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On the q-meromorphic Weyl algebra

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Abstract. We introduce a q-analogue MW_q for the meromorphic Weyl algebra, and study the normalization problem and the symmetric powers $\mathrm{Sym}^n(MW_q)$ for such algebra from a combinatorial viewpoint.

1. Introduction

Pioneered by Euler, Jacobi, and Jackson among others, the results and applications of q-calculus [4, 10] have grown both in depth and scope, touching by now most branches of mathematics, including partition theory [3], combinatorics [30, 31], number theory [26], hypergeometric functions [4], quantum groups [25], knot theory [21], q-probabilities [28], Gaussian q-measure [20], Feynman q-integrals [13, 14], homological algebra [5, 24], and category theory [9]. Our goal in this work is to bring yet another mathematical object into the field of q-calculus, namely, we provide a q-analogue for the meromorphic Weyl algebra MW introduced in [15]. Roughly speaking MW is the algebra generated by x^{-1} and the derivative ∂. The q-analogue MW_q of the meromorphic Weyl algebra is essentially the algebra generated by x^{-1} and the q-derivative ∂_q. We focus on the normal polynomials for MW_q which arise in the problem of writing arbitrary monomials in MW_q as linear combination of monomials written in normal form; we provide both explicit formulae and a combinatorial interpretation for the normal polynomials. We also study the symmetric powers of MW_q using the methodology developed in [15] and further applied in [16, 19].

Let us say a few words on q-combinatorics. As explained by Zeilberger in [31] a combinatorial interpretation for a sequence $n_0, n_1, n_2, \ldots.$ of non-negative integers, is a sequence of finite sets $x_0, x_1, x_2, \ldots$ such that $|x_k| = n_k$ for $k \in \mathbb{N}$. Each sequence of non-negative integers admits a wide variety of combinatorial interpretations; the art of combinatorics consists in finding patterns that yield, systematically, combinatorial interpretations for families of sequences of non-negative integers.

The field of q-combinatorics provides another approach for the study of natural numbers by combinatorial methods. Let $\mathbb{N}[q]$ be the semi-ring of polynomials in the variable q with coefficients in $\mathbb{N}$. Instead of working with sequences of finite sets the main object of study in q-combinatorics are sequences $(x_0, \omega_0), (x_1, \omega_1), (x_2, \omega_2), \ldots$ of pairs (x, ω) where x is a finite set and $\omega : x \longrightarrow \mathbb{N}[q]$ is an arbitrary map. The cardinality of such a pair (x, ω) is defined to be

$$|x, \omega| = \sum_{i \in x} \omega(i) \in \mathbb{N}[q].$$

Notice that the cardinality $|x, \omega|$ of the pair (x, ω) is not an integer, but rather a polynomial in the variable q with non-negative integer coefficients. We say that a sequence of pairs $(x_0, \omega_0), (x_1, \omega_1), (x_2, \omega_2), \cdots$ provides a combinatorial interpretation for a sequence of non-negative integers $n_0, n_1, n_2, \cdots$ if $|x_k, \omega_k|(1) = n_k$ for $k \in \mathbb{N}$, where $|x_k, \omega_k|(1)$ is the evaluation of the polynomial $|x_k, \omega_k|$ at 1. Of course the additional value of q-combinatorics comes from the fact that it is suited to handle not just sequences in $\mathbb{N}$, but more generally sequences in $\mathbb{N}[q]$. We say that a sequence $(x_0, \omega_0), (x_1, \omega_1), (x_2, \omega_2), \cdots$ provides a combinatorial interpretation for a sequence of polynomials $p_1, p_2, p_3, \cdots$ in $\mathbb{N}[q]$ if $|x_k, \omega_k| = p_k$ for $k \in \mathbb{N}$. One of the most prominent examples is the q-combinatorial interpretation for the q-analogues $[n]! \in \mathbb{N}[q]$ of the factorial numbers $n!$ given by

$$[n]! = \prod_{k=1}^{n} [k] \quad \text{where} \quad [k] = 1 + \cdots + q^{k-1}.$$

Consider the pair (S_n, i_n) where S_n is the set of permutations of $[[1, n]] = \{1, 2, \cdots, n\}$ and $i_n : S_n \longrightarrow \mathbb{N}[q]$ is the map given by $i_n(\sigma) = q^{|I_n(\sigma)|}$ where

$$I_n(\sigma) = \{(i, j) \ \ | \ \ 1 \leq i < j \leq n \ \text{ and } \ \sigma(i) > \sigma(j)\}.$$

An inductive argument [3, 14] shows that $|S_n, i_n| = [n]!$, therefore the sequence (S_n, i_n) provides a combinatorial interpretation for $[n]!$.

