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The inverse problem of variational calculus and the problem of mixed endpoint conditions

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Abstract. P. A. Griffiths established the so-called mixed endpoint conditions for variational problems with non-holonomic constraints. We will present some results in this context and discuss the inverse problem of calculus of variations.
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1. **Introduction**

The study of Calculus of Variations for multiple integrals was first developed by Caratheodory [1929], while Weil-De Donder [1936], [1935] advanced a different theory later. The two approaches were unified by Lepage [1936-1942], Dedecker [1953-1977] and Liesen [1967] in a framework using the n-Grassmannian manifold of a C^∞ manifold. Important contributions in the Calculus of Variations on smooth manifolds were made by R. Hermann [1966], H. Goldschmidt and S. Sternberg [1973] with their Hamilton-Cartan formalism, as well as by Ouzilou [1972], D. Krupka [1970-1975] and I. M. Anderson [1980]. The symplectic approach of P. L. Garcia and A. Pérez-Rendón [1969-1978], the multisymplectic version of Kijowski and Tulczyjew [1979] based on the theory of Dedecker, the polysymplectic approach of C. Günther [1987], Edelen [1961] and Rund [1966] are also important references in this field. Here we deal with the broader problem of finding extrema of a functional on a set of n-dimensional integral manifolds of a Pfaffian differential system.

In 1983, Griffiths proposed a new approach to variational problems based on techniques from the theory of exterior differential systems. His work dealt with the problem of finding extrema for a functional ϕ defined on the set of one-dimensional integral manifolds of a differential system (I^*, L^*).

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This approach was established using intrinsic entities. In this work we present a general setting based on [25] (sections 2 to 8), and we deal with the inverse problem (section 9).

In 1887 Helmholtz addressed the following problem: given

$$P_i = P_i(x, u^j, u^j_x, u^j_{xx}),$$

is there a Lagrangian $L(x, u^j, u^j_x)$ such that

$$E_i(L) = \partial L/\partial u^i - D_x \partial L/\partial u^i_x = P_i$$

where

$$D_x = \partial/\partial x + u^i_x \partial/\partial u^i + u^i_{xx} \partial/\partial u^i_x?$$

He found necessary conditions for P_i to form an Euler-Lagrange system of equations (see (9.1), (9.2) and (9.3)). Some years later, these conditions where proved to be locally sufficient. I. M. Anderson [1992], [1980], P. J. Olver [1986], F. Takens [1979], W. M. Tulczyjew [1980] and A. M. Vinagradov [1984] generalized Helmholtz's conditions for both higher order systems of partial differential equations and multiple integrals.

2. Integral manifolds of a differential system and valued differential systems

We assume that a Pfaffian differential system (I^*, L^*) is given on a real-manifold X by:

i) a subbundle $I^* \subset T^*X$,
ii) another subbundle $L^* \subset T^*X$ with $I^* \subset L^* \subset T^*X$,

such that the rank $(L^*/I^*) = n$ (with n being a natural number).

An integral manifold of (I^*, L^*) is given by an oriented connected compact n-dimensional smooth manifold N (possibly with a piecewise smooth boundary ∂N) together with a smooth mapping

$$f : N \to X$$

satisfying

$${I^*_{f(x)}}^\perp = {L^*_{f(x)}}^\perp + f_*(TN), \tag{2.1}$$

for all $x \in N$, where $f_* : T_xN \to T_{f(x)}X$ is the differential of f at x.

We denote by $V(I^*, L^*)$ the collection of integral manifolds f of (I^*, L^*).

A valued differential system is a triple (I^*, L^*, φ), where (I^*, L^*) is a Pfaffian differential system and φ is an n-form on X.

We define the functional ϕ associated with (I^*, L^*, φ) in $V(I^*, L*)$ by:

$$\phi : V(I^*, L^*) \to R,$$

$$f \to \phi[f] = \int f^{*}\varphi. \tag{2.2}$$

3. Local embeddability

The following definition is a general setting for the study of problems in the Calculus of Variations. In [25] we proved that there exist locally defined mappings that induce (I^*, L^*) from the canonical system in $J^1(R^n, R^s)$ possibly with some constraints, establishing a local coorespondence between these differential systems. Let us assume that $d(C^\infty(X, L^*)) \subset C^\infty(X, L^* \wedge T^*X)$, and let $d' = \dim X$; $s = \mathrm{rank} I^*$ ($d(C^\infty(X, L^*))$ is the set of images produced by the exterior derivative of $C^\infty(X, L^*)$). Using the Frobenius theorem, we can set for every $p \in X$ a chart coordinate system $\{u^1, ..., u^{s+n}, v^1, ..., v^{d'-s-n}\}$ so that

i)
$$L^* = \ \mathrm{span}\{du^\alpha | 1 \le \alpha \le s+n\}, \tag{3.1}$$
ii)
$$L^{*\perp} = \ \mathrm{span}\{\frac{\partial}{\partial v^i} | 1 \le i \le d' - s - n\} \tag{3.2}$$

for an open subset U of X with $p \in U$.

