

Cohen-Macaulay clutters with combinatorial optimization properties and parallelizations of normal edge ideals

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Abstract. Let \mathcal{C} be a uniform clutter and let $I = I(\mathcal{C})$ be its edge ideal. We prove that if \mathcal{C} satisfies the packing property (resp. max-flow min-cut property), then there is a uniform Cohen-Macaulay clutter \mathcal{C}_1 satisfying the packing property (resp. max-flow min-cut property) such that \mathcal{C} is a minor of \mathcal{C}_1 . For arbitrary edge ideals of clutters we prove that the normality property is closed under parallelizations. Then we show some applications to edge ideals and clutters which are related to a conjecture of Conforti and Cornuéjols and to max-flow min-cut problems.

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1. Introduction

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and let I be an ideal of R minimally generated by a finite set $F = \{x^{v_1}, \dots, x^{v_q}\}$ of square-free monomials. As usual we use the notation $x^a := x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \dots, a_n)$ is in \mathbb{N}^n . The *support* of a monomial x^a is given by $\text{supp}(x^a) = \{x_i \mid a_i > 0\}$. For technical reasons we shall assume that each variable x_i occurs in at least one monomial of F .

A *clutter* with finite vertex set X is a family of subsets of X , called edges, none of which is included in another. The set of vertices of a clutter \mathcal{C} is denoted by $V(\mathcal{C})$ and the set of edges of \mathcal{C} is denoted by $E(\mathcal{C})$. A clutter is called *d-uniform* if all its edges have exactly d vertices. We associate to the ideal I a *clutter* \mathcal{C} by taking the set of indeterminates $X = \{x_1, \dots, x_n\}$ as vertex set and $E = \{S_1, \dots, S_q\}$ as edge set, where S_k is the support of x^{v_k} . The vector v_k is called the *characteristic vector* of S_k . The assignment $I \mapsto \mathcal{C}$ gives a natural one to one correspondence between the family of square-free monomial ideals and the family of clutters. The ideal I is called the *edge ideal* of \mathcal{C} . To stress the relationship between I and \mathcal{C} we will use the notation $I = I(\mathcal{C})$. Edge ideals of graphs were introduced and studied in [24, 29]. Edge ideals of clutters also correspond to simplicial complexes via the Stanley-Reisner correspondence [25] and to facet ideals [9, 32]. The Cohen-Macaulay property of edge ideals has been recently studied in [3, 10, 16, 22, 27] using a combinatorial approach based on the notions of shellability, linear quotients, unmixedness, acyclicity and transitivity of digraphs, and the König property.

The aim of this note is to study the behavior, under certain operations, of various algebraic and combinatorial optimization properties of edge ideals and clutters such as the Cohen-Macaulay property, the normality, the torsion freeness, the packing and the max-flow min-cut properties. The study of edge ideals from the combinatorial optimization point of view was initiated in [1, 26] and continued in [6, 8, 13, 14, 15, 31], see also [17]. The Cohen-Macaulay and normality properties are two of the most interesting properties an edge ideal can have, see [2, 10, 25, 30] and [19, 28] respectively.

Recall that the *integral closure* of I^i , denoted by $\overline{I^i}$, is the ideal of R given by

$$\overline{I^i} = (\{x^a \in R \mid \exists p \geq 1; (x^a)^p \in I^{pi}\}).$$

An ideal I is called *normal* if $I^i = \overline{I^i}$ for all i . A clutter obtained from \mathcal{C} by a sequence of deletions and duplications of vertices is called a *parallelization* of \mathcal{C} and a clutter obtained from \mathcal{C} by a sequence of deletions and contractions of vertices is called a *minor* of \mathcal{C} , see Section 2. It is known that the normality of $I(\mathcal{C})$ is closed under minors [8]. One of our main results shows that the normality of $I(\mathcal{C})$ is closed under parallelizations:

Theorem 2.3 *Let \mathcal{C} be a clutter and let \mathcal{C}' be a parallelization of \mathcal{C} . If $I(\mathcal{C})$ is normal, then $I(\mathcal{C}')$ is normal.*

The ideal $I = I(\mathcal{C})$ is called *normally torsion free* if $I^i = I^{(i)}$ for all i , where $I^{(i)}$ is the i th symbolic power of I . As an application we prove that if $I(\mathcal{C})$ is normally torsion free and \mathcal{C}' is a parallelization of \mathcal{C} , then $I(\mathcal{C}')$ is normally torsion free (Corollary 2.12). Let A be the incidence matrix of \mathcal{C} , i.e., A is the matrix with column vectors v_1, \dots, v_q . A clutter \mathcal{C} satisfies the *max-flow min-cut* (MFMC) property if both sides of the LP-duality equation

$$\min\{\langle w, x \rangle \mid x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1} \rangle \mid y \geq 0; Ay \leq w\}$$

have integral optimum solutions x and y for each non-negative integral vector w . A remarkable result of [15] (cf. [14, Theorem 4.6]) shows that $I(\mathcal{C})$ is normally torsion free if and only if \mathcal{C} has the max-flow min-cut property. This fact makes a strong connection between commutative algebra and combinatorial optimization. It is known [23, Chapter 79] that a clutter \mathcal{C} satisfies the max-flow min-cut property if and only if all parallelizations of the clutter \mathcal{C} satisfy the König property (see Definition 2.7). As another application we give a proof of this fact using that the integrality of the polyhedron $\{x \mid x \geq 0; xA \geq \mathbf{1}\}$ is closed under parallelizations and minors and using that the normality of $I(\mathcal{C})$ is preserved under parallelizations and minors (Corollary 2.16).

