rP_j}}^{\partial_\Lambda(i)}$ satisfying the equality $h_{\mathbb{Z}\mathbf{D}_N}(X_i) = h_\Lambda(i)$.*
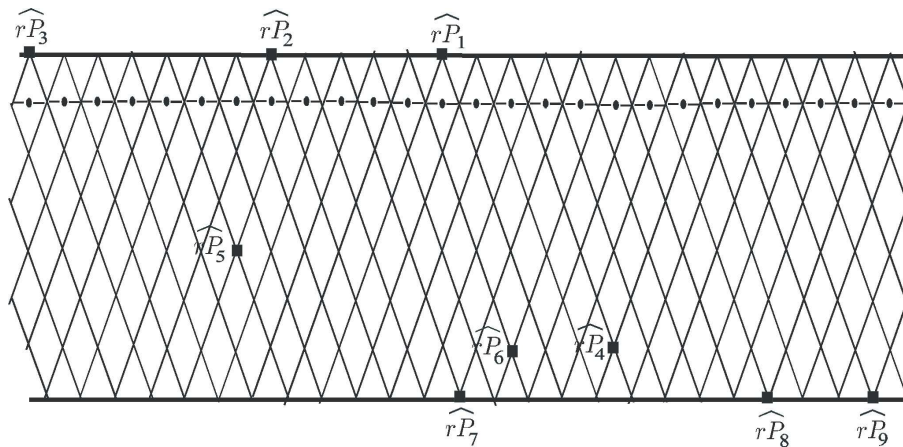
Example 6.3. *Let Λ be the trivial extension of Class II and Cartan class $\mathbf{D}_9$ given by the quiver*

In the following table, we compute the values of the height function and the border map associated to Λ.

$i \in (Q_\Lambda)_0$	1	2	3	4	5	6	7	8	9
$\partial_\Lambda(i)$	$*$	$*$	$*$	$+$	$+$	$-$	$+$	$+$	$+$
$h_\Lambda(i)$	8	8	8	2	4	2	1	1	1

In the picture below, we indicate the vertices of $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_9)$ with small black squares

Let $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)[\mathbb{Z}] := \{\tau^{s(2N-3)}(x) \ : \ x \in r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N), \ s \in \mathbb{Z}\}$. In what follows, we prove that such a set is the configuration $\widetilde{\mathcal{C}_\Lambda}$ of $\mathbb{Z}\mathbf{D}_N$ associated to Λ. This will be the main result of the paper.

Theorem 6.8. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. Fix a vertex u in $\mathbb{Z}\mathbf{D}_N$ and take $i_0 \in Q_\Lambda$ with $i_0 := 1$ if Λ is of Class I and $i_0 := 3$ otherwise. Consider the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ and a lifting of ${}_S\Gamma_\Lambda[0]$ to $\mathbb{Z}\mathbf{D}_N$ as in 2.10 (c), 2.11, 2.12, 5.3 and such that $rP_{i_0}[0] = u$. Let $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)$ be the set associated to these data (see 6.7). Then*

(a) $\widetilde{\mathcal{C}_\Lambda} = r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)[\mathbb{Z}]$;

(b) $\pi(\widehat{rP_i}) = rP_i$ *for any vertex $i \in Q_\Lambda$*;

(c) $h_{\mathbb{Z}\mathbf{D}_N}(rP_i[0]) = h_\Lambda(i)$ *for any vertex $i \in Q_\Lambda$.*

The proof of 6.8 will be carried out by induction on the number of minimal oriented cycles of Q_Λ. To do that, we delete an elemental cycle C (see 1.4) of Q_Λ obtaining a trivial extension Γ having less elemental cycles than Λ. So, we get the functions: $h_\Lambda, h_\Gamma, \partial_\Lambda, \partial_\Gamma$. In order to prove that theorem, we have to stablish, firstly, the relationship between such functions. To start with, it is easy to see that the restriction of ∂_Λ to $(Q_\Gamma)_0$ is ∂_Γ. However, the relationship between h_Λ and h_Γ is more complicated. Hence, to solve this problem, we introduce the notions of C-linked and C-free vertices of Q_Λ.

Definition 6.9. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$, and C be an elemental cycle of Q_Λ which is not contained in $Q_{\Lambda_{min}}$. It can be seen easily, that there exists only one chain $C = C_1, C_2, \cdots, C_\ell$ consisting of elemental cycles of Q_Λ such that: (a) $(C_i)_0 \cap (C_{i+1})_0 = \{x_i\}$ where x_i is a vertex in Q_Λ for $1 \le i < \ell$, (b) $C_i \nsubseteq Q_{\Lambda_{min}}$ for $1 \le i < \ell$ and (c) $C_\ell \subseteq Q_{\Lambda_{min}}$. The vertices $x_1, x_2, \cdots, x_{\ell-1}$ are said to be C-**linked**; and the remaining ones of Q_Λ, which are non-initial vertices, will be called C-**free**.*

Example 6.4. *Consider the trivial extension Λ of Class II_2 and Cartan class $\mathbf{D}_9$, having as quivers Q_Λ and $Q_{\Lambda_{min}}$ the following ones*
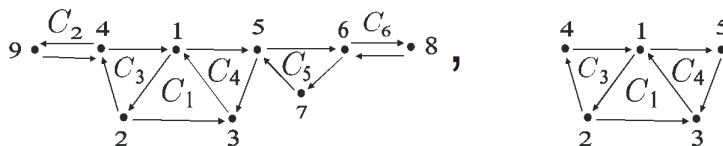
The elemental cycles of Q_Λ are: C_1, C_2, C_3, C_4, C_5 and C_6; and can be seen in the above figures. In this case, we have