The rest of this work is organized as follows. In Section 2 we summarize some facts on the meromorphic Weyl algebra; we do not include proofs since all the stated results are consequences, setting $q = 1$, of the corresponding q-analogue results proved in the subsequent sections. The main results of this work are given in Sections 3 and 4 where we introduce MW_q the q-analogue of the meromorphic Weyl algebra, discuss its basic properties, provide a couple of representations for it, study the normal polynomials that arise in the process of writing monomials in MW_q in normal form, and begin the study of the symmetric powers $\mathrm{Sym}^n(MW_q)$ of the q-meromorphic Weyl algebra.

2. **The meromorphic Weyl algebra**

The Weyl algebra is the associative algebra over the field of complex numbers $\mathbb{C}$ given by

$$W = \mathbb{C}\langle x, y\rangle / \langle yx - xy - 1\rangle$$

where $\mathbb{C}\langle x, y\rangle$ is the free associative algebra over $\mathbb{C}$ generated by formal variables x and y, and $\langle yx - xy - 1\rangle$ is the ideal generated by $yx - xy - 1$. The Weyl algebra comes with a natural representation

$$\rho : W \longrightarrow End(\mathbb{C}[x]),$$

where $\mathbb{C}[x]$ is the vector space of polynomials in the variable x and $End(\mathbb{C}[x])$ is the algebra of endomorphisms of $\mathbb{C}[x]$, which explain why it appears so often in many branches of mathematics and physics. The map ρ is given on the generators of W by

$$\rho(x)f = xf \quad \text{and} \quad \rho(y)f = \frac{\partial f}{\partial x}.$$

Notice that in the definition above the letter x on the left-hand side is a non-commutative variable, while on the right-hand side the letter x denotes the generator of $\mathbb{C}[x]$. This sort of abuse of notation is common in the literature and we hope it causes no confusion.

The meromorphic Weyl algebra MW is the associative algebra over $\mathbb{C}$ given by

$$MW = \mathbb{C}\langle x, y\rangle / \langle yx - xy - x^2\rangle.$$

MW comes with a natural representation ρ which justifies its name. Let $C^\infty(\mathbb{R}^*)$ be the space of smooth complex valued functions on the punctured real line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The representation

$$\rho : MW \longrightarrow \mathrm{End}(C^\infty(\mathbb{R}^*))$$

is defined by letting the generators of MW act on $f \in C^\infty(\mathbb{R}^*)$ as follows:

$$\rho(x)f = x^{-1}f \quad \text{and} \quad \rho(y)f = -\frac{\partial f}{\partial x}.$$

An integral analogue of the Weyl algebra is obtained by considering the operators $l(x)$ and $l(y)$ acting on $f \in C^{\infty}(\mathbb{R})$ as follows:

$$l(x)f = xf \quad \text{and} \quad l(y)f = \int_0^x f(t)dt.$$

It is not hard to see that l extends naturally to yield a representation

$$l : \mathbb{C}\langle x, y\rangle/\langle yx - xy + y^2\rangle \longrightarrow \mathrm{End}(C^{\infty}(\mathbb{R}))$$

of the algebra

$$\mathbb{C}\langle x, y\rangle/\langle yx - xy + y^2\rangle,$$

which is isomorphic to the meromorphic Weyl algebra via the isomorphism

$$t : MW \longrightarrow \mathbb{C}\langle x, y\rangle/\langle yx - xy + y^2\rangle$$

given on generators by $t(x) = y$ and $t(y) = x$. Thus the map $\iota : MW \longrightarrow \mathrm{End}(C^{\infty}(\mathbb{R}))$ given on generators by

$$\iota(x)f = \int_0^{\infty} f(t)dt \quad \text{and} \quad \iota(y)f = xf$$

defines a representation of the meromorphic Weyl algebra.

We will use the following notation. For $A = (A_1, \cdots, A_n) \in (\mathbb{N}^2)^n$ where $A_i = (a_i, b_i)$, we set $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$, and $|A| = (|a|, |b|) = (a_1 + \cdots + a_n, b_1 + \cdots + b_n)$.

The normal coordinates $N(A, k)$ of the monomial $\prod_{i=1}^n x^{a_i} y^{b_i} \in MW$ are given by

$$\prod_{i=1}^{n} x^{a_i} y^{b_i} = \sum_{k=0}^{|b|} N(A, k) x^{|a|+k} y^{|b|-k}.$$

For $k > |b|$ we set $N(A, k) = 0$.