A clutter \mathcal{C} satisfies the *packing property* (PP for short) if all minors of \mathcal{C} satisfy the König property. We say that a clutter \mathcal{C} is *Cohen-Macaulay* if $R/I(\mathcal{C})$ is a Cohen-Macaulay ring, see [21]. The other main result of this note is:

Theorem 3.3 *Let \mathcal{C} be a d -uniform clutter on the vertex set X . Let*

$$Y = \{y_{ij} \mid 1 \leq i \leq n; 1 \leq j \leq d - 1\}$$

be a set of new variables, and let \mathcal{C}' be the clutter with vertex set $V(\mathcal{C}') = X \cup Y$ and edge set

$$E(\mathcal{C}') = E(\mathcal{C}) \cup \{\{x_1, y_{11}, \dots, y_{1(d-1)}\}, \dots, \{x_n, y_{n1}, \dots, y_{n(d-1)}\}\}.$$

Then the edge ideal $I(\mathcal{C}')$ is Cohen-Macaulay. If \mathcal{C} satisfies PP (resp. max-flow min-cut), then \mathcal{C}' satisfies PP (resp. max-flow min-cut).

It is well known that if \mathcal{C} satisfies the max-flow min-cut property, then \mathcal{C} satisfies the packing property [5] (see Corollary 2.10). Conforti and Cornuéjols [4] conjecture that the converse is also true. Theorem 3.3 is interesting because it says that for uniform clutters it suffices to prove the conjecture for Cohen-Macaulay clutters, which have a rich structure. The

Conforti-Cornuéjols conjecture has been studied in [7, 14, 15] using an algebraic approach based on certain algebraic properties of blowup algebras.

2. Normality is preserved under parallelizations

Let \mathcal{C} be a clutter on the vertex set $X = \{x_1, \dots, x_n\}$ and let $I = I(\mathcal{C}) = (x^{v_1}, \dots, x^{v_q})$ be its edge ideal. The *incidence matrix* of \mathcal{C} , denoted by $A = (a_{ij})$, is the $n \times q$ matrix whose (i, j) entry is given by $a_{ij} = 1$ if $x_i \in g_j$ and $a_{ij} = 0$ otherwise, where g_1, \dots, g_q are the edges of \mathcal{C} . Notice that the column vectors of A are v_1, \dots, v_q , where $v_i = \sum_{x_j \in g_i} e_j$ for $1 \leq i \leq q$. As usual e_j denotes the j th unit vector in \mathbb{R}^n . Recall that the *Rees algebra* of I is given by:

$$R[It] := R \oplus It \oplus \dots \oplus I^i t^i \oplus \dots \subset R[t],$$

where t is a new variable. The Rees algebra of I can be written as

$$R[It] = K[\{x^a t^b \mid (a, b) \in \mathbb{N}\mathcal{A}'\}]$$

where $\mathcal{A}' = \{(v_1, 1), \dots, (v_q, 1), e_1, \dots, e_n\}$ and $\mathbb{N}\mathcal{A}'$ is the subsemigroup of \mathbb{N}^{n+1} spanned by \mathcal{A}' . In other words $R[It]$ is equal to $K[\mathbb{N}\mathcal{A}']$, the semigroup ring of $\mathbb{N}\mathcal{A}'$, see [12]. On the other hand according to [30, Theorem 7.2.28] the integral closure of $R[It]$ in its field of fractions can be expressed as

$$\begin{aligned} \overline{R[It]} &= K[\{x^a t^b \mid (a, b) \in \mathbb{Z}\mathcal{A}' \cap \mathbb{R}_+\mathcal{A}'\}] \\ &= R \oplus \overline{I}t \oplus \overline{I}^2 t^2 \oplus \dots \oplus \overline{I}^i t^i \oplus \dots, \end{aligned}$$

where \overline{I}^i is the integral closure of I^i , $\mathbb{R}_+\mathcal{A}'$ is the cone spanned by \mathcal{A}' , and $\mathbb{Z}\mathcal{A}'$ is the subgroup spanned by \mathcal{A}' . Notice that $\mathbb{Z}\mathcal{A}' = \mathbb{Z}^{n+1}$. Hence $R[It]$ is normal if and only if any of the following two equivalent conditions hold:

- (a) $\mathbb{N}\mathcal{A}' \cong \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$.
- (b) $I^i = \overline{I}^i$ for all $i \geq 1$.

If the second condition holds we say that I is a *normal* ideal.