$\bullet$) *vertices* $C_2 - \begin{cases} \textit{ligados: } 4. \\ \textit{libres: } 5, 6, 7, 8, 9. \end{cases}$ $\bullet$) *vertices* $C_6 - \begin{cases} \textit{ligados: } 5, 6 \\ \textit{libres: } 4, 7, 8, 9. \end{cases}$

Proposition 6.10. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$ and C an elimination cycle of Q_Λ (see 2.1). Consider the trivial extension Γ obtained from Λ by eliminating the cycle C. Then, for any vertex $t \in Q_\Gamma$ which is a non-initial one, we have that*

$$h_\Lambda(t) = \begin{cases} h_\Gamma(t) & \textit{if } t \textit{ is } C\textit{-free,} \\ h_\Gamma(t) + |(\underline{C})_0| - 1 & \textit{if } t \textit{ is } C\textit{-linked;} \end{cases}$$

where $|(\underline{C})_0|$ is the number of vertices in the cycle C.

Proof.   It is straightforward. $\square$

In the following proposition, we prove a result that will be essential to do the inductive step in the proof of 6.8. Here, we make use of the partition $\{C_y^-[i], C_y^+[i] \ : \ i \in \mathbb{Z}\}$ in $\mathbb{Z}\mathbf{D}_N$ induced by a vertex y in the bottom border of $\mathbb{Z}\mathbf{D}_N$ (see 4.2).
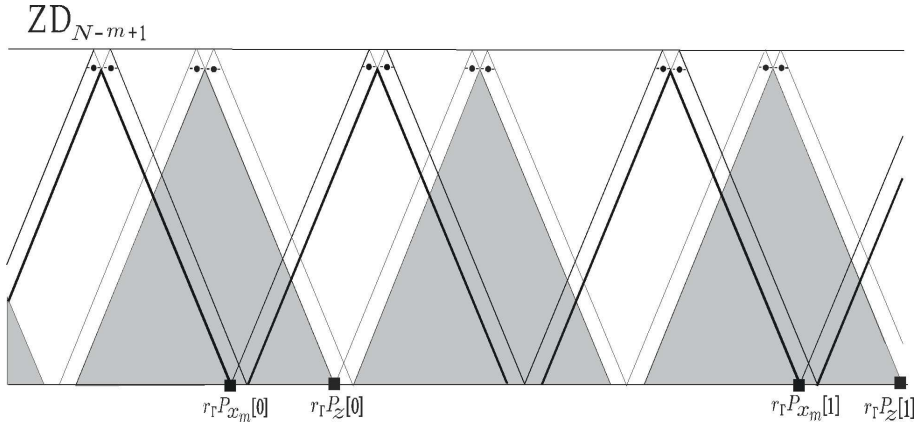
Proposition 6.11. *Let Λ be a trivial extension of Cartan class $\mathbf{D}_N$. Consider the universal covering $\pi : \mathbb{Z}\mathbf{D}_N \to {}_S\Gamma_\Lambda$ and a lifting of ${}_S\Gamma_\Lambda[0]$ to $\mathbb{Z}\mathbf{D}_N$ as in 2.10 (c), 2.11, 2.12 and 5.3. Then, for any insertion vertex z of Q_Λ and any vertex $x \in Q_\Lambda$, the following statements hold.*

(a) *If $z \in Q_{\Lambda_{min}}$ and $\partial_\Lambda(x) \neq *$, then $rP_x[0] \in C_{rP_z[0]}^-$.*

(b) *If $\mathcal{C}$ is an elemental cycle of Q_Λ which is not contained in $Q_{\Lambda_{min}}$ and z is a vertex in the cycle $\mathcal{C}$, then*

(b1) *$rP_x[0] \in C_{rP_z[0]}^+$ if x is $\mathcal{C}$-linked,*

(b2) *$rP_x[0] \in C_{rP_z[0]}^-$ if x is $\mathcal{C}$-free.*

Proof.   We proceed by induction on the number of minimal cycles of Q_Λ.

(a) Let $z \in Q_{\Lambda_{min}}$ be an insertion vertex of Q_Λ and $\partial_\Lambda(x) \neq *$. If Λ is minimal, then (a) follows from 2.11, 2.12, 2.14 and 2.15. Suppose that Λ is not minimal, and let $\mathcal{C} = x_m \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow x_m$ be an elimination cycle of Q_Λ. Consider the trivial extension Γ of Cartan

class $\mathbf{D}_{N-m+1}$ obtained from Λ by eliminating the cycle C. We make use of 4.11, and so, we assume all the hypothesis needed there. Hence, we have a full an faithful functor $\Phi := \Phi_{x_m, i_0} : k(\mathbb{Z}\mathbf{D}_{N-m+1}) \rightarrow k(\mathbb{Z}\mathbf{D}_N)$. Thus, by 4.9 and 4.11, we are able to compare the partition $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i] : i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$ with the partition $\{C^-_{r_\Lambda P_z[0]}[i], C^+_{r_\Lambda P_z[0]}[i] : i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_N$ (see the following picture).