Given vector $a = (a_1, \cdots, a_n)$ then for $i \in [[1, n-1]]$ we let $a_{>i}$ be the vector $(a_{i+1}, \cdots, a_n)$. The increasing factorial [29] is given by

$$n^{(k)} = n(n+1)(n+2)\cdots(n+k-1)$$

for $n \in \mathbb{N}$ and $k \geq 1$ an integer. In the statement of the Theorem 1 the notation $p \vdash k$ means that p is a vector $(p_1, \cdots, p_{n-1}) \in \mathbb{N}^{n-1}$ such that $|p| = \sum_{i=1}^{n-1} p_i = k$.

Theorem 1. For $(A, k) \in (\mathbb{N}^2)^n \times \mathbb{N}$ the following identity holds

$$N(A, k) = \sum_{p \vdash k} \binom{b}{p} \prod_{i=1}^{n-1} (|a_{>i}| + |p_{>i}|)^{(p_i)},$$

where

$$\binom{b}{p} = \prod_{i=1}^{n-1} \binom{b_i}{p_i}.$$

The numbers $N(A,k)$ have a nice combinatorial meaning. Let $E_1,\ldots,E_n,F_1,\ldots,F_n$ be disjoint sets such that $|F_i| = a_i$, $|E_i| = b_i$ for $i \in [[1,n]]$, and set $E = \sqcup E_i$, $F = \sqcup F_i$. Let M_k be the set whose elements are maps $f : F \longrightarrow \{$ subsets of $E\}$ such that:

- $f(x) \cap f(y) = \emptyset$ for $x,y \in F$;
- if $y \in f(x),\ x \in F_i,\ y \in E_j$, then $j < i$;
- $\sum_{a\in F} |f(a)| = k$.

The sets M_k provide a combinatorial interpretation for the numbers $N(A,k)$, that is

$$|M_k| = N(A,k).$$

Figure 1 illustrates the combinatorial interpretation for $N(((2,3),(3,3),(3,4)),6)$: it shows an example of a map contributing to $N(((2,3),(3,3),(3,4)),6)$.

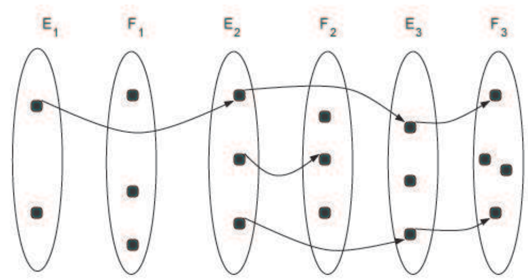

FIGURE 1. Combinatorial interpretation of $N(((2,3),(3,3),(3,4)),6)$.

Applying Theorem 1, specialized in the representation ρ, to $x^{-t} \in C^{\infty}(\mathbb{R}^*)$ we obtain for $(a,b,t) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}_+$ the following identity:

$$\prod_{i=1}^{n} (t + |a_{>i}| + |b_{>i}|)^{(b_i)} = \sum_{p \vdash k} \binom{b}{p} \prod_{i=1}^{n-1} (|a_{>i}| + |p_{>i}|)^{(p_i)} t^{(|b|-k)}.$$

This identity is thus an easy corollary of Theorem 1; however guessing or even proving it directly could be a bit of a pain. Applying Theorem 1, specialized in the representation ι, to x^t we get another quite intriguing identity:

$$\frac{1}{\prod_{i=1}^{n}(t+|a_{>i}|+|b_{\geq i}|+1)^{(a_i)}} = \sum_{p\vdash k}\binom{b}{p}\prod_{i=1}^{n-1}\frac{(|a_{>i}|+|p_{>i}|)^{(p_i)}}{(t+|b|-k+1)^{(|a|+k)}}.$$

A fundamental yet not fully appreciated fact in algebra is that one can associate with each associative algebra A a family of associative algebras $Sym^n(A)$ indexed by the natural numbers $n \in \mathbb{N}$. Formally, let $\mathbb{C}$-alg be the category of associative complex algebras. For $n \geq 1$ consider

$$\mathrm{Sym}^n : \mathbb{C}\text{-}alg \longrightarrow \mathbb{C}\text{-}alg$$

the functor sending an algebra A into its n-th symmetric power given by

$$\mathrm{Sym}^n(A) = A^{\otimes n}/\langle a_1 \otimes \cdots \otimes a_n - a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} \mid a_i \in A,\ \sigma \in S_n \rangle.$$

Given $a_1 \otimes ... \otimes a_n \in A^{\otimes n}$ we denote by $\overline{a_1 \otimes ... \otimes a_n}$ the corresponding element in $\mathrm{Sym}^n(A)$. The rule for the product of m elements in $\mathrm{Sym}^n(A)$, see [15], is given as follows: let $a_{ij} \in A$ for $(i,j) \in [[1,m]] \times [[1,n]]$, then we have that

$$n!^{m-1}\prod_{i=1}^{m}\overline{\bigotimes_{j=1}^{n} a_{ij}} = \sum_{\sigma \in \{1\}\times S_n^{m-1}} \overline{\bigotimes_{j=1}^{n}\prod_{i=1}^{m} a_{i\sigma_i^{-1}(j)}},$$

where 1 denotes the identity permutation.