Let \mathcal{C} be a clutter on the vertex set $X = \{x_1, \dots, x_n\}$ and let $x_i \in X$. Then *duplicating* x_i means extending X by a new vertex x'_i and replacing $E(\mathcal{C})$ by

$$E(\mathcal{C}) \cup \{(e \setminus \{x_i\}) \cup \{x'_i\} \mid x_i \in e \in E(\mathcal{C})\}.$$

The *deletion* of x_i , denoted by $\mathcal{C} \setminus \{x_i\}$, is the clutter formed from \mathcal{C} by deleting the vertex x_i and all edges containing x_i . A clutter obtained from \mathcal{C} by a sequence of deletions and duplications of vertices is called a *parallelization*. If $w = (w_i)$ is a vector in \mathbb{N}^n , we denote by \mathcal{C}^w the clutter obtained from \mathcal{C} by deleting any vertex x_i with $w_i = 0$ and duplicating $w_i - 1$ times any vertex x_i if $w_i \geq 1$. The map $w \mapsto \mathcal{C}^w$ gives a one to one correspondence between \mathbb{N}^n and the parallelizations of \mathcal{C} .

Example 2.1. Let G be the graph whose only edge is $\{x_1, x_2\}$ and let $w = (3, 3)$. Then $G^w = \mathcal{K}_{3,3}$ is the complete bipartite graph with bipartition $V_1 = \{x_1, x_1^2, x_1^3\}$ and $V_2 = \{x_2, x_2^2, x_2^3\}$. Notice that x_i^k is a vertex, i.e., k is an index not an exponent.

The following notion of minor comes from combinatorial optimization [4, 23] and it is not apparently related to the minors (subdeterminants) of A .

Definition 2.2. Let $X' = \{x_{i_1}, \dots, x_{i_r}, x_{j_1}, \dots, x_{j_s}\}$ be a subset of X . A minor of I is a proper ideal I' of $R' = K[X \setminus X']$ obtained from I by making $x_{i_k} = 0$ and $x_{j_\ell} = 1$ for all k, ℓ . The ideal I is considered itself a minor. A minor of \mathcal{C} is a clutter \mathcal{C}' whose edge ideal is I' .

Notice that the generators of I' are obtained from the generators of I by making $x_{i_k} = 0$ and $x_{j_\ell} = 1$ for all k, ℓ . This means that \mathcal{C}' is obtained from \mathcal{C} by shrinking some edges and deleting some other edges. Also notice that \mathcal{C}' is obtained from I' by considering the unique set of square-free monomials of R' that minimally generate I' . If I' is the ideal obtained from I by making $x_i = 0$, then $I' = I(\mathcal{C} \setminus \{x_i\})$, i.e., making a variable equal to zero corresponds to a deletion. If I' is the minor obtained from I by making $x_i = 0$ for $1 \leq i \leq r$ and $x_i = 1$ for $r + 1 \leq i \leq s$, then in algebraic terms I' can be expressed as

$$(I \cap K[x_{r+1}, \dots, x_n])_{\mathfrak{p}} = I'K[x_{r+1}, \dots, x_n]_{\mathfrak{p}},$$

where $(I \cap K[x_{r+1}, \dots, x_n])_{\mathfrak{p}}$ and $K[x_{r+1}, \dots, x_n]_{\mathfrak{p}}$ are localizations at the prime ideal \mathfrak{p} generated by the variables x_{s+1}, \dots, x_n .

It is known that the normality of $I(\mathcal{C})$ is closed under minors [8]. A main result of this section shows that the normality of $I(\mathcal{C})$ is closed under parallelizations.

Theorem 2.3. Let \mathcal{C} be a clutter and let \mathcal{C}' be a parallelization of \mathcal{C} . If $I(\mathcal{C})$ is normal, then $I(\mathcal{C}')$ is normal.

Proof. From [8] we obtain that if $I(\mathcal{C})$ is normal and \mathcal{C}' is a minor of \mathcal{C} , then $I(\mathcal{C}')$ is also normal. Thus we need only show that the normality of $I(\mathcal{C})$ is preserved when we duplicate a vertex of \mathcal{C} . Let $V(\mathcal{C}) = \{x_2, \dots, x_n\}$ be the vertex set of \mathcal{C} and let \mathcal{C}' be the clutter obtained from \mathcal{C} by duplicating the vertex x_2 , i.e., we extend $V(\mathcal{C})$ by a new vertex x'_2 as explained before Example 2.1. For convenience we set $x_1 = x'_2$. We may assume that

$$I = I(\mathcal{C}) = (x_2x^{w_1}, \dots, x_2x^{w_r}, x^{w_{r+1}}, \dots, x^{w_q}),$$

where $x^{w_i} \in K[x_3, \dots, x_n]$ for all i . We must show that the ideal

$$I(\mathcal{C}') = I + (x_1x^{w_1}, \dots, x_1x^{w_r})$$

is normal. Consider the sets

$$\begin{aligned}\mathcal{A} &= \{e_2, \dots, e_n, (0, 1, w_1, 1), \dots, (0, 1, w_r, 1), (0, 0, w_{r+1}, 1), \dots, \\ &\quad (0, 0, w_q, 1)\}, \\ \mathcal{A}' &= \mathcal{A} \cup \{e_1, (1, 0, w_1, 1), \dots, (1, 0, w_r, 1)\}.\end{aligned}$$