Definition 3.1. *Let (I^*, L^*) be a Pfaffian differential system with*
$$d(C^\infty(X, L)) \subset C^\infty(X, L^* \wedge T^*X).$$
We say that (I^*, L^*) is locally embeddable if for every $p \in X$ there exist an open neighborhood U of p and local coframes
$$CF = \{\theta_1, ..., \theta_s\} \tag{3.3}$$
for I^* and
$$CF' = \{\theta_1, ..., \theta_s, du''^s + 1, du''^s + n\} \tag{3.4}$$
for L^*_U, satisfying the following conditions:

(i)
$$\delta(I^*_U \wedge \Omega) \subset T^* \wedge \Lambda^n(L^*_U)/(T^*U \wedge I^*_U \wedge \Lambda^{n-1}(L^*)) \tag{3.5}$$
(ii) *Ker δ is a constant rank subbundle of $I^* \wedge \Omega$,*

where $\Omega = span\{du''^{s+1} \wedge ... \wedge \widehat{du''^{s+\beta}} \wedge ... \wedge du''^{s+n}\}$; $\widehat{du''^{s+\beta}}$ -means deletion of the $s+b$ factor (for $n = 1, \widehat{du''^{s+1}} = 1$). We use u'' since we may have to reorder these coordinates.

The map $\delta : I^* \wedge \Omega \to \Lambda^{n+1}(T^*U)/I_u^* \wedge (\Lambda^n(T*U))$ is induced by

$$d : C^\infty(U, I^* \wedge \Omega) \to C^\infty(U, \Lambda^{n+1}(T^*U))$$

on $I^* \wedge \Omega$.

This definition means that if I^* has no Cauchy characteristics, the structure equations are locally:

$$d\theta^i \equiv \pi^i_j \wedge du''^{s+j} + A^{ij'}_{i'\alpha}\pi^{i'}_{j'} \wedge \theta^\alpha + B^i_{\alpha\beta}\theta^\alpha \wedge du''^{s+\beta} mod I \wedge I \tag{3.6}$$

$1 \leq i, i', \alpha \leq s, 1 \leq j, j', \beta \leq n, I = C^\infty(X, I^*)$.

4. The Cartan system of Ψ

Let (I^*, L^*, φ) be a valued differential system on X, and W be the total space of I^*. Let χ be the canonical form on T^*X, and i the inclusion map $W \overset{i}{\hookrightarrow} T^*X$.

Let us assume that there exists a local n-form ω inducing a nonzero section of $\Lambda^n(L^*/I^*)$ and has the following form:

$$\omega = \omega^1 \wedge ... \wedge \omega^n. \tag{4.1}$$

We define:

$$\omega_i = (-1)^{i-1}\omega^1 \wedge ... \wedge \widehat{\omega^i} ... \wedge \omega^n. \tag{4.2}$$

Let W^n be the n-Cartesian power of W, and Z be a subset of W^n defined by $Z = \{z \in W^n : \pi'(z) \in \Delta X^n\}$, where π' is the natural projection $\pi' : W^n \to X^n$, and ΔX^n is the diagonal submanifold of X^n. The subset Z is a vector subbundle over X and $dim Z = d + sn$. We define

$$\Psi = d\psi \tag{4.3}$$

where ψ is given by

$$\psi = \pi^*\varphi + (\pi^j \circ i')^*[i^*(\chi)] \wedge \pi^*\omega_j. \tag{4.4}$$

π^j is the natural projection into the j^{th} component $\pi^j : W^n \to W$, i' is the inclusion map $Z \to W^n$ and π is the natural projection $\pi : Z \to X$.

Definition 4.1. *Given the $n+1$-form Ψ, the Cartan system $C(\Psi)$ is the ideal generated by the set of n-forms*

$$\{v \lrcorner \Psi \quad where \quad v \in C^\infty(Z, TZ)\}.$$

An integral manifold of $(C(\Psi), \omega)$ is given by an oriented connected compact n-dimensional smooth manifold N (possibly with a piecewise smooth boundary ∂N) together with a smooth mapping

$$f : N \to X$$

satisfying:

$$f^*\theta = 0 \quad \textit{for every} \quad \theta \in C(\Psi) \tag{4.5}$$

and

$$f^*(\omega) \neq 0. \tag{4.6}$$

A solution of $(C(\Psi), \omega)$ projected in X will give an extremum of ϕ.

5. The momentum space, prolongation of $(C(\Psi), \pi^*\omega)$ in the momentum space, non-degeneracy

The momentum space is constructed in the following way. Suppose we are given on Z (see section 4):

(i) a closed $(n+1)$-form Ψ with the associated Cartan system $C(\Psi)$,

(ii) $\pi^{'*}$ the pull back to Z of the ω n-form which induces a nonzero section on $\Lambda^n(L^*/I^*)$.