$\mathbb{Z}\mathbf{D}_{N-m+1}$

$$r_\Gamma P_{x_m}[0] \qquad r_\Gamma P_z[0] \qquad\qquad\qquad r_\Gamma P_{x_m}[1] \quad r_\Gamma P_z[1]$$

The shaded areas in the preceding picture correspond to $C^-_{r_\Gamma P_z[0]}$. Since $x_m \neq z$ we get by 4.11 (b1) that $\Phi(r_\Gamma P_z[0]) = r_\Lambda P_z[0]$. Furthermore, by definition of Φ and the above picture, it can be seen that $\Phi(C^-_{r_\Gamma P_z[0]}) \subseteq C^-_{\Phi(r_\Gamma P_z[0])}$. Hence, we get

$$(E1) \qquad \Phi(C^-_{r_\Gamma P_z[0]}) \subseteq C^-_{r_\Lambda P_z[0]}.$$

If x is a vertex in the cycle C, then by the picture above, 3.1 (a) and 4.11, we obtain that: (i) $r_\Lambda P_{x_m}[0] \in C^-_{r_\Lambda P_z[0]}$ and (ii) for $x \neq x_m$, the lifted radical $r_\Lambda P_x[0]$ is in the bottom border of $\mathbb{Z}\mathbf{D}_N$ and $r_\Lambda P_x[0] \notin \Phi(C^-_{r_\Gamma P_{x_m}[0]})$. Thus $r_\Lambda P_x[0] \in C^-_{r_\Lambda P_z[0]}$, proving (a) in this case.

Let, now, x be a vertex in Q_Γ different from x_m. Then by induction $r_\Gamma P_x[0] \in C^-_{r_\Gamma P_z[0]}$. Hence $r_\Lambda P_x[0] = \Phi(r_\Gamma P_x[0]) \in \Phi(C^-_{r_\Gamma P_z[0]}) \subseteq C^-_{r_\Lambda P_z[0]}$ by 4.11 (b) and (E1); proving (a).

(b) Let C be an elemental cycle of Q_Λ which is not contained in $Q_{\Lambda_{min}}$ and z be a vertex in C. In what follows, we analyze two possibilities.

Case 1: Suppose that $\mathcal{C}$ is an elimination cycle of Q_Λ.
Let $\mathcal{C} = y \leftarrow y_1 \leftarrow y_2 \leftarrow \cdots \leftarrow y_{m-1} \leftarrow y$. By assumption, we have that $z \in \{y_1, y_2, \cdots, y_{m-1}\}$. Let Γ be the trivial extension of Cartan class $\mathbf{D}_{N-m+1}$ obtained from Λ by eliminating the cycle $\mathcal{C}$. We assume all the hypothesis needed in 4.11, and so, we have a full an faithful functor $\Phi := \Phi_{y,i_0} : k(\mathbb{Z}\mathbf{D}_{N-m+1}) \to k(\mathbb{Z}\mathbf{D}_N)$. Consider the partition $\{C^-_{r_\Gamma P_y[0]}[i], C^+_{r_\Gamma P_y[0]}[i] \; : \; i \in \mathbb{Z}\}$ of $\mathbf{ZD}_{N-m+1}$, that can be illustrated by the shaded areas in the following picture.
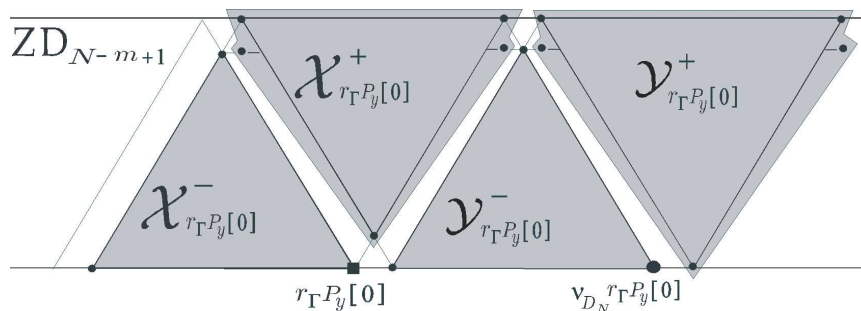
Applying the functor $\Phi = \Phi_{y,i_0} : k(\mathbb{Z}\mathbf{D}_{N-m+1}) \to k(\mathbb{Z}\mathbf{D}_N)$ to the partition above we get, by 4.10 and 4.11, that such a partition can be embedded in $k(\mathbb{Z}\mathbf{D}_N)$ as can be seen in the following picture; furthermore, in this picture, we also illustrate the partition $\{C^-_{r_\Lambda P_z[0]}[i], C^+_{r_\Lambda P_z[0]}[i] \; : \; i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_N$ for $z = y_1$.
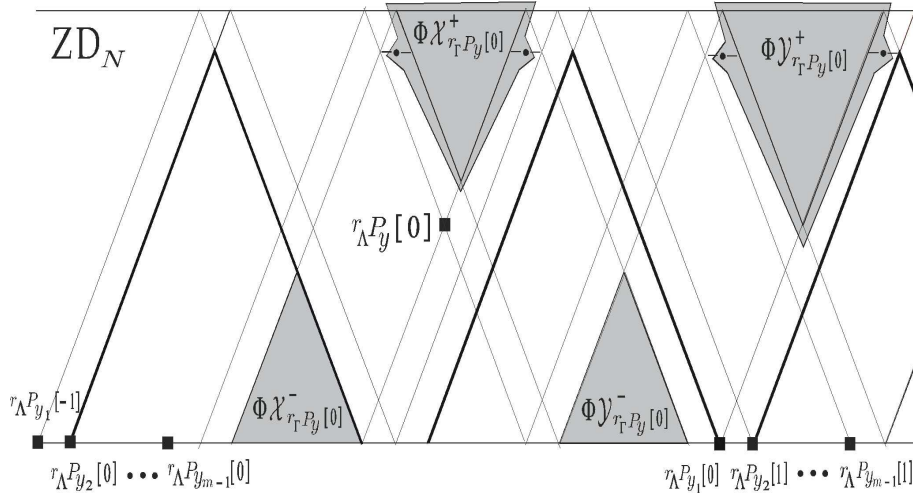
By using the two pictures above, it is not difficult to prove the following lemma.