By hypothesis $\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A} = \mathbb{N}\mathcal{A}$. We must prove that $\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}' = \mathbb{N}\mathcal{A}'$. It suffices to show that the left hand side is contained in the right hand side because the other inclusion always holds. Take an integral vector (a, b, c, d) in $\mathbb{R}_+\mathcal{A}'$, where $a, b, d \in \mathbb{Z}$ and $c \in \mathbb{Z}^{n-2}$. Then

$$\begin{aligned}(a, b, c, d) &= \sum_{i=1}^r \alpha_i(0, 1, w_i, 1) + \sum_{i=r+1}^q \alpha_i(0, 0, w_i, 1) + \sum_{i=1}^r \beta_i(1, 0, w_i, 1) \\ &\quad + \sum_{i=1}^n \gamma_i e_i\end{aligned}$$

for some $\alpha_i, \beta_i, \gamma_i$ in \mathbb{R}_+ . Comparing entries one has

$$\begin{aligned}a &= \beta_1 + \dots + \beta_r + \gamma_1, \\ b &= \alpha_1 + \dots + \alpha_r + \gamma_2, \\ c &= \sum_{i=1}^r (\alpha_i + \beta_i)w_i + \sum_{i=r+1}^q \alpha_i w_i + \sum_{i=3}^n \gamma_i e_i, \\ d &= \sum_{i=1}^r (\alpha_i + \beta_i) + \sum_{i=r+1}^q \alpha_i.\end{aligned}$$

Consequently we obtain the equality

$$\begin{aligned}(0, a + b, c, d) &= \sum_{i=1}^r (\alpha_i + \beta_i)(0, 1, w_i, 1) + \sum_{i=r+1}^q \alpha_i(0, 0, w_i, 1) + (\gamma_1 + \gamma_2)e_2 \\ &\quad + \sum_{i=3}^n \gamma_i e_i,\end{aligned}$$

that is, the vector $(0, a + b, c, d)$ is in $\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A} = \mathbb{N}\mathcal{A}$. Thus there are λ_i, μ_i in \mathbb{N} such that.

$$(0, a + b, c, d) = \sum_{i=1}^r \mu_i(0, 1, w_i, 1) + \sum_{i=r+1}^q \mu_i(0, 0, w_i, 1) + \sum_{i=2}^n \lambda_i e_i.$$

Comparing entries we obtain the equalities

$$\begin{aligned} a + b &= \mu_1 + \cdots + \mu_r + \lambda_2, \\ c &= \mu_1 w_1 + \cdots + \mu_q w_q + \lambda_3 e_3 + \cdots + \lambda_n e_n, \\ d &= \mu_1 + \cdots + \mu_q. \end{aligned}$$

Case (I): $b \leq \sum_{i=1}^r \mu_i$. If $b < \mu_1$, we set $b = \mu'_1$, $\mu'_1 < \mu_1$, and define $\mu''_1 = \mu_1 - \mu'_1$. Otherwise pick $s \geq 2$ such that

$$\mu_1 + \cdots + \mu_{s-1} \leq b \leq \mu_1 + \cdots + \mu_s$$

Then $b = \mu_1 + \cdots + \mu_{s-1} + \mu'_s$, where $\mu'_s \leq \mu_s$. Set $\mu''_s = \mu_s - \mu'_s$. Notice that

$$\begin{aligned} a + b &= \mu_1 + \cdots + \mu_r + \lambda_2 = a + \mu_1 + \cdots + \mu_{s-1} + \mu'_s, \\ a &= \mu_s + \cdots + \mu_r + \lambda_2 - \mu'_s = \mu_{s+1} + \cdots + \mu_r + \mu''_s + \lambda_2. \end{aligned}$$

Then

$$\begin{aligned} (a, b, c, d) &= \sum_{i=1}^{s-1} \mu_i(0, 1, w_i, 1) + \mu'_s(0, 1, w_s, 1) + \sum_{i=r+1}^q \mu_i(0, 0, w_i, 1) \\ &\quad + \mu''_s(1, 0, w_s, 1) + \sum_{i=s+1}^r \mu_i(1, 0, w_i, 1) + \lambda_2 e_1 + \sum_{i=3}^n \lambda_i e_i, \end{aligned}$$

that is, $(a, b, c, d) \in \mathbb{N}\mathcal{A}'$.

Case (II): $b > \sum_{i=1}^r \mu_i$. Then $b = \sum_{i=1}^r \mu_i + \lambda'_2$. Since

$$a + b = \mu_1 + \cdots + \mu_r + \lambda_2 = a + \mu_1 + \cdots + \mu_r + \lambda'_2$$

we get $a = \lambda_2 - \lambda'_2$. In particular $\lambda_2 \geq \lambda'_2$. Then

$$(a, b, c, d) = \sum_{i=1}^r \mu_i(0, 1, w_i, 1) + \sum_{i=r+1}^q \mu_i(0, 0, w_i, 1) + a e_1 + \lambda'_2 e_2 + \sum_{i=3}^n \lambda_i e_i$$

that is, $(a, b, c, d) \in \mathbb{N}\mathcal{A}'$. □

Our next goal is to present some applications of this result, but first we need to prove a couple of lemmas and we need to recall some notions and results.

Definition 2.4. *A subset $C \subset X$ is a minimal vertex cover of the clutter \mathcal{C} if: (i) every edge of \mathcal{C} contains at least one vertex of C , and (ii) there is no proper subset of C with the first property. If C satisfies condition (i) only, then C is called a vertex cover of \mathcal{C} .*

Definition 2.5. Let A be the incidence matrix of \mathcal{C} . The clutter \mathcal{C} satisfies the max-flow min-cut (MFMC) property if both sides of the LP-duality equation

$$\min\{\langle w, x \rangle \mid x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1} \rangle \mid y \geq 0; Ay \leq w\} \quad (1)$$

have integral optimum solutions x and y for each non-negative integral vector w .