To our knowledge the symmetric powers have been fully studied only for a few algebras: for the algebra of polynomials whose symmetric powers may be identified with the algebra of symmetric polynomials; and for the algebra of matrices whose symmetric powers may be identified with the so called Schur algebras [15]. The symmetric powers of the Weyl algebra and its q-analogues are studied in [15, 16], the symmetric powers of the linear Boolean algebras are studied in [19].

Let $\mathrm{Sym}^n(MW)$ be the n-symmetric power of the meromorphi5c Weyl algebra. An explicit formulae for the product of m elements in $\mathrm{Sym}^n(MW)$ is provided next. We denote the element

$$\overline{x^{a_1}y^{b_1} \otimes ... \otimes x^{a_n}y^{b_n}} \in \mathrm{Sym}^n(MW) \ \text{ by } \ \overline{\prod_{j=1}^{n} x_j^{a_j} y_j^{b_j}}.$$

Theorem 2. For each map $(a,b) : [[1,m]]\times[[1,n]] \longrightarrow \mathbb{N}^2$ the following identity holds in $\mathrm{Sym}^n(MW)$:

$$(n!)^{m-1}\prod_{i=1}^{m}\overline{\prod_{j=1}^{n} x_j^{a_{ij}} y_j^{b_{ij}}} =$$

$$= \sum_{\sigma,k,p} \left(\prod_{l=1}^{m-1} \prod_{j=1}^{n} \binom{b_j^\sigma}{p^j} (|(a_j^\sigma)_{>l}| + |p_{>l}^j|)^{(p_l^j)} \right) \overline{\prod_{j=1}^{n} x_j^{|a_j^\sigma|+k_j} y_j^{|b_j^\sigma|-k_j}}.$$

In the formula above we are using the following conventions:
$\sigma \in \{1\} \times S_n^{m-1}$, $k \in \mathbb{N}^n$ is such that $k_j \leq |b_j^\sigma|$, $p = (p^1, ..., p^n) \in (\mathbb{N}^{m-1})^n$, $p^j = (p_1^j, ..., p_{m-1}^j)$, $a_j^\sigma = (a_{1\sigma_1^{-1}(j)}, ..., a_{m\sigma_m^{-1}(j)})$, and
$b_j^\sigma = (b_{1\sigma_1^{-1}(j)}, ..., b_{m\sigma_m^{-1}(j)})$

The next example shows the high computational power required to compute even the simplest products in the symmetric powers of the meromorphic Weyl algebra.

Example 3. For $n = 2, m = 2$ we have

$$2(x_1y_1^2x_2^2y_2^2)(x_1^2y_1x_2y_2^2) =$$
$$= x_1^3y_1^4x_2^3y_2^4 + 6x_1^3y_1^4x_2^4y_2^3 + 8x_1^3y_1^4x_2^5y_2^2 + +8x_1^4y_1^3x_2^4y_2^3 + 20x_1^4y_1^3x_2^5y_2^2 +$$
$$+ 6x_1^5y_1^2x_2^3y_2^4 + 12x_1^5y_1^2x_2^5y_2^2 + x_1^3y_1^4x_2^4y_2^4 + 2x_1^3y_1^4x_2^5y_2^3 + 6x_1^3y_1^4x_2^6y_2^2 +$$
$$+ 2x_1^4y_1^3x_2^4y_2^4 + 4x_1^4y_1^3x_2^5y_2^3 + 12x_1^4y_1^3x_2^6y_2^3 + 6x_1^5y_1^2x_2^4y_2^4 + 12x_1^5y_1^2x_2^5y_2^3 +$$
$$+ 36x_1^5y_1^2x_2^6y_2^2.$$

3. The q-meromorphic Weyl algebra

In this section we introduce the q-meromorphic Weyl algebra and discuss some of its basic properties. Let us first review a few basic notions of q-calculus; the interested reader may consult [10, 11, 20] for further information. Let $M(\mathbb{R}^*)$ be the space of complex value functions defined on the punctured real line $\mathbb{R} \setminus \{0\}$ and fix a positive real number $0 < q < 1$. The q-derivative

$$\partial_q : M(\mathbb{R}^*) \longrightarrow M(\mathbb{R}^*)$$

is given by

$$\partial_q f = \frac{I_q f - f}{(q-1)x},$$

where $I_q f(x) = f(qx)$ for $x \in \mathbb{R}^*$.