Integral elements of $(C(\Psi), \pi'^*\omega)$ are defined in a similar way as the integral elements of (I^*, L^*). The set of integral elements $[x_0, E_0^n]$ gives a subset

$$V_n(C(\Psi), \pi^*\omega)) \subset G_n(Z) \quad (G_n(Z) \text{ is the } n\text{-Grassmanian}).$$

Denoting by π'' the projection $G_n(Z) \to Z$ and assuming regularity at each step, one inductively defines:

$$\begin{aligned} Z_1 = \pi''(V_n(C(\Psi), \pi^*\omega), V_n'(C(\Psi), \pi^*\omega)) = \\ \{E \in V_n(C(\Psi), \pi^*\omega) : E \text{ tangent to } Z_1\}, \end{aligned} \tag{5.1}$$

$$\begin{aligned} Z_2 = \pi''(V_n'(C(\Psi), \pi^*\omega), V_n''(C(\Psi), \pi^*\omega)) = \\ \{E \in V_n'(C(\Psi, \pi^*\omega)) : E \text{ tangent to } Z_2\}. \end{aligned} \tag{5.2}$$

Definition 5.1. *Suppose (I^*, L^*, φ) is a valued differential system, with (I^*, L^*) being a locally embeddable differential system and $\omega = \omega^1 \wedge ... \wedge \omega^n$. If there exists a $k_0 \in N$ such that $Z_{k_0} = Z_{k_0+1} = ... = Z_{k_0+n'} (n' \in N)$ in the above construction, with*

(i) *Z_{k_0} being a manifold of dimension $(n+1)m + n$ for $m \in N$, and*

(ii) *$(C(\Psi), \pi^*\omega)_{Z_{k_0}}$ being a differential system in Z_{k_0} with $r_n = 0$ (Cartan number in Cartan-Kähler Theorem) for all $V_{n-1}(C(\Psi), \pi^*\omega)$; (for $n = 1$ we follow* [23] *and replace this condition by $\psi \wedge \Psi^n \neq 0 \quad on \quad Z_{k_0}$).*

Then (I^*, L^*, φ) is a non-degenerate valued differential system, and $Z = Y$ is called the momentum space.

We call $(C(\Psi), \pi^*\omega)_Y$ the prolongation of $(C(\Psi), \pi^*\omega)$ in the momentum space. By construction, the differential system $(C(\Psi), \pi^*\omega)_Y$ satisfies:

(i) the projection $(C(\Psi), \pi^*\omega) \to Y$ is surjective,
(ii) the integral manifolds of $(C(\Psi), \pi'^*\omega)$ on Z coincide with those of $(C(\Psi), \pi^*\omega)$ on Y.

6. Well-posed valued differential systems

Definition 6.1. *$(I^*, L^*, \varphi, P^*, M^*)$ is a well-posed valued differential system, if the following conditions are satisfied:*

(i) *(I^*, L^*, φ) is a non-degenerate valued differential system (with $dimY = (n+1)m + n$) and $\varphi = L\omega$ for a smooth function L on X;*
(ii) *there exists a subbundle P^* of I^* of rank m and a subbundle M^* of L^* of rank $m+n$, such that:*
  (a) $$\begin{array}{ccccc} I^* & \subset & L^* & \subset & T^*X \\ \cup & & \cup & & \\ P^* & \subset & M^*, & & \end{array}$$
  (b) *the locally given n-form ω also induces a nonzero section on $\Lambda^n(M^*/P^*)$,*
  (c) *$Y \subset (P^*)^n|_{\Delta X^n}$, with Y a subbundle of $(P^*)^n|_{\Delta X^n}$,*
(iii) *$\pi"^* M^*$ =span$\{\pi^*\theta | \theta \in C^\infty(X, M^*)\}$ is completely integrable on Y, where $\pi" = \pi \circ i$. As before i denotes the inclusion mapping $Y \to Z$ and π the projection $Z \to X$.*

Let us assume that there exists a coframe $CF = \{\theta^\alpha, du^{s+j}, \pi^{i'}_{j'}, \pi^{i''}_j | 1 \leq \alpha \leq s, 1 \leq i' \leq s_l, j' \in L_{i'}, s_{l+1} \leq i'' \leq s, 1 \leq j \leq n\}$ for T^*X with $L_{i'} \subset \{k \in N, 1 \leq k \leq n\}$ such that

(i)
$$I^* = \text{ span}\{\theta^\alpha | 1 \leq \alpha \leq s\}; \tag{6.1}$$
(ii)
$$L^* = \text{ span}\{\theta^\alpha, du^{s+j} | 1 \leq \alpha \leq s, 1 \leq j \leq n\}; \tag{6.2}$$
(iii) $T^*X = L^* \oplus R^*$ ($\oplus$ denotes a direct sum) with $R^* = \text{span}\{\pi^{i'}_{j'}, \pi^{i''}_j | 1 \leq i' \leq s_l, j' \in L_{i'}, s_{l+1} \leq i'' \leq s, 1 \leq j \leq n\}$;
(iv)
$$d\theta^{i'}_{j''} \equiv 0 \text{ mod } I, \text{ for } j" \notin L_{i'}; \tag{6.3}$$
(v)
$$d\theta^{i'}_{j'} \equiv \pi^{i'}_{j'} \wedge \omega \text{ mod } I, \text{ for } j' \in L_{i'}; \tag{6.4}$$
(vi)
$$d\theta^{i''}_j \equiv \pi^{i''}_j \wedge \omega \text{ mod } I, \text{ when } 1 \leq j \leq n; \tag{6.5}$$

(vii) $\pi^{i'}_{j'}, \pi^{i"}_{j}$ are linearly independent mod L.

We define $\theta^{\alpha}_{j} \doteq \theta^{\alpha} \wedge \omega_j$.