Let A be the incidence matrix of \mathcal{C} whose column vectors are v_1, \dots, v_q . The set covering polyhedron of \mathcal{C} is given by:

$$Q(A) = \{x \in \mathbb{R}^n \mid x \geq 0; xA \geq \mathbf{1}\},$$

where $\mathbf{1} = (1, \dots, 1)$. This polyhedron was studied in [14, 15] to characterize the max-flow min-cut property of \mathcal{C} and to study certain algebraic properties of blowup algebras. A clutter \mathcal{C} is said to be *ideal* if $Q(A)$ is an *integral polyhedron*, i.e., it has only integral vertices. The integral vertices of $Q(A)$ are precisely the characteristic vectors of the minimal vertex covers of \mathcal{C} [14, Proposition 2.2].

Theorem 2.6 ([8, 14, 15, 18]). *The following conditions are equivalent:*

- (i) $gr_I(R) = R[It]/IR[It]$ is reduced, i.e., $gr_I(R)$ has no non-zero nilpotent elements.
- (ii) $R[It]$ is normal and $Q(A)$ is an integral polyhedron.
- (iii) $I^i = I^{(i)}$ for $i \geq 1$, where $I^{(i)}$ is the i th symbolic power of I .
- (iv) \mathcal{C} has the max-flow min-cut property.

If condition (iii) is satisfied we say that I is *normally torsion free*. A set of edges of the clutter \mathcal{C} is *independent* or *stable* if no two of them have a common vertex. We denote the smallest number of vertices in any minimal vertex cover of \mathcal{C} by $\alpha_0(\mathcal{C})$ and the maximum number of independent edges of \mathcal{C} by $\beta_1(\mathcal{C})$. These numbers are related to min-max problems because they satisfy:

$$\begin{aligned} \alpha_0(\mathcal{C}) &\geq \min\{\langle \mathbf{1}, x \rangle \mid x \geq 0; xA \geq \mathbf{1}\} \\ &= \max\{\langle y, \mathbf{1} \rangle \mid y \geq 0; Ay \leq \mathbf{1}\} \geq \beta_1(\mathcal{C}). \end{aligned}$$

Notice that $\alpha_0(\mathcal{C}) = \beta_1(\mathcal{C})$ if and only if both sides of the equality have integral optimum solutions. These two numbers can be interpreted in terms of invariants of I . By [14] the height of the ideal I , denoted by $ht(I)$, is equal to the *vertex covering number* $\alpha_0(\mathcal{C})$ and the *edge independence number* $\beta_1(\mathcal{C})$ is equal to the maximum r such that there exists a regular sequence of r monomials inside I .

Definition 2.7. *If $\alpha_0(\mathcal{C}) = \beta_1(\mathcal{C})$ we say that the clutter \mathcal{C} (or the ideal I) has the König property.*

Definition 2.8. *The clutter \mathcal{C} (or the ideal I) satisfy the packing property (PP for short) if all its minors satisfy the König property, i.e., $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for any minor \mathcal{C}' of \mathcal{C} .*

Theorem 2.9. *(A. Lehman; see [5, Theorem 1.8]) If \mathcal{C} has the packing property, then $Q(A)$ is integral.*

Corollary 2.10 ([5]). *If the clutter \mathcal{C} has the max-flow min-cut property, then \mathcal{C} has the packing property.*

Proof. Assume that the clutter \mathcal{C} has the max-flow min-cut property. This property is closed under taking minors. Thus it suffices to prove that \mathcal{C} has the König property. We denote the incidence matrix of \mathcal{C} by A . By hypothesis the LP-duality equation

$$\min\{\langle \mathbf{1}, x \rangle \mid x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1} \rangle \mid y \geq 0; Ay \leq \mathbf{1}\}$$

has optimum integral solutions x, y . To complete the proof notice that the left hand side of this equality is $\alpha_0(\mathcal{C})$ and the right hand side is $\beta_1(\mathcal{C})$. \square

Conforti and Cornuéjols conjecture that the converse is also true:

Conjecture 2.11. ([4]) *If the clutter \mathcal{C} has the packing property, then \mathcal{C} has the max-flow min-cut property.*

To the best of our knowledge this conjecture is open, see [5, Conjecture 1.6].

Corollary 2.12. *Let \mathcal{C} be a clutter and let \mathcal{C}' be a parallelization of \mathcal{C} . If $I(\mathcal{C})$ is normally torsion free, then $I(\mathcal{C}')$ is normally torsion free.*

Proof. Let A and A' be the incidence matrices of \mathcal{C} and \mathcal{C}' respectively. By Theorem 2.6 the ideal $I(\mathcal{C})$ is normal and $Q(A)$ is integral. From Theorem 2.3 the ideal $I(\mathcal{C}')$ is normal, and since the integrality of $Q(A)$ is closed under minors and parallelizations (see [14] and [23]) we get that $Q(A')$ is again integral. Thus applying Theorem 2.6 once more we get that $I(\mathcal{C}')$ is normally torsion free. \square