Lemma (E2) For any $z \in \{y_1, y_2, \cdots, y_{m-1}\}$, we have that

(i) $r_\Lambda P_y[0] \in C^+_{r_\Lambda P_z[0]}$,

(ii) $\Phi(C^\pm_{r_\Gamma P_y[0]}) \subseteq C^\pm_{r_\Lambda P_z[0]}$,

(iii) $r_\Lambda P_w[0] \in C^-_{r_\Lambda P_z[0]}$ for all $w \in \{y_1, y_2, \cdots, y_{m-1}\}$.

Now, we are ready to prove (b1) and (b2) in case $\mathcal{C}$ is an elimination cycle of Q_Λ. Let x be a non-initial vertex in Q_Λ, that is, $\partial_\Lambda(x) \neq *$.

(b1) Assume that x is $\mathcal{C}$-linked. If $x = y$ then, by (E2) (i), it follows that $r_\Lambda P_x[0] \in C^+_{r_\Lambda P_z[0]}$, otherwise $x \in (Q_\Gamma)_0 \setminus \{y\}$; and so, by induction we have $r_\Gamma P_x[0] \in C^+_{r_\Gamma P_z[0]}$. Hence, from (E2) (ii) and 4.11 (b1), we conclude that
$r_\Lambda P_x[0] = \Phi(r_\Gamma P_x[0]) \in \Phi(C^+_{r_\Gamma P_z[0]}) \subseteq C^+_{r_\Lambda P_z[0]}$; proving (b1).

(b2) Assume that x is $\mathcal{C}$-free. If x is a vertex in the cycle $\mathcal{C}$ then, by (E2) (iii), it follows that $r_\Lambda P_x[0] \in C^-_{r_\Lambda P_z[0]}$. Suppose that $x \in (Q_\Gamma)_0$

with $x \neq y$. Then by induction $r_\Gamma P_x[0] \in C^-_{r_\Gamma P_z[0]}$. Therefore, from (E2) (ii) and 4.11 (b1), we get $r_\Lambda P_x[0] = \Phi(r_\Gamma P_x[0]) \in \Phi(C^-_{r_\Gamma P_z[0]}) \subseteq C^-_{r_\Lambda P_z[0]}$; proving (b2).

Case 2: Suppose that $\mathcal{C}$ is not an elimination cycle of Q_Λ. Let $C' = x_m \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow x_m$ be an elimination cycle of Q_Λ having a chain of elemental cycles $\mathcal{C} = C'_1, C'_2, \cdots, C'_{\ell-1}, C'_\ell = C'$ as in 6.9 (see the picture below).
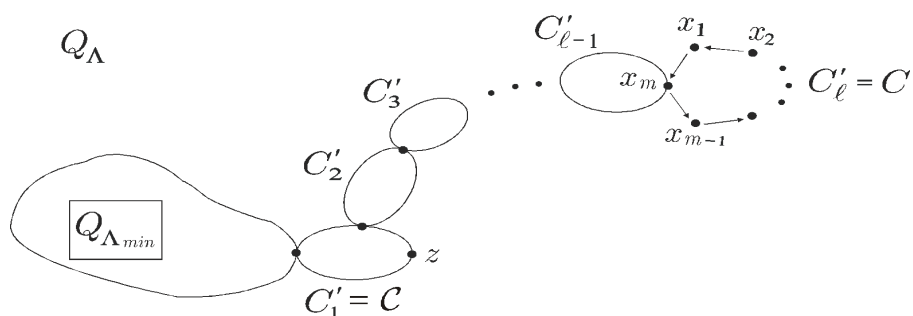
Consider the trivial extension Γ of Cartan class D_{N-m+1} obtained from Λ by eliminating the cycle C'. In particular, we have that x_m is an insertion vertex of Q_Γ. We assume all the hypothesis needed in 4.11; and so, we have a full an faithful functor $\Phi := \Phi_{x_m, i_0} : k(\mathbb{Z}\mathbf{D}_{N-m+1}) \to k(\mathbb{Z}\mathbf{D}_N)$. Furthermore, we have a partition $\{C^-_{r_\Gamma P_{x_m}[0]}[i], C^+_{r_\Gamma P_{x_m}[0]}[i] : i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$ (see the first picture of this proof). We use the functor Φ, to compare the partition $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i] : i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$ with the partition $\{C^-_{r_\Lambda P_z[0]}[i], C^+_{r_\Lambda P_z[0]}[i] : i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_N$. Hence, we get the following lemma.

Lemma (E3) With the preceding notations and hypothesis, we have

(i) $\Phi(C^\pm_{r_\Gamma P_z[0]}) \subseteq C^\pm_{r_\Lambda P_z[0]}$,

(ii) $r_\Lambda P_x[0] \in C^-_{r_\Lambda P_z[0]}$ for any vertex w in C'.

Now, we are ready to prove (b1) and (b2) in case $\mathcal{C}$ is not an elimination cycle of Q_Λ. Let x be a non-initial vertex in Q_Λ.

(b1) Assume that x is $\mathcal{C}$-linked. So $x \in (Q_\Gamma)_0$, and by induction, $r_\Gamma P_x[0] \in C^+_{r_\Gamma P_z[0]}$. Hence, from (E3) (i) and 4.11 (b1), it follows

$$r_\Lambda P_x[0] = \Phi(r_\Gamma P_x[0]) \in \Phi(C^+_{r_\Gamma P_z[0]}) \subseteq C^+_{r_\Lambda P_z[0]};$$

proving (b1).