Let $d\varphi \equiv L^{j}_{i''} \wedge \pi^{i''}_{j} + L^{j'}_{i'} \wedge \pi^{i'}_{j'}$ mod I and $dL^{\alpha}_{\nu} \equiv L^{\alpha\alpha'}_{\nu\nu'} \pi^{\nu'}_{\alpha'}$ mod πL^* $1 \leq \alpha, \alpha' \leq s \quad \nu \in L_{\alpha}$ and $\nu' \in L_{\alpha'}$.

Quadratic form A**:** Let $(I^*, L^*, \varphi, P^*, M^*)$ be a well-posed valued differential system and A be a quadratic form defined in T^*X given by $A(v,w) = L^{\alpha\alpha'}_{\nu\nu'} v^{\alpha}_{\nu} w^{\alpha'}_{\nu'}$, where $v = v_{\theta^{\alpha}} \partial/\partial\theta^{\alpha} + v_{\pi^{\nu}_{\alpha}} \partial/\partial\pi^{\nu}_{\alpha}$ and $w = w_{\theta^{\alpha}} \partial/\partial\theta^{\alpha} + w_{\pi^{\nu}_{\alpha}} \partial/\partial\pi^{\nu}_{\alpha}$. This quadratic form plays an important role in establishing necessary conditions for a local extremum.

6.1. **Generalized Lagrange Problem.** Let us describe the following problem:

Generalized Lagrange Problem. Let $X = J^1(R^n, R^m)$ (the 1 jet manifold), with the canonical system I^* defined on X (i.e. $I^* = \text{span}\{\theta^{\alpha} = dy^{\alpha} - y^{\alpha}_{x^i} dx^i\}$). Let $\varphi = L\omega$ with $\omega = dx^1 \wedge ... \wedge dx^n$. We choose $x^1, ..., x^n$ to be coordinates for R^n, and $y^1, ..., y^m$ to be coordinates for R^m.

We proved in [26] that a Lagrange problem for $n = 1$ with $L det L^{\alpha\alpha'}_{\nu\nu'} \neq 0$, and with constraints not envolving more than one variable $\dot{y}$ in each equation of restriction is a well posed valued differential system.

7. The Euler-Lagrange differential system for a well-posed valued differential system

When we compute the first variation of ϕ, we find an integral over N and another over the boundary ∂N. The volume integral will vanish for projections of integral manifolds of the Cartan system $(C(\Psi), \pi^*\omega)$ into X. Choosing suitably the set of boundary conditions we can make the integral over the boundary to vanish as well, providing stationary integral manifolds for generalized Lagrange problems (see [25]).

7.1. The Euler-Lagrange differential system.

Definition 7.1. *Let* (I^*, L^*, φ) *be a valued differential system. The Cartan system* $(C(\Psi), \pi^*\omega)$ *is called the Euler-Lagrange differential system associated with* (I^*, L^*, φ).

Assuming that (I^*, L^*, φ) is non-degenerate, we now consider the restriction to Y of the Euler-Lagrange differential system associated with (I^*, L^*, φ). The following proposition is easy to prove (see [25]):

Proposition 7.1. *If* g *is an integral manifold of* $(C(\Psi), \pi^*\omega)$, *then* $\pi \circ g \in V(I^*, L^*)$, *where* π *is the natural projection* $\pi : Z \to X$.

We denote by $(V(C(\Psi),\pi^*\omega)$ the set of integral manifolds of $(C(\Psi),\pi^*\omega))$.

8. **Examples**

Example 1. *Strings* [41], [42]

Let $X = J^1(N, R^m)$, N being a two-dimensional manifold. In this case $I^* = span\{dx^\alpha - x'^\alpha d\sigma - \dot{x}^\alpha d\tau | 0 \leq \alpha \leq m-1, x^\alpha\}$ are coordinates in R^m, and σ, τ are coordinates of N, $x'^\alpha = \frac{\partial x^\alpha}{\partial \sigma}$, $\dot{x}^\alpha = \frac{\partial x^\alpha}{\partial \tau}$. In R^m we take a metric defined in TR^m by $g^{00} = -g^{11} = 1, 1 \leq i \leq m$ and $g^{ij} = 0$ for $i \neq j$. The set X is given by: $X = \{x \in X_0 | (\dot{x}\cdot\dot{x}) \geq 0$ and $(x'\cdot x') \leq 0\}$ (where $(\cdot)$ denotes the inner product with respect to the metric g). The form ω is $\omega = d\sigma \wedge d\tau$. We have

$$\varphi = L\omega = [(x'\cdot\dot{x})^2 - (\dot{x}\cdot\dot{x})(x'\cdot x')]^{1/2} d\sigma \wedge d\tau. \tag{8.1}$$

Note: L is a function of $\dot{x}$ and x' only.

First variation of ϕ.. *Let $\phi = \int f^*(\varphi)$, where $f \in V(I^*, L^*)$. Then*

$$\delta\phi = \int f^*(v \lrcorner d\varphi + d(v \lrcorner \varphi)), \tag{8.2}$$

where $v(\sigma,\tau) = F_*(\partial/\partial t)(t,\sigma,\tau)|_{t=0}, (\sigma,\tau) \in N, t \in [0,1]$ and F is the one parameter variation of f i.e, $F(t,\sigma,\tau)|_{t=t_1} \in V(I^*, L^*)$ for all $0 \leq t_1 \leq 1$. Hence the Lie derivative of $dx^\alpha - x'^\alpha d\sigma - \dot{x}^\alpha d\tau$ by v along $f(N)$ vanishes, $(d(v \lrcorner (dx^\alpha - x'^\alpha d\sigma - \dot{x}^\alpha d\tau)) + (v \lrcorner (-dx'^\alpha \wedge d\sigma - d\dot{x}^\alpha \wedge d\tau)))|_{f(N)} = 0$.