Corollary 2.13. *Let \mathcal{C} be a clutter and let \mathcal{C}' be a parallelization of \mathcal{C} . If \mathcal{C} has the max-flow min-cut property, then \mathcal{C}' has the König property. In particular \mathcal{C}^w has the König property for all $w \in \mathbb{N}^n$.*

Proof. By Corollary 2.12 the clutter \mathcal{C}' has the max-flow min-cut property. Thus applying Corollary 2.10 we obtain that \mathcal{C}' has the König property. \square

Lemma 2.14. *Let \mathcal{C} be a clutter and let A be its incidence matrix. If $w = (w_i)$ is a vector in \mathbb{N}^n , then*

$$\beta_1(\mathcal{C}^w) \leq \max\{\langle y, \mathbf{1} \rangle \mid y \in \mathbb{N}^q; Ay \leq w\}.$$

Proof. We may assume that $w = (w_1, \dots, w_m, 0, \dots, 0)$, where $w_i \geq 1$ for $i = 1, \dots, m$. Recall that for each i the vertex x_i is duplicated $w_i - 1$ times. We denote the duplications of x_i by $x_i^2, \dots, x_i^{w_i}$ and set $x_i^1 = x_i$. Thus the vertex set of \mathcal{C}^w is equal to

$$V(\mathcal{C}^w) = \{x_1^1, \dots, x_1^{w_1}, \dots, x_i^1, \dots, x_i^{w_i}, \dots, x_m^1, \dots, x_m^{w_m}\}.$$

There are f_1, \dots, f_{β_1} independent edges of \mathcal{C}^w , where $\beta_1 = \beta_1(\mathcal{C}^w)$. Each f_i has the form

$$f_k = \{x_{k_1}^{j_{k_1}}, x_{k_2}^{j_{k_2}}, \dots, x_{k_r}^{j_{k_r}}\} \quad (1 \leq k_1 < \dots < k_r \leq m; 1 \leq j_{k_i} \leq w_{k_i}).$$

We set $g_k = \{x_{k_1}^1, x_{k_2}^1, \dots, x_{k_r}^1\} = \{x_{k_1}, x_{k_2}, \dots, x_{k_r}\}$. By definition of \mathcal{C}^w we get that $g_k \in E(\mathcal{C})$ for all k . We may re-order the f_i so that

$$\underbrace{g_1 = g_2 = \dots = g_{s_1}}_{s_1}, \underbrace{g_{s_1+1} = \dots = g_{s_2}}_{s_2-s_1}, \dots, \underbrace{g_{s_{r-1}+1} = \dots = g_{s_r}}_{s_r-s_{r-1}}$$

and g_{s_1}, \dots, g_{s_r} distinct, where $s_r = \beta_1$. Let v_i be the characteristic vector of g_{s_i} . Set $y = s_1 e_1 + (s_2 - s_1)e_2 + \dots + (s_r - s_{r-1})e_r$. We may assume that the incidence matrix A of \mathcal{C} has column vector v_1, \dots, v_r . Then y satisfies $\langle y, \mathbf{1} \rangle = \beta_1$. For each k_i the number of variables of the form $x_{k_i}^\ell$ that occur in f_1, \dots, f_{β_1} is at most w_{k_i} because the f_i are pairwise disjoint. Hence for each k_i the number of times that the variable $x_{k_i}^1$ occurs in g_1, \dots, g_{β_1} is at most w_{k_i} . Then

$$Ay = s_1 v_1 + (s_2 - s_1)v_2 + \dots + (s_r - s_{r-1})v_r \leq w.$$

Therefore we obtain the required inequality. \square

Let \mathcal{C} be a clutter. For use below we denote the set of minimal vertex covers of \mathcal{C} by $\Upsilon(\mathcal{C})$.

Lemma 2.15. *Let \mathcal{C} be a clutter and let A be its incidence matrix. If $w = (w_i)$ is a vector in \mathbb{N}^n , then*

$$\min \left\{ \sum_{x_i \in C} w_i \mid C \in \Upsilon(\mathcal{C}) \right\} = \alpha_0(\mathcal{C}^w).$$

Proof. We may assume that $w = (w_1, \dots, w_m, w_{m+1}, \dots, w_{m_1}, 0, \dots, 0)$, where $w_i \geq 2$ for $i = 1, \dots, m$, $w_i = 1$ for $i = m+1, \dots, m_1$, and $w_i = 0$ for $i > m_1$. Thus for $i = 1, \dots, m$ the vertex x_i is duplicated $w_i - 1$ times. We denote the duplications of x_i by $x_i^2, \dots, x_i^{w_i}$ and set $x_i^1 = x_i$.