(b2) Assume that x is $\mathcal{C}$-free. If x is a vertex in the cycle C' then, by (E3) (ii), we obtain that $r_\Lambda P_x[0] \in C^-_{r_\Lambda P_z[0]}$.
Suppose that $x \in (Q_\Gamma)_0$ with $x \neq x_m$. So, we have by induction that $r_\Gamma P_x[0] \in C^-_{r_\Gamma P_z[0]}$. Therefore, from (E3) (i) and 4.11 (b1), we get that

$$r_\Lambda P_x[0] = \Phi(r_\Gamma P_x[0]) \in \Phi(C^-_{r_\Gamma P_z[0]}) \subseteq C^-_{r_\Lambda P_z[0]};$$

proving the result. $\square$

Now, we are in position to prove the main result of this section.

Proof of Theorem 6.8 It is enough to prove (b) since it implies (a) and (c). The proof will be carried out by induction on the number of minimal oriented cycles of Q_Λ. If Λ is minimal then from 2.11, 2.12, 2.14 and 2.15 the proof of (b) is complete.

Suppose that Λ is not minimal. Let $C' = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots z_{m-1} \leftarrow z$ be an elimination cycle of Q_Λ. Consider the trivial extension Γ of Cartan class $\mathbf{D}_{N-m+1}$ obtained from Λ eliminating the cycle C'. We assume all the hypothesis needed in 4.11; and so, we have a full an faithful functor

$$\Phi := \Phi_{z,i_0} : k(\mathbb{Z}\mathbf{D}_{N-m+1}) \to k(\mathbb{Z}\mathbf{D}_N).$$

Using such a functor, we will compare $r\mathcal{P}(\Gamma, \mathbb{Z}\mathbf{D}_{N-m+1})$ with $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{D}_N)$. Consider the partition $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i] \ : \ i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$ (see 4.10 (a)). Then, the functor Φ embeds such a partition in $\mathbb{Z}\mathbf{D}_N$ as can be seen in 4.10 (b).

By induction, we have that Theorem 6.8 is true for Γ; therefore $\pi(\widehat{r_\Gamma P_i}) = r_\Gamma P_i$ for any vertex $i \in Q_\Gamma$, where $\pi : \mathbb{Z}\mathbf{D}_{N-m+1} \to {}_S\Gamma_\Gamma$ is the universal covering of ${}_S\Gamma_\Gamma$.

Given a vertex x in $\mathbb{Z}\mathbf{D}_{N-m+1}$, we start by finding a relationship between $h_{\mathbb{Z}\mathbf{D}_N}(\Phi(x))$ and $h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(x)$.

Lemma A For any vertex x in $\mathbb{Z}\mathbf{D}_{N-m+1}$, we have that

(a1) $\Phi(\mathcal{S}_x^{\pm}) \subseteq \mathcal{S}_{\Phi(x)}^{\pm}$ if x and $\tau^{-1}x$ belong to the same component in the partition $\{C_{r_\Gamma P_z[0]}^{-}[i], C_{r_\Gamma P_z[0]}^{+}[i] \ : \ i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$;

(a2) $\Phi(\mathcal{S}_{r_\Gamma P_z[0]}^{-}) \subseteq \mathcal{S}_{r_\Lambda P_z[0]}^{-}$;

(a3) if $x = r_\Gamma P_j[d]$ for $j \in (Q_\Gamma)_0 \setminus \{z\}$ and $d \in \mathbb{Z}$, then x and $\tau^{-1}x$ belong to the same component in the partition $\{C_{r_\Gamma P_z[0]}^{-}[i],$ $C_{r_\Gamma P_z[0]}^{+}[i] \ : \ i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$.

(a4) $h_{\mathbb{Z}\mathbf{D}_N}(\Phi(x)) = \begin{cases} h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(x) & \text{if } x \in C_{r_\Gamma P_z[0]}^{-}, \\ h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(x) + m - 1 & \text{if } x \in C_{r_\Gamma P_z[0]}^{+}. \end{cases}$

Proof. We apply the full an faithful functor $\Phi = \Phi_{z,i_0}$ to the desired partition $\{C_{r_\Gamma P_z[0]}^{-}[i], C_{r_\Gamma P_z[0]}^{+}[i] \ : \ i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$; afterwards, by 4.11 and 6.1, we get the items (a1), (a2) and (a4) as can be seen by using the following two pictures.
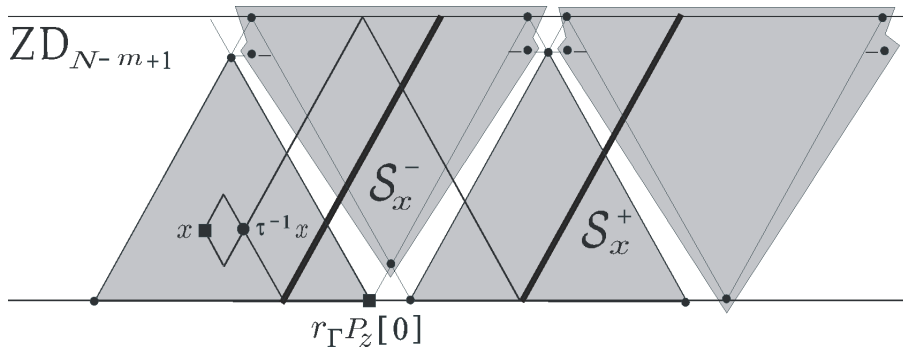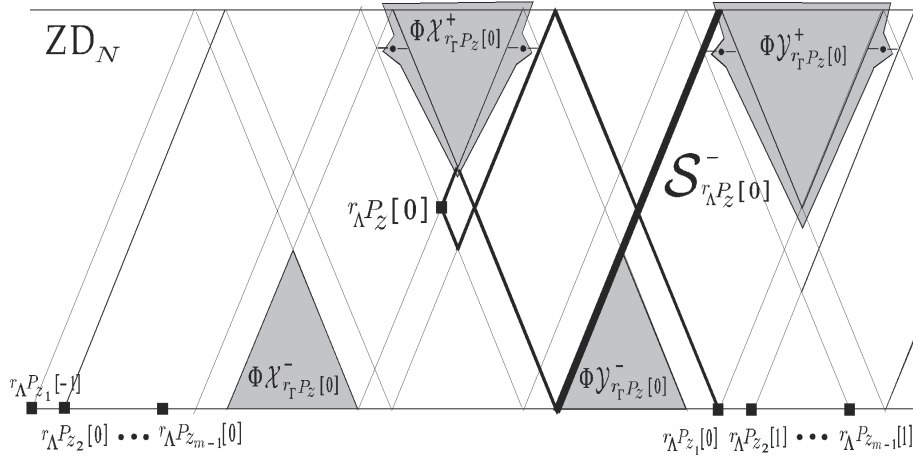
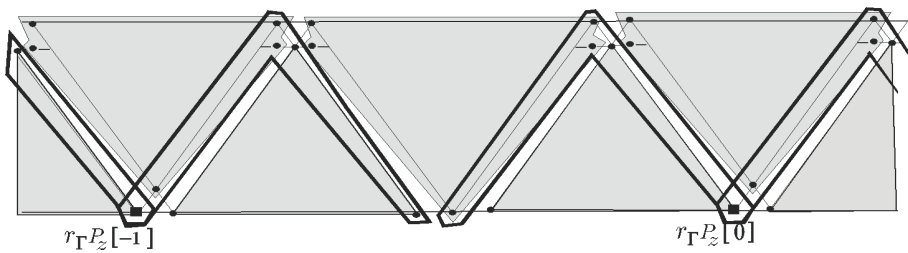(a3) Denote by $\wp$ to the set of vertices y in $\mathbb{Z}\mathbf{D}_{N-m+1}$ such that y and $\tau^{-1}y$ do not belong to the same component in the partition $\{C^-_{r_\Gamma P_z[0]}[i],$
$C^+_{r_\Gamma P_z[0]}[i] \,:\, i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$. We assert that