The form Ψ_Z is given by

$$\Psi_Z = (L_{\dot{x}^\alpha} - \dot{\lambda}_\alpha)\pi^*(d\dot{x}^\alpha \wedge \omega) + (L_{x'^\alpha} - \lambda'_\alpha)\pi^*(dx'^\alpha \wedge \omega) +$$
$$(d\dot{\lambda}_\alpha \wedge \pi^* d\sigma - d\lambda'_\alpha \wedge \pi^* d\tau) \wedge \pi^* dx^\alpha + (-\dot{x}^\alpha d\dot{\lambda}_\alpha - x'^\alpha d\lambda'_\alpha) \wedge \pi^*\omega \tag{8.3}$$

The Cartan system in Z is:

(i)
$$\partial/\partial\dot{\lambda}_\alpha \lrcorner \Psi_Z = -\pi^*((dx^\alpha - \dot{x}^\alpha d\tau) \wedge \pi * d\sigma) = 0, \tag{8.4}$$

(ii)
$$\partial/\partial\lambda'_\alpha \lrcorner \Psi_Z = -\pi^*((dx^\alpha - x'^\alpha d\tau) \wedge \pi * d\sigma) = 0, \tag{8.5}$$

(iii)
$$\partial/\partial\dot{x}^\alpha \lrcorner \Psi_Z = -\pi^*(L_{\dot{x}^\alpha} - \dot{\lambda}_\alpha)\omega = 0, \tag{8.6}$$

(iv)
$$\partial/\partial x'^\alpha \lrcorner \Psi_Z = -\pi^*(L_{x'^\alpha} - \lambda'_\alpha)\omega = 0, \tag{8.7}$$

(v)
$$\partial/\partial x^\alpha \lrcorner \Psi_Z = -\pi^* d\dot{\lambda}_\alpha \wedge \pi^* d\sigma - d\lambda'_\alpha \wedge \pi^* d\tau = 0. \tag{8.8}$$

Hence

$$Z_1 = Z|L_{\dot{x}^\alpha - \dot{\lambda}_\alpha}, L_{x'^\alpha - \lambda'_\alpha}. \tag{8.9}$$

Note that from (i) and (ii) we have $\theta^\alpha = 0$;

from (iii), (iv) and (v) we have $E[L]\omega = (\partial L/\partial x^\alpha - D_\sigma \partial L/\partial x'^\alpha - D_\tau \partial L/\partial \dot{x}^\alpha)\omega = 0$ *for* $D_\tau = \partial/\partial\tau + \dot{x}^\alpha \partial/\partial x^\alpha + \ddot{x}^\alpha \partial/\partial \dot{x}^\alpha$ *and* $D_\sigma = \partial/\partial\sigma + x'^\alpha \partial/\partial x^\alpha + x''^\alpha \partial/\partial x'^\alpha$.

The generalized momenta are given by

$$\dot{\lambda}_\alpha = \frac{x'^\alpha (x' \cdot \dot{x}) - (x' \cdot\ x')\dot{x}^\alpha}{[(x' \cdot \dot{x})^2 - (\dot{x} \cdot \dot{x})(x' \cdot x')]^{1/2}}, \tag{8.10}$$

$$\lambda'_\alpha = \frac{\dot{x}^\alpha (x' \cdot \dot{x}) - (\dot{x} \cdot \dot{x}) x'^\alpha}{[(x' \cdot \dot{x})^2 - (\dot{x} \cdot \dot{x})(x' \cdot x')]^{1/2}}. \tag{8.11}$$

Let $R^{2m}|(\dot{x} \cdot \dot{x}) \geq 0, (x' \cdot x') \leq 0 \ \ \overset{F'}{\to} \ \ R^{2m}$ *be given by*

$$F'(\dot{x}^\alpha, x'^\alpha) = (\lambda'_\alpha(\dot{x}^\alpha, x'^\alpha), \dot{\lambda}_\alpha(\dot{x}^\alpha, x'^\alpha)).$$

In this case F' *has an inverse in* $R^{2m}|(\dot{x} \cdot \dot{x}) \geq 0, (x' \cdot x') \leq 0$ *and* F'^{-1} *is given by:*

$$\dot{x}^\alpha = \frac{\lambda'_\alpha (\lambda' \cdot \dot{\lambda}) - (\lambda' \cdot \lambda')\dot{\lambda}_\alpha}{[(\lambda' \cdot \dot{\lambda})^2 - (\dot{\lambda} \cdot \dot{\lambda})(\lambda' \cdot \lambda')]^{1/2}}, \tag{8.12}$$

$$x'^\alpha = \frac{\dot{\lambda}_\alpha (\lambda' \cdot \dot{\lambda}) - (\dot{\lambda} \cdot \dot{\lambda})\lambda'_\alpha}{[(\lambda' \cdot \dot{\lambda})^2 - (\dot{\lambda} \cdot \dot{\lambda})(\lambda' \cdot \lambda')]^{1/2}}. \tag{8.13}$$