We first prove that the left hand side is less or equal than the right hand side. Let C be a minimal vertex cover of \mathcal{C}^w with α_0 elements, where $\alpha_0 = \alpha_0(\mathcal{C}^w)$. We may assume that $C \cap \{x_1, \dots, x_{m_1}\} = \{x_1, \dots, x_s\}$. Note

that $x_i^1, \dots, x_i^{w_i}$ are in C for $i = 1, \dots, s$. Indeed since C is a minimal vertex cover of \mathcal{C}^w , there exists an edge e of \mathcal{C}^w such that $e \cap C = \{x_i^1\}$. Then $(e \setminus \{x_i^1\}) \cup \{x_i^j\}$ is an edge of \mathcal{C}^w for $j = 1, \dots, w_i$. Consequently $x_i^j \in C$ for $j = 1, \dots, w_i$. Hence

$$w_1 + \dots + w_s \leq |C| = \alpha_0. \tag{2}$$

On the other hand the set $C' = \{x_1, \dots, x_s\} \cup \{x_{m_1+1}, \dots, x_n\}$ is a vertex cover of \mathcal{C} . Let D be a minimal vertex cover of \mathcal{C} contained in C' . Let e_D denote the characteristic vector of D . Then, since $w_i = 0$ for $i > m_1$, using Eq. (2) we get

$$\langle w, e_D \rangle = \sum_{x_i \in D} w_i = \sum_{x_i \in D \cap \{x_1, \dots, x_s\}} w_i \leq \sum_{x_i \in \{x_1, \dots, x_s\}} w_i \leq \alpha_0.$$

This completes the proof of the asserted inequality.

Next we show that the right hand side of the inequality is less or equal than the left hand side. Let C be a minimal vertex cover of \mathcal{C} . Note that the set

$$C' = \cup_{x_i \in C} \{x_i^1, \dots, x_i^{w_i}\}$$

is a vertex cover of \mathcal{C}^w . Indeed any edge e^w of \mathcal{C}^w has the form $e^w = \{x_{i_1}^{j_1}, \dots, x_{i_r}^{j_r}\}$ for some edge $e = \{x_{i_1}, \dots, x_{i_r}\}$ of \mathcal{C} and since e is covered by C , we have that e^w is covered by C' . Hence $\alpha_0(\mathcal{C}^w) \leq |C'| = \sum_{x_i \in C} w_i$. As C was an arbitrary vertex cover of \mathcal{C} we get the asserted inequality. \square

Corollary 2.16. [23, Chapter 79] *Let \mathcal{C} be a clutter. Then \mathcal{C} satisfies the max-flow min-cut property if and only if $\beta_1(\mathcal{C}^w) = \alpha_0(\mathcal{C}^w)$ for all $w \in \mathbb{N}^n$.*

Proof. If \mathcal{C} has the max-flow min-cut property, then \mathcal{C}^w has the König property by Corollary 2.13. Conversely if \mathcal{C}^w has the König property for all $w \in \mathbb{N}^n$, then by Lemmas 2.14 and 2.15 both sides of the LP-duality equation

$$\min\{\langle w, x \rangle \mid x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1} \rangle \mid y \geq 0; Ay \leq w\}$$

have integral optimum solutions x and y for each non-negative integral vector w , i.e., \mathcal{C} has the max-flow min-cut property. \square

3. Cohen-Macaulay ideals with max-flow min-cut

One of the aims here is to show how to construct Cohen-Macaulay clutters satisfying max-flow min-cut, PP, and normality properties. Let \mathcal{C} be a uniform clutter. A main result of this section proves that if \mathcal{C} satisfies PP (resp. max-flow min-cut), then there is a uniform Cohen-Macaulay clutter \mathcal{C}_1 satisfying PP (resp. max-flow min-cut) such that \mathcal{C} is a minor

of \mathcal{C}_1 . In particular for uniform clutters we prove that it suffices to show Conjecture 2.11 for Cohen-Macaulay clutters (see Corollary 3.4).

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and let \mathcal{C} be a clutter on the vertex set X . As usual, in what follows, we denote the edge ideal of \mathcal{C} by $I = I(\mathcal{C})$. Recall that \mathfrak{p} is a minimal prime of $I = I(\mathcal{C})$ if and only if $\mathfrak{p} = (C)$ for some minimal vertex cover C of \mathcal{C} [30, Proposition 6.1.16]. Thus the primary decomposition of the edge ideal of \mathcal{C} is given by

$$I(\mathcal{C}) = (C_1) \cap (C_2) \cap \cdots \cap (C_p),$$

where C_1, \dots, C_p are the minimal vertex covers of \mathcal{C} . In particular observe that the *height* of $I(\mathcal{C})$, denoted by $ht I(\mathcal{C})$, is equal to the minimum cardinality of a minimal vertex cover of \mathcal{C} . Also notice that the associated primes of $I(\mathcal{C})$ are precisely the minimal primes of $I(\mathcal{C})$.

Proposition 3.1. *Let $R[z_1, \dots, z_\ell]$ be a polynomial ring over R . If I is a normal ideal of R , then $J = (I, x_1 z_1 \cdots z_\ell)$ is a normal ideal of $R[z_1, \dots, z_\ell]$.*

Proof. By induction on p we will show $\overline{J^p} = J^p$ for all $p \geq 1$. If $p = 1$, then $\overline{J} = J$ because J is square-free (see [30, Corollary 7.3.15]). Assume $\overline{J^i} = J^i$ for $i < p$ and $p \geq 2$. Let y be a monomial in $\overline{J^p}$, then $y^m \in J^{pm}$, for some $m > 0$. Since $\overline{J^p} \subset \overline{J^{p-1}} = J^{p-1}$ we can write

$$y = z_1^{t_1} \cdots z_\ell^{t_\ell} (x_1 z_1 \cdots z_\ell)^r M f_1 \cdots f_{p-r-1},$$

where M is a monomial with $z_i \notin \text{supp}(M)$ for all i and the f_i 's are monomials in J with $z_i \notin \text{supp}(f_j)$ for all i, j . We set $h = M f_1 \cdots f_{p-r-1}$. It suffices to show that $y \in J^p$. Since $y^m \in J^{pm}$ we have

$$y^m = z_1^{mt_1} \cdots z_\ell^{mt_\ell} (x_1 z_1 \cdots z_\ell)^{rm} h^m = N (x_1 z_1 \cdots z_\ell)^s g_1 \cdots g_{mp-s}, \quad (3)$$

where N is a monomial, $z_i \notin \text{supp}(g_j)$ for all i, j , and the g_i 's are monomials in J . We distinguish two cases:

Case (a): Assume $t_i = 0$ for some i , then $s \leq rm$ because z_i^{rm} is the maximum power of z_i that divides y^m . Making $z_j = 1$ for $j = 1, \dots, \ell$ in Eq. (3) we get

$$x_1^{rm-s} h^m = N' g_1 \cdots g_{mp-s}.$$

Thus $h^m \in I^{(mp-s)-(rm-s)} = I^{m(p-r)}$. Therefore we get $h \in \overline{I^{p-r}} = I^{p-r}$ and $y = z_1^{t_1} \cdots z_\ell^{t_\ell} (x_1 z_1 \cdots z_\ell)^r h \in J^p$.

Case (b): If $t_i > 0$ for all i , we may assume $x_1 \notin \text{supp}(M)$, otherwise $y \in J^p$. We may also assume $x_1 \notin \text{supp}(f_i)$ for all i , otherwise it is not hard to see that we are back in case (a). Notice that $s \leq rm$, because $x_1 \notin \text{supp}(h)$. From Eq. (3) it follows that $h \in \overline{I^{p-r}} = I^{p-r}$ and $y = z_1^{t_1} \cdots z_\ell^{t_\ell} (x_1 z_1 \cdots z_\ell)^r h \in J^p$. \square

Lemma 3.2. *Let $R[z_1, \dots, z_\ell]$ be a polynomial ring over R and let I_1 be the ideal obtained from I by making $x_1 = 0$. Then: (a) if I and I_1 satisfy the König property, then the ideal $J = (I, x_1 z_1 \cdots z_\ell)$ satisfies the König property, and (b) if I satisfies PP, then J satisfies PP.*

Proof. (a): If $ht(I) = ht(J)$, then J satisfies König because I does. Assume that $g = ht(I) < ht(J)$. Then $ht(J) = g + 1$. Notice that every associated prime ideal of I of height g cannot contain x_1 . We claim that $ht(I_1) = g$. If $r = ht(I_1) < g$, pick a minimal prime \mathfrak{p} of I_1 of height r . Then $\mathfrak{p} + (x_1)$ is a prime ideal of height at most g containing both I and x_1 , a contradiction. This proves the claim. Since I_1 satisfies König, there are g independent monomials in I_1 . Hence $h_1, \dots, h_g, x_1 z_1 \cdots z_\ell$ are $g + 1$ independent monomials in J , as required. Part (b) follows readily from part (a). □

Theorem 3.3. *Let \mathcal{C} be a d -uniform clutter on the vertex set X . Let*

$$Y = \{y_{ij} \mid 1 \leq i \leq n; 1 \leq j \leq d - 1\}$$

be a set of new variables, and let \mathcal{C}' be the clutter with vertex set $V(\mathcal{C}') = X \cup Y$ and edge set

$$E(\mathcal{C}') = E(\mathcal{C}) \cup \{\{x_1, y_{11}, \dots, y_{1(d-1)}\}, \dots, \{x_n, y_{n1}, \dots, y_{n(d-1)}\}\}.$$

Then the edge ideal $I(\mathcal{C}')$ is Cohen-Macaulay. If \mathcal{C} satisfies PP (resp. max-flow min-cut), then \mathcal{C}' satisfies PP (resp. max-flow min-cut).

Proof. Set $S = K[X \cup Y]$ and $I' = I(\mathcal{C}')$. The clutter \mathcal{C}' is a grafting of \mathcal{C} as defined by Faridi in [10]. Then I' is Cohen-Macaulay by [10, Theorem 8.2]. If \mathcal{C} satisfies PP, then from Lemma 3.2(b) it follows that \mathcal{C}' satisfies PP. Assume that \mathcal{C} satisfies MFMC. By Proposition 3.1 $S[I't]$ is normal. Since \mathcal{C}' satisfies PP, by Lehman's theorem we get that $Q(A')$ is integral, where A' is the incidence matrix of \mathcal{C}' . Therefore using Theorem 2.6 we conclude that \mathcal{C}' has MFMC. □

Recall that a clutter \mathcal{C} is called *Cohen-Macaulay* (CM for short) if $R/I(\mathcal{C})$ is a Cohen-Macaulay ring. Since \mathcal{C} is a minor of \mathcal{C}' we obtain:

Corollary 3.4. *Let \mathcal{C} be a uniform clutter. If \mathcal{C} satisfies PP (resp. max-flow min-cut), then there is a uniform Cohen-Macaulay clutter \mathcal{C}_1 satisfying PP (resp. max-flow min-cut) such that \mathcal{C} is a minor of \mathcal{C}_1 .*

This result is interesting because it says that for uniform clutters it suffices to prove Conjecture 2.11 for Cohen-Macaulay clutters, which have a rich structure.

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