$$\wp = \bigcup_{i\in\mathbb{Z}} \mathrm{Supp}(r_\Gamma P_z[i], -) \cup \mathrm{Supp}(-, r_\Gamma P_z[i]).$$

Indeed, by 1.11 (a), we know the shape in $\mathbb{Z}\mathbf{D}_{N-m+1}$ of $\mathrm{Supp}(-, w)$ and $\mathrm{Supp}(w, -)$ for any vertex w in the bottom border of $\mathbb{Z}\mathbf{D}_{N-m+1}$; and so, our assertion follows from the picture below.

On the other hand, since $j \neq z$ we have that

$$\underline{\mathrm{Hom}}\,(r_\Gamma P_j, r_\Gamma P_z) = 0 = \underline{\mathrm{Hom}}\,(r_\Gamma P_j, r_\Gamma P_z).$$

Thus, by the assertion above, we conclude that $r_\Gamma P_j[d] \notin \wp$; proving (a3). $\square$

Lemma B Let $i \to j$ be an arrow in Q_Γ satisfying $\partial_\Gamma(i) \neq *$. Then

(b1) if $i \neq z$ and $j \neq z$, then $\Phi(\widehat{r_\Gamma P_i}) \in \mathcal{S}^{\partial_\Lambda(i)}_{\Phi(\widehat{r_\Gamma P_j})}$ with $h_{\mathbb{Z}\mathbf{D}_N}(\Phi(\widehat{r_\Gamma P_i}))$
       $= h_\Lambda(i)$;

(b2) if $j = z$ then $\Phi(\widehat{r_\Gamma P_i}) \in \mathcal{S}^-_{r_\Lambda P_z[0]}$ with $h_{\mathbb{Z}\mathbf{D}_N}(\Phi(\widehat{r_\Gamma P_i})) = h_\Lambda(i)$;

(b3) if $i = z$ then $r_\Lambda P_z[d] \in \mathcal{S}^{\partial_\Lambda(z)}_{\Phi(\widehat{r_\Gamma P_j})}$ with $h_{\mathbb{Z}\mathbf{D}_N}(r_\Lambda P_z[d]) = m-1 =$
       $h_\Lambda(z)$ for some integer d.

Proof.   (b1) By the definition of $r\mathcal{P}(\Gamma, \mathbb{Z}\mathbf{D}_{N-m+1})$, we have that $\widehat{r_\Gamma P_i} \in \mathcal{S}^{\partial_\Gamma(i)}_{\widehat{r_\Gamma P_j}}$ with $h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(\widehat{r_\Gamma P_i}) = h_\Gamma(i)$. On the other hand, by Lemma A (a1) and (a3), it follows $\Phi(\widehat{r_\Gamma P_i}) \in \Phi(\mathcal{S}^{\partial_\Gamma(i)}_{\widehat{r_\Gamma P_j}}) \subseteq \mathcal{S}^{\partial_\Lambda(i)}_{\Phi(\widehat{r_\Gamma P_j})}$ since $\partial_\Gamma(i) = \partial_\Lambda(i)$, $j \neq z$ and $\widehat{r_\Gamma P_j} = r_\Gamma P_j[0]$. In order to prove that $h_{\mathbb{Z}\mathbf{D}_N}(\Phi(\widehat{r_\Gamma P_i})) = h_\Lambda(i)$, we consider two cases and use the equality $h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(\widehat{r_\Gamma P_i}) = h_\Gamma(i)$ that holds by induction.