The Cartan system in $Z'_1 = Z_1|(\dot{\lambda} \cdot \dot{\lambda}) \geq 0, (\lambda' \cdot \lambda') \leq 0$ *is given by (i),(ii), (iv) and (v) of the Cartan system in* Z. *Let* $Y = Z'_1$. *The prolongation of* $(C(\Psi), \pi^*\omega)$ *ends at* Z'_1. *The dimension of* Y *is* $dimY = 3m+2$. *Every point in* Y *is a zero-dimensional integral element of* $(C(\Psi), \pi^*\omega)$, *and* $r_1 = 2m+1$. *The Cartan system is in involution at* x *if* $detC(v)|_{X_0} \neq 0$, *and*

$$C(v) = \begin{bmatrix} < v, d\tau > I & < v, d\sigma > I \\ m \times m & m \times m \\ A & B \\ m \times m & m \times m \end{bmatrix} \tag{8.14}$$

for every $v \neq 0$ *along* E^1, *with* $[x_0, E^1]$ *being any integral element of* $(C(\Psi), \pi^*\omega)$, *where*

$$A =< v, d\sigma > L_{\dot{x}^\alpha \dot{x}^\beta} - < v, d\tau > L_{x'^\alpha \dot{x}^\beta} \tag{8.15}$$

and

$$B =< v, d\sigma > L_{\dot{x}^\alpha x'^\beta} - < v, d\tau > L_{x'^\alpha x'^\beta}, \text{ with } 0 \leq \beta \leq m-1. \tag{8.16}$$

Let us define the energy momentum current $P = (P^0, ..., P^{m-1})$ *on the surface* $\gamma = \{x^\alpha(\sigma,\tau), \sigma, \tau | 0 \le \alpha \le m-1\}$ *by*

$$P^\alpha = \int \dot{P}^\alpha d\tau + P'^\alpha d\sigma \tag{8.17}$$

where $\dot{P}^\alpha = -L_{\dot{x}^\alpha}, P'^\alpha = -L_{x'^\alpha}$.

Case 1. Open strings. *Let* $N = [0,\pi] \times [t_1, t_2], (t_1, t_2) \in R^2, t_1 < t_2$. *We will impose the following constraints on variations of* $f \in V(I^*, L^*)$:

a)
$$g^*(v \lrcorner \pi^* \omega)_{\partial N} = 0, \tag{8.18}$$

b)
$$g^*(v \lrcorner \pi^*(dx^\alpha - \dot{x}^\alpha d\tau - x'^\alpha d\sigma))_B = 0 \tag{8.19}$$

where $B = [0,\pi] \times t_1 \cup [0,\pi] \times t_2$,

c)
$$\lambda'_\alpha = 0 \text{ on } g(A) \text{ where } A = N \setminus B. \tag{8.20}$$

In this case, G *is any smooth lift of* F *to* Y *with* $G|_{t=0} = g, (\pi \circ g = f)$, *and* v *is a vector field defined along* g *with* $v = G_*(\partial/\partial t)|_{t=0}$. *The constraint c) forces the boundary term in the first variation of* $\phi(f)$ *vanish.*

Case 2. Closed strings. *Let* $N = S_1 \times [t_1, t_2]$, *with* S_1 *being the unit circle. Its coordinate* $\sigma \in [0, 2\pi]$, *and* $(t_1, t_2) \in R^2, t_1 < t_2$. *We will replace the constraints on variations of* $f \in V(I^*, L^*)$ *of the previous case with the following:*

a)
$$g^*(v \lrcorner \pi^* \omega)_{\partial N} = 0, \tag{8.21}$$

b)
$$g^*(v \lrcorner \pi^*(dx^\alpha - \dot{x}^\alpha d\tau - x'^\alpha d\sigma))_B = 0 \tag{8.22}$$

where $B = S_1 \times t_1 \cup [0,\pi] \times t_2$.

The quadratic form A.. *The cone* $X' = X | (\dot{x} \cdot \dot{x}) \ge 0, (x' \cdot x') \le 0$ *is convex.* F' *has an inverse in* X' *with* $F' : X" \overset{F^{-1}}{\rightarrow} R^{2m}$ *where* $X'' = R^{2m} | (\dot{\lambda} \cdot \dot{\lambda}) \ge 0, (\lambda' \lambda') \le 0$. *Hence the matrix*

$$A' = \begin{bmatrix} L_{\dot{x}^\alpha \dot{x}^\beta} & L_{\dot{x}^\alpha x'^\beta} \\ L_{x'^\alpha \dot{x}^\beta} & L_{x'^\alpha x'^\beta} \end{bmatrix} \tag{8.23}$$

has an inverse. Therefore, the eigenvalues of A' *do not vanish on* X'. *Thus, it suffices to know the eigenvalues of* A' *at an interior point of* X' *to determine the number of positive eigenvalues of* A' *in every point of* X'.

Let

$$a = \{\dot{x}^0 = 1, \dot{x}^i = 0, x'^1 = 1, x'^j = 0 \quad with \quad 1 \leq i \leq m-1, j = 0 \\ or \quad 2 \leq j \leq m-1\}.$$

Then

$$L_{\dot{x}^0 x'^1}(a) = -L_{\dot{x}^1 x'^0}(a) = -L_{\dot{x}^i x'^i}(a) = L_{x'^i x'^i}(a) = 1, 2 \leq i \leq m-1, \tag{8.24}$$

and all the other elements of A' are zero. We conclude that the matrix has m-positive eigenvalues and m-negative eigenvalues in X' and the quadratic form A is neither positive nor negative definite.