Definition 4. The q-meromorphic Weyl is the algebra given by

$$MW_q = \mathbb{C}\langle x, y\rangle[q]/\langle yx - qxy - x^2\rangle,$$

where $\mathbb{C}\langle x, y\rangle[q]$ is the free associative algebra generated by the non-commuting variables x, y and the commutative variable q.

Notice that in the definition above q is used as a formal variable rather than a number. It should always be clear from the context whether we are using q as a formal variable or as a number. Next result explains how the algebra MW_q arises in q-calculus. For our next result we make use of the q-Leibnitz rule

$$\partial_q(fg) = f\partial_q g + I_q g\partial_q f.$$

Theorem 5. a The map $\rho : MW_q \longrightarrow \mathrm{End}(M(\mathbb{R}^*))$ given on generators by

$$\rho(x)f = x^{-1}f, \quad \rho(y)f = -q^{-1}\partial_{q^{-1}}f, \text{ and } \rho(q)f = qf$$

for $f \in M(\mathbb{R}^*)$ defines a representation of MW_q.

Proof. We must prove that

$$\rho(y)\rho(x)f = q\rho(x)\rho(y)f + \rho(x^2)f.$$

Since $\partial_{q^{-1}}x^{-1} = -qx^{-2}$ we find that

$$\begin{aligned} \rho(y)\rho(x)f &= \rho(y)(x^{-1}f) = -q^{-1}\partial_{q^{-1}}(x^{-1}f) \\ &= -q^{-1}(q^{-1}x)^{-1}\partial_{q^{-1}}f - q^{-1}f\partial_{q^{-1}}(x^{-1}) \\ &= -x^{-1}\partial_{q^{-1}}f + x^2 f \\ &= q\rho(x)\rho(y)f + \rho(x^2)f. \end{aligned}$$

□

Recall [10] that the Jackson integral of a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$\int_0^x f(t)d_qt = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x).$$

A non-fully exploited feature of the Jackson integral is that it satisfies a twisted form of the Rota-Baxter identity [9, 12, 29]; indeed one can show that

$$\left(\int_0^x f(s)d_qs\right)\left(\int_0^x g(t)d_qt\right) = \int_0^x\left(\int_0^t f(s)d_qs\right)g(t)d_qt + \\ + \int_0^x f(t)\left(\int_0^{qt} g(s)d_qs\right)d_qt.$$

It is not hard to check that the Jackson integral is a right inverse operator for the q-derivative, that is

$$\partial_q\int_0^x f(t)d_qt = f(x).$$

From the q-Leibniz rule and the fundamental theorem of q-calculus one obtains the q-integration by parts formula

$$\int_0^x I_q f \partial_q g d_q t = f(x)g(x) - f(0)t(0) - \int_0^x g\partial_q f d_q t.$$

In particular setting

$$f(x) = x \quad \text{and} \quad g(x) = \int_0^x f(t) d_q t$$

we obtain the relation

$$x\int_0^x f d_q t = q\int_0^x t f d_q t + \int_0^x \int_0^t f d_q s d_q t.$$

Let $I(\mathbb{R})$ be a space of functions on the real line closed under Jackson integration and under multiplication by polynomial functions. The previous considerations give the following result.

Theorem 6. The map

$$\iota : MW_q \longrightarrow \mathrm{End}(I(\mathbb{R}))$$

given on generators by

$$\iota(x)f = \int_0^x f d_q t, \quad \iota(y)f = xf, \quad \text{and} \quad \iota(q)f = qf,$$

for $f \in I(\mathbb{R})$ defines a representation of MW_q.

We order the generators of MW_q as $q < x < y$. A monomial in MW_q of the form $q^a x^b y^c$ is said to be in normal form. One can show that the set monomials in normal form is a basis for MW_q. Recall from the introduction that we are writing $[n] = 1 + ... + q^{n-1}$ for an integer $n \geq 1$.

Lemma 7. For $n \geq 1$ the identity $yx^n = q^n x^n y + [n]x^{n+1}$ holds in MW_q.