    Case I: z is a vertex in $Q_{\Lambda_{min}}$. So, in this case, z is the unique vertex which is C'-linked. In particular, the vertex i is C'-free; and then, by 6.10, it follows that $h_\Gamma(i) = h_\Lambda(i)$. On the other hand, from 6.11 (a), we get that $r_\Gamma P_i[0] \in C^-_{r_\Gamma P_z[0]}$. Thus, by Lemma A (a4), we obtain that $h_{\mathbb{Z}\mathbf{D}_N}(\Phi(\widehat{r_\Gamma P_i})) = h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(\widehat{r_\Gamma P_i})$. Therefore

$$h_{\mathbb{Z}\mathbf{D}_N}(\Phi(\widehat{r_\Gamma P_i})) = h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(\widehat{r_\Gamma P_i}) = h_\Gamma(i) = h_\Lambda(i).$$

    Case II: z does not belongs to $Q_{\Lambda_{min}}$. So, the vertex i is either C'-linked or C'-free. If i is C'-free, we get by 6.11 (b2) and 6.10 that $r_\Gamma P_i[0] \in C^-_{r_\Gamma P_z[0]}$ and $h_\Lambda(i) = h_\Gamma(i)$; and hence

$$h_{\mathbb{Z}\mathbf{D}_N}(\Phi(\widehat{r_\Gamma P_i})) = h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(\widehat{r_\Gamma P_i}) = h_\Gamma(i) = h_\Lambda(i).$$

Assume that the vertex i is C'-linked. Then, by 6.11 (b1) and 6.10, it follows that $r_\Gamma P_i[0] \in C^+_{r_\Gamma P_z[0]}$ and $h_\Lambda(i) = h_\Gamma(i) + m - 1$. Therefore,

$$h_{\mathbb{Z}\mathbf{D}_N}(\Phi(\widehat{r_\Gamma P_i})) = h_{\mathbb{Z}\mathbf{D}_{N-m+1}}(\widehat{r_\Gamma P_i}) + m - 1 = h_\Gamma(i) + m - 1 = h_\Lambda(i);$$

proving (b1).

(b2) By the definition of ∂_Γ (see 6.4), we have that $\partial_\Gamma(i) = -$ since by assumption there is an arrow $i \to z$ and z is C'-linked. Then, by using Lemma A (a2), the proof of (b2) follows similarly as the one given in (b1).

(b3) Suppose $i = z$, that is, we have an arrow $z \to j$ in Q_Γ with $\partial_\Gamma(z) \neq *$. We may assume that $d = 0$, since otherwise we apply an appropriate shifting. Then, by induction, we have that $r_\Gamma P_z[0] \in \mathcal{S}^{\partial_\Gamma(z)}_{\widehat{r_\Gamma P_j}}$. Since $j \neq z$, we conclude, from Lemma A (a3), that $r_\Gamma P_j[0]$ and $\tau^{-1} r_\Gamma P_j[0]$ belong to the same component in the partition $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i] \ : \ i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{D}_{N-m+1}$ as can be seen in the following two pictures.
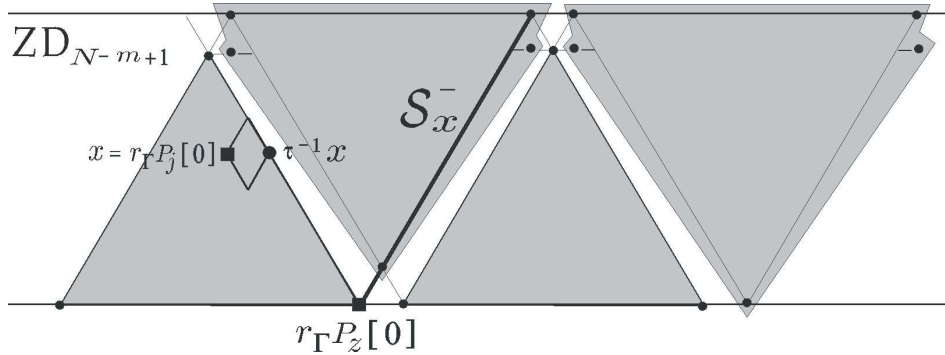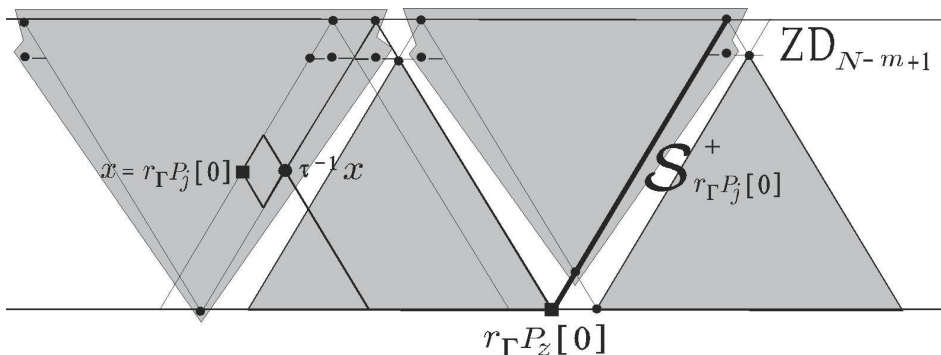
But $h_\Lambda(z) = m - 1$; and so, by 4.11, we get that $r_\Lambda P_z[0] \in \mathcal{S}^{\partial_\Lambda(z)}_{\Phi(\widehat{r_\Gamma P_j})}$ with $h_{\mathbb{Z}\mathbf{D}_N}(r_\Lambda P_z[0]) = m - 1 = h_\Lambda(z)$; proving (b3). $\square$

We are now in position to finish the proof of (b) in the Theorem 6.8. For the initial vertices t of Q_Λ, by 6.6 and 6.7 (a), we conclude that $\pi(\widehat{r_\Lambda P_t}) = r_\Lambda P_t$. On the other hand, since $\Phi(\widehat{r_\Gamma P_t}) = \Phi(r_\Gamma P_t[0]) = r_\Lambda P_t[0]$, we choose the coordinates of $\widehat{r_\Lambda P_t}$ in $\mathbb{Z}\mathbf{D}_N$ in such a way that $\Phi(\widehat{r_\Gamma P_t}) = \widehat{r_\Lambda P_t}$. So, we assert the following:

(C) $\Phi(\widehat{r_\Gamma P_i}) = \widehat{r_\Lambda P_i}$ for any $i \in (Q_\Gamma)_0 \setminus \{z\}$.