Example 2. *Let $X_0 = J^1(R^2, R^m), N \subset R^2$, with N being a two-dimensional manifold with boundary. Let also*
$I^* = span\{dx^\alpha - x'^\alpha d\sigma - \dot{x}^\alpha d\tau | 1 \leq \alpha \leq m\}$, x^α are coordinates in R^m and $x'^\alpha = \frac{\partial x^\alpha}{\partial \sigma}, \dot{x}^\alpha = \frac{\partial x^\alpha}{\partial \tau}$. Moreover, let

$$\varphi = L\omega = [\sum_{\alpha=1}^{m} (x'^\alpha)^2 + (\dot{x}^\alpha)^2] d\sigma \wedge d\tau. \tag{8.25}$$

The Cartan system in Z is

(i)
$$\partial/\partial\dot{\lambda}_\alpha \lrcorner \Psi_Z = -\pi^*((dx^\alpha - \dot{x}^\alpha d\tau) \wedge \pi * d\sigma) = 0, \tag{8.26}$$

(ii)
$$\partial/\partial\lambda'_\alpha \lrcorner \Psi_Z = -\pi^*((dx^\alpha - x'^\alpha d\tau) \wedge \pi * d\sigma) = 0, \tag{8.27}$$

(iii)
$$\partial/\partial\dot{x}^\alpha \lrcorner \Psi_Z = -\pi^*(2\dot{x}^\alpha - \dot{\lambda}_\alpha)\omega = 0, \tag{8.28}$$

(iv)
$$\partial/\partial x'^\alpha \lrcorner \Psi_Z = -\pi^*(2x'^\alpha - \lambda'_\alpha)\omega = 0, \tag{8.29}$$

(v)
$$\partial/\partial x^\alpha \lrcorner \Psi_Z = -\pi^* d\dot{\lambda}_\alpha \wedge \pi^* d\sigma - d\lambda'_\alpha \wedge \pi^* d\tau = 0. \tag{8.30}$$

Hence

$$Z_1 = Z | L_{\dot{x}^\alpha} = \dot{\lambda}_\alpha, L_{x'^\alpha} = \lambda'_\alpha. \tag{8.31}$$

The prolongation ends at Z_1 with $(C(\Psi), \pi^*\omega)$ on Z_1 given by (8.26), (8.27) and (8.30). It is easy to prove that $(C(\Psi), \pi^*\omega)$ in Y is in involution and $(I^*, L^*, \varphi, I^*, L^*)$ is a well-posed valued differential system.

Boundary conditions. *The constraints on one-parameter variations F of f in $V(I^*, L^*)$ are:*

a)
$$g^*(v \lrcorner \pi^* \omega)_{\partial N} = 0, \tag{8.32}$$
b)
$$g^*(v \lrcorner \pi^*(dx^\alpha - \dot{x}^\alpha d\tau - x'^\alpha d\sigma))_{\partial N} = 0. \tag{8.33}$$

In this case, too, G is any smooth lift of F to Y with $G|_{t=0} = g, (\pi \circ g = f)$, and v is a vector field defined along g with $v = G|_{t=0*}(\partial/\partial t)$.

The quadratic form A.. *A simple computation yields*

$$L_{\dot{x}^\alpha \dot{x}^\beta} = 2\delta_{\alpha\beta}, L_{\dot{x}^\alpha x'^\beta} = 0, L_{x'^\alpha x'^\beta} = 2\delta_{\alpha\beta}. \tag{8.34}$$

Thus, the quadratic form A is positive definite.

Example 3. *Let $X_0 = J^1(R^2, R^m)$. We associate coordinates σ, τ to R^2, x^i, $1 \leq i \leq m$ to R^m, and $x'^i = \frac{\partial x^i}{\partial \sigma}, \dot{x}^i = \frac{\partial x^i}{\partial \tau}$. Let $X = X_0 | g_1 = 0$, where $g_1(\dot{x}^1, x^2) = \dot{x}^1 - x^2 = 0$. Let $N = B_1$ be a ball with radius 1 centered at $(0,0)$. Then*

$$x^1(t,b) - x^1(a,b) = \int_a^t \frac{\partial x^1}{\partial \tau} d\tau = \int_a^t x^2 d\tau, \tag{8.35}$$

where $a \leq 0$ and $a^2 + b^2 = 1$.