Proof. For $n = 1$ we get $yx = qxy + x^2$. By induction we have that

$$\begin{aligned} yx^{n+1} &= yx^n x = (q^n x^n y + [n]x^{n+1})x = q^n x^n (yx) + [n]x^{n+1}x = \\ &= q^{n+1}x^{n+1}y + [n+1]x^{n+2}. \end{aligned}$$

□

Definition 8. Let $(a, b) \in \mathbb{N}$ and $0 \leq k \leq a$. The normal coordinates $c(a, b, k)$ are the elements of $\mathbb{N}[q]$ given by the following identity in MW_q:

$$y^a x^b = \sum_{k=0}^{a} c(a, b, k)x^{b+k}y^{a-k}.$$

For $k > a$ we set $c(a, b, k) = 0$. Notice that by definition $c(0, b, k) = \delta_{0,k}$ where δ is Kronecker's delta function.

Proposition 9. The following identities hold in MW_q:

(1) $c(a+1, b, k) = c(a, b, k)q^{b+k} + c(a, b, k-1)[b+k-1]$ for $1 \leq k \leq a$.
(2) $c(a+1, b, 0) = c(a, b, 0)q^b$.
(3) $c(a+1, b, a+1) = c(a, b, a)[b+a]$.

Proof. By Lemma 7 and Definition 8 we have

$$yx^b = \sum_{k=0}^{1} c(1, b, k)x^{b+k}y^{1-k} = q^b x^b y + [b]x^{b+1},$$

thus $c(1, b, 0) = q^b$ and $c(1, b, 1) = [b]$. On the other hand we compute

$$\begin{aligned} y^{a+1}x^b &= \sum_{k=0}^{a} c(a, b, k)(yx^{b+k})y^{a-k} \\ &= \sum_{k=0}^{a} c(a, b, k)(q^{b+k}x^{b+k}y + [b+k]x^{b+k+1})y^{a-k} \\ &= c(a, b, 0)q^b x^b y^{a+1} + \sum_{k=1}^{a} c(a, b, k)q^{b+k}x^{b+k}y^{a+1-k} \\ &\quad + \sum_{k=1}^{a} c(a, b, k-1)[b+k-1]x^{b+k}y^{a+1-k} + \\ &\quad + c(a, b, a)[b+a]x^{a+b+1}. \end{aligned}$$

By definition we have that

$$y^{a+1}x^b = \sum_{k=0}^{a+1} c(a+1, b, k)x^{b+k}y^{a+1-k}.$$

Therefore we have shown that

$$\begin{aligned} &\sum_{k=0}^{a+1} c(a+1, b, k)x^{b+k}y^{a+1-k} = c(a, b, 0)q^b x^b y^{a+1} \\ &+ \sum_{k=1}^{a} \Big(c(a, b, k)q^{b+k} + c(a, b, k-1)[b+k-1] \Big) x^{b+k}y^{a+1-k} \\ &+ c(a, b, a)[b+a]x^{a+b+1}. \end{aligned}$$

Considering this equality termwise gives the desired identities.

□

Notice that the first identity from Proposition 9 together with the initial conditions $c(0,b,k) = \delta_{0,k}$ completely determine the function $c(a,b,k)$. We shall use this fact in the proof of Theorem 11. Our next result shows that $c(a,b,a)$ is the q-analogue of the increasing factorial.

Lemma 10. (1) $c(a,b,0) = q^{ab}$.
(2) $c(a,b,a) = [b][b+1]\dots[b+a-1] = [b]^{(a)}$.

Proof. Clearly $c(1,b,0) = q^b$. Moreover by Proposition 9 we have that

$$c(a+1,b,0) = c(a,b,0)q^b = q^{ab}q^b = q^{(a+1)b}.$$

For $a = 1$ we have $c(1,b,1) = [b]^{(1)} = [b]$, and using again Proposition 9 we get

$$c(a+1,b,a+1) = c(a,b,a)[b+a] = [b]^{(a)}[b+a] = [b]^{(a+1)}.$$

□

We are ready to discuss the combinatorial interpretation of the normal polynomials $c(a,b,k)$. Let $P_k[[1,a]]$ be the set of subsets of $[[1,a]]$ with k elements. We define a q-weight

$$\omega_b : P_k[[1,a]] \longrightarrow \mathbb{N}[q]$$

which sends $A \in P_k[[1,a]]$ into

$$\omega_b(A) = [b]^{(k)}q^{(a-k)b}q^{\sum_{i\in A^c}|A_{<i}|}.$$

Theorem 11. For $(a,b) \in \mathbb{N}\times\mathbb{N}_+$ and $0 \le k \le a$, we have that $c(a,b,k) = |P_k[[1,a]],\omega_b|$.