Indeed, let $i \to j$ be an arrow in Q_Γ with $\partial_\Gamma(j) \neq *$ and $i \neq z$. Since $\Phi(\widehat{r_\Gamma P_t}) = \widehat{r_\Lambda P_t}$ for $\partial_\Lambda(t) = *$, to get (C), it is enough to proof the following:

1) if $j \neq z$ and $\Phi(\widehat{r_\Gamma P_j}) = \widehat{r_\Lambda P_j}$, then $\Phi(\widehat{r_\Gamma P_i}) = \widehat{r_\Lambda P_i}$;

2) if $j = z$ then $\Phi(\widehat{r_\Gamma P_i}) \in \mathcal{S}^-_{r_\Lambda P_z[0]}$ with $h_{\mathbb{Z}\mathbf{D}_N}(\Phi(\widehat{r_\Gamma P_i})) = h_\Lambda(i)$.

But now, the statements 1) and 2) follow from Lemma B items (b1) and (b2); proving (C).

Let now i be any vertex in Q_Λ. We consider several cases as follows.

Case I: Assume that $i \in (Q_\Gamma)_0 \setminus \{z\}$. Then, by 4.11 (b1), we have that $\Phi(r_\Gamma P_i[0]) = r_\Lambda P_i[0]$. On the other hand, from (C), we know

that $\Phi(\widehat{r_\Gamma P_i}) = \widehat{r_\Lambda P_i}$. Hence $r_\Lambda P_i[0] = \Phi(r_\Gamma P_i[0]) = \Phi(\widehat{r_\Gamma P_i}) = \widehat{r_\Lambda P_i}$, proving (b) in this case.

Case II: Let $i = z$ and $z \to j$ be an arrow in Q_Γ. Hence, by (C), we have that $\Phi(\widehat{r_\Gamma P_j}) = \widehat{r_\Lambda P_j}$ since $j \neq z$. Therefore, Lemma B (b3) implies that $r_\Lambda P_z[d] \in \mathcal{S}^{\partial_\Lambda(z)}_{\widehat{r_\Lambda P_j}}$ with $h^{\partial_\Lambda(z)}_{\mathbb{Z}\mathbf{D}_N}(r_\Lambda P_z[d]) = h_\Lambda(z)$; and hence $\widehat{r_\Lambda P_z} = r_\Lambda P_z[d]$ for some integer d. We may assume that $d = 0$, since otherwise we apply an appropriate shifting.

Case III: Suppose that $i \in \{z_1, z_2, \cdots, z_{m-1}\}$. We already know that $\widehat{r_\Lambda P_z} = r_\Lambda P_z[0]$. Then, from the second picture appearing in the proof of Lemma A, it can be seen that $\widehat{r_\Lambda P_{z_1}} = r_\Lambda P_{z_1}[0]$ and so $\widehat{r_\Lambda P_i} = r_\Lambda P_i[1]$ for $i = z_2, \cdots, z_{m-1}$; proving (b) in this case and ending the proof of the Theorem 6.8.

$\square$

ACKNOWLEDGMENTS

References

[1]   H. Asashiba. *The derived Equivalence Classification of Representation-Finite Selfinjective Algebras.* Journal of Algebra 214, 182-221 (1999).

[2]   I. Assem. *Tilting Theory - an introduction.* Topics in algebra. Banach Center publications, volume 26, part 1(1990).

[3]   M. Auslander, I. Reiten, S. Smalo. *Representation Theory of Artin Algebras.* Cambridge University Press, (1995).

[4]   Otto Bretscher, Chr. Läser, Chr. Riedtmann. *Selfinjective and Simply Connected Algebras.* Manuscripta Math. 36, 253-307, (1981).

[5]   E. Fernández. Ph.D. Thesis. Universidad Nacional del Sur, Argentina (1999).

[6]   E. Fernández and M.I. Platzeck. *Presentations of trivial extensions of finite dimensional algebras and a theorem of S. Brenner.* Journal of Algebra, 249, 326-344 (2002).

[7]   O. Mendoza. Symetric Quasi-Schurian Algebras. *Lecture. Notes in pure and applied Mathematics*, Marcel Dekker Inc. Octubre 2002, Vol. 224, 99-116.

[8]   O. Mendoza and M.I. Platzeck.*Embedding of the AR-quiver of an iterated tilted algebra of Dynkin type Δ in $\mathbb{Z}\Delta$.* Journal of algebra 265(2003)247-263.

[9]   O. Mendoza and M.I. Platzeck.*Configuration and trivial extension of Dynkin type* $\mathbf{A}_n$. Journal of algebra 281(2004)167-199.

[10]   Chr. Riedtmann. *Representation-Finite Selfinjective Algebras of Class* $\mathbf{A}_n$. LNM 832, 449-520, (1980).

[11]   Chr. Riedtmann. *Configurations of* $\mathbb{Z}\mathbf{D}_n$. Journal of Algebra, vol. 82, num. 2, June 1983, 309-327.

[12]   Chr. Riedtmann. *Representation-Finite Selfinjective Algebras of Class* $\mathbf{D}_n$. Compositio Math, 49 (1983) 231-282.