Proof. We have to show that

$$c(a,b,k) = |P_k(a),\omega_b| = [b]^{(k)}q^{(a-k)b}\sum_{A\in P_k[[1,a]]} q^{\sum_{i\in A^c}|A_{<i}|}.$$

Let $\overline{c}(a,b,k)$ be given by the right hand side of formula above for $a \ge 1$ and $\overline{c}(0,b,k) = \delta_{0,k}$. We must show that $\overline{c}(a,b,k) = c(a,b,k)$. Since $\overline{c}(0,b,k) = c(0,b,k)$, it is enough to show that $\overline{c}(a,b,k)$ satisfies, for $1 \le k \le a$, the recursion

$$\overline{c}(a+1,b,k) = \overline{c}(a,b,k)q^{b+k} + \overline{c}(a,b,k-1)[b+k-1].$$

Sets $A \in P_k[[1, a+1]]$ are classified in two blocks according to whether $a+1 \notin A$ or $a+1 \in A$. Thus we obtain that

$$\overline{c}(a+1,b,k) = |P_k(a+1), \omega_b| = [b]^{(k)} q^{(a-k+1)b} \sum_{A \in P_k[[1,a+1]]} q^{\sum_{i \in A^c} |A_{<i}|}$$

is equal to the sum of two terms

$$\left([b]^{(k)} q^{(a-k)b} \sum_{A \in P_k[[1,a]]} q^{\sum_{i \in A^c} |A_{<i}|} \right) q^{b+k} \;\; +$$

$$\left([b]^{(k-1)} q^{(a-k+1)b} \sum_{A \in P_{k-1}[[1,a]]} q^{\sum_{i \in A^c} |A_{<i}|} \right) [b+k-1].$$

Thus the numbers $\overline{c}(a,b,k)$ satisfy the required recursion. $\square$

Let us remark that writing $A \in P_k[[1,a]]$ as $A = \{t_1 < t_2 < \cdots < t_k\}$, using the elementary identity

$$\sum_{i \in A^c} |A_{<i}| = \sum_{s=1}^{k} s(t_{s+1} - t_s - 1)$$

and setting $t_{k+1} = a+1$ we obtain that:

$$c(a,b,k) = [b-1]^{(k)} q^{(a-k)b} \sum_{1 \leq t_1 < \cdots < t_k \leq a} q^{\sum_{s=1}^{k} s(t_{s+1} - t_s - 1)}.$$

4. Normal polynomials and symmetric powers of MW_q

In this section we find explicit formulae for the normal polynomials of the algebra MW_q. We also begin the study of the symmetric power of that algebra.

Definition 12. Let $A = (A_1, \cdots, A_n) \in (\mathbb{N}^2)^n$ with $A_i = (a_i, b_i)$. The normal polynomial $N(A,k,q) \in \mathbb{N}[q]$ is defined by the following identity in MW_q:

$$\prod_{i=1}^{n} x^{a_i} y^{b_i} = \sum_{k=0}^{|b|} N(A,k,q) x^{|a|+k} y^{|b|-k}.$$

For $k > |b|$ we set $N(A,k,q) = 0$.

Recall from Section 2 that the notation $p \vdash k$ means that p is a vector $(p_1, \cdots, p_{n-1}) \in \mathbb{N}^{n-1}$ such that $|p| = \sum_{i=1}^{n-1} p_i = k$. Our next result is obtained using Definition 8 several times.

Theorem 13. For $(A, k) \in (\mathbb{N}^2)^n \times \mathbb{N}$ we have that

$$N(A,k,q) = \sum_{p \vdash k} \prod_{i=1}^{n-1} c(b_i, |a_{>i}| + |p_{>i}|, p_i),$$

where the partition p of k must be such that $0 \leq p_i \leq b_i$ for $i \in [[1, n-1]]$.

It is not hard to show that the normal polynomial may also be computed via the identity

$$N(A,k,q) = \sum_{p \vdash k} \prod_{i=1}^{n-1} c(|b_{\leq i}| - |p_{<i}|, a_{i+1}, p_i),$$

where $0 \leq p_i \leq |b_{\leq i}| - |p_{<i}|$ for $i \in [[1, n-1]]$.

Applying Theorem 13, specialized in the representation ρ, to x^{-t} we obtain that if $(a, b, t) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}_+$ then

$$\prod_{i=1}^{n} [t+|b_{\geq i}|+|a_{>i}|-1] = \sum_{k=0}^{|b|} \left(\sum_{p \vdash k} \prod_{i=1}^{n-1} c(b_i, |a_{>i}| + |p_{>i}|, p_i) \right) [t+|b|-k-1],$$

where $0 \leq p_i \leq b_i$ for $i \in [[1, n-1]]$.

Using the alternative expression for $N(A,k,q)$ given above, one obtains that:

$$\prod_{i=1}^{n} [t+|b_{\geq i}|+|a_{>i}|-1] = \sum_{k=0}^{|b|} \left(\sum_{p \vdash k} \prod_{i=1}^{n-1} c(|b_{\leq i}| - |p_{<i}|, a_{i+1}, p_i) \right) [t+|b|-k-1],$$

where $0 \leq p_i \leq |b_{\leq i}| - |p_{<i}|$ for $i \in [[1, n-1]]$.

If instead of ρ we use the representation ι applied to x^t we get the identity:

$$\frac{1}{\prod_{i=1}^{n} [t + |a_{\geq i}| + |b_{\geq i}| + 1]^{(a_i)}} =$$
$$= \sum_{k=0}^{|b|} \left(\sum_{p \vdash k} \prod_{i=1}^{n-1} c(b_i, |a_{>i}| + |p_{>i}|, p_i) \right) \frac{1}{[t + |a| + |b|]^{(|a|+k)}},$$

where $0 \leq p_i \leq b_i$ for $i \in [[1, n-1]]$.

Also with the alternative expression for $N(A,k,q)$ we get:

$$\frac{1}{\prod_{i=1}^{n}[t+|a_{\geq i}|+|b_{\geq i}|+1]^{(a_i)}}=$$
$$=\sum_{k=0}^{|b|}\left(\sum_{p\vdash k}\prod_{i=1}^{n-1}c(|b_{\leq i}|-|p_{<i}|,a_{i+1},p_i)\right)\frac{1}{[t+|a|+|b|]^{(|a|+k)},}$$

where $0\leq p_i\leq |b_{\leq i}|-|p_{<i}|$ for $i\in[[1,n-1]]$.

Next we provide explicit formulae for the products of several elements in the n-th symmetric power $\mathrm{Sym}^n(MW_q)$ of the q-meromorphic Weyl algebra MW_q.

Theorem 14. For each map $(a,b):[[1,m]]\times[[1,n]]\longrightarrow\mathbb{N}^2$ the following identity holds in $\mathrm{Sym}^n(MW)$:

$$(n!)^{m-1}\prod_{i=1}^{m}\overline{\prod_{j=1}^{n}x_j^{a_{ij}}y_j^{b_{ij}}}=$$
$$=\sum_{\sigma,k,p}\left(\prod_{l=1}^{m-1}\prod_{j=1}^{n}c((b_j^{\sigma})_l,|(a_j^{\sigma})_{>l}|+|p_{>l}^j|,p_l^j)\right)\overline{\prod_{j=1}^{n}x_j^{|a_j^{\sigma}|+k_j}y_j^{|b_j^{\sigma}|-k_j}}.$$

In the formula above we are using the following conventions:
$\sigma\in\{1\}\times S_n^{m-1}$, $k\in\mathbb{N}^n$ is such that $k_j\leq|b_j^{\sigma}|$, $p=(p^1,...,p^n)\in(\mathbb{N}^{m-1})^n$, $p^j=(p_1^j,...,p_{m-1}^j)$, $a_j^{\sigma}=(a_{1\sigma_1^{-1}(j)},...,a_{m\sigma_m^{-1}(j)})$, and
$b_j^{\sigma}=(b_{1\sigma_1^{-1}(j)},...,b_{m\sigma_m^{-1}(j)})$

The explicit computation of products in $\mathrm{Sym}^n(MW_q)$ is rather difficult as the following example shows.

Example 15. For $n=2, m=2$ we have

$$2(x_1y_1x_2^2y_2)(x_1^2y_1^2x_2y_2)=x_1y_1x_2^2y_2x_1^2y_1^2x_2y_2+x_1y_1x_2^2y_2x_1y_1x_2^2y_2^2=$$
$$q^3x_1^3y_1^3x_2^3y_2^2+q^2x_1^3y_1^3x_2^4y_2+(q^2+q)x_1^4y_1^2x_2^3y_2^2+(q+1)x_1^4y_1^2x_2^4y_2+$$
$$q^3x_1^2y_1^2x_2^4y_2^3+(q^2+q)x_1^2y_1^2x_2^5y_2^2+q^2x_1^3y_1x_2^4y_2^3+(q+1)x_1^3y_1x_2^5y_2^2.$$

We close this work mentioning a couple of research problems. First, it would be interesting to study the Hochschild cohomology of the meromorphic and q-meromorphic Weyl algebras and their corresponding symmetric powers along the lines developed in [1, 2]. Second, using techniques introduced in [18] and further developed in [6, 7, 8, 9] we have constructed a categorification of the Weyl algebra, and more generally of the Kontsevich star product [27] for Poisson structures on $\mathbb{R}^n$. It would be interesting to study the categorification of the meromorphic and q-meromorphic Weyl algebras.

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