Boundary condition $h_{A'}$.. *We have the following system for $v = F_*(\partial/\partial t)(t,x)|_{t=0}$ where F is a one-parameter variation of f :*

$$\frac{\partial v_{x^1}}{\partial \tau} - v_{\dot{x}^1} = 0, \tag{8.36}$$
$$\frac{\partial v_{x^1}}{\partial \sigma} - v_{x'^1} = 0, \tag{8.37}$$
$$\frac{\partial v_{x^1}}{\partial \tau} - v_{x^2} = 0, \tag{8.38}$$
$$\frac{\partial v_{x^1}}{\partial \sigma} - v_{x'^1} = 0. \tag{8.39}$$

Let $A' = \{(\tau, \sigma) \in R^2 | (\tau)^2 + (\sigma)^2 = 1 \quad \text{and} \quad \tau \leq 0\}$. A' is nowhere characteristic for (8.38) and the values of v_{x^1} at A' and v_{x^2} in N determine uniquely a solution in N for the system of equations. Let $h^1_{A'} : A' \to R$ and $h^j_{\partial N} : \partial N \to R \quad (2 \leq j \leq m)$ be a smooth function. Assume $f \in V(I^*, L^*)$, and let I^*, L^* be as before. Then, f satisfies the boundary condition $[h_{A'}]$ if

$$x^1_{A'} = h^1_{A'} \quad \text{and} \quad x^j_{\partial N} = h^j_{\partial N}. \tag{8.40}$$

In this case,

$$\phi[f] = \int f^*\varphi, \text{ where } f \in V(I^*, L^*, [h^1_{A'}]), \tag{8.41}$$

and

$$\varphi = Lw = [(x'^1)^2 + \sum_j (\dot{x}^j)^2 + \sum_j (x'^j)^2] d\sigma \wedge d\tau. \tag{8.42}$$

Momentum space. *The Cartan system in Z is:*

(i)

$$\partial/\partial\dot{\lambda}_i \lrcorner \Psi_Z = -\pi^*((dx^i - \dot{x}^i d\tau) \wedge \pi * d\sigma) = 0, \tag{8.43}$$

(ii)

$$\partial/\partial\lambda'_i \lrcorner \Psi_Z = -\pi^*((dx^i - x'^i d\tau) \wedge \pi^* d\sigma) = 0, \tag{8.44}$$

(iii)

$$\partial/\partial\dot{x}^j \lrcorner \Psi_Z = -\pi^*(2\dot{x}^j - \dot{\lambda}_j)\omega = 0, \tag{8.45}$$

(iv)

$$\partial/\partial x'^i \lrcorner \Psi_Z = -\pi^*(2x'^i - \lambda'_i)\omega = 0, \tag{8.46}$$

(v)

$$\partial/\partial x^i \lrcorner \Psi_Z = -\pi^* d\dot{\lambda}_i \wedge \pi^* d\sigma - d\lambda'_i \wedge \pi^* d\tau = 0. \tag{8.47}$$

From (8.46) and (8.47) we also have $Y = Z_1 = Z|_{2\dot{x}^j=\dot{\lambda}_j, 2x'^i=\lambda'_i}$. This Cartan system $(C(\Psi), \pi^*\omega)$ is non-degenerate. Let us transfer the boundary condition to $Q_i = Y|_{\pi^* L_i}$, where $L^*_i = span\{dx^i - \dot{x}^i d\tau - x'^i d\sigma, d\sigma, d\tau\}$. Then, $f \in V(I^*, L^*)$ satisfies the boundary condition $h_{A'}$, if for any lift g of f to Y we have:

$$(\omega'_1 \circ g)|_{A'} = h^1_{A'} \quad \text{and} \quad (\omega'_j \circ g)|_{\partial N} = h^j_{\partial N}, \tag{8.48}$$

where $h^1_{A'} : A' \to Q_1$ with $\pi_1 \circ h^1_{A'} = h^1_{A'}$ and the projection $\pi_i : Q_i \to R$ given by $\pi_i(q) = x^i(q)$.

Furthermore, g is a solution to the Euler-Lagrange system satisfying the mixed boundary condition $[h_{A'}]$, if g satisfies (8.43), (8.44) and (8.47), and

$$v \lrcorner (\dot{\lambda}_i \pi^*[dx^i - \dot{x}^i d\tau - x'^i d\sigma] \wedge d\tau +$$
$$\lambda'_i \pi^*[dx^i - \dot{x}^i d\tau - x'^i d\sigma] \wedge d\sigma)_{g(\partial N \setminus A')} \equiv 0 \tag{8.49}$$

for any element, $v = F_*(\partial/\partial t)(t, x)|_{t=0}$ where F is a one parameter variation of $\pi \circ g$ satisfying $v_{x^1}|_{A'=0}$ and $v_{x^2}|_{N=0}$.

Finally, the quadratic form A is positive definite.

9. Inverse problem for calculus of variations

Example 4. *In 1887, Helmholtz solved the following problem:*

It is given $P_i = P_i(x, u^j, u^j_x, u^j_{xx})$. *Is there a Lagrangian* $L(x, u^j, u^j_x)$ *such that* $E_i(L) = \partial L/\partial u^i - D_x \partial L/\partial u^i_x = P_i$, *where* $D_x = \partial/\partial x + u^i_x \partial/\partial u^i + u^i_{xx}\partial/\partial u^i_x$? *He found the following necessary conditions for* P_i :

(i)

$$\partial P_i/\partial u^j_{xx} = \partial P_j/\partial u^i_{xx}, \tag{9.1}$$

(ii)

$$\partial P_i/\partial u^j_x = \partial P_j/\partial u^i_x + 2D_x \partial P_j/\partial u^i_{xx}, \tag{9.2}$$

(iii)

$$\partial P_i/\partial u^j = \partial P_j/\partial u^i - D_x \partial P_j/\partial u^i_x + D_{xx} \partial P_j/\partial u^i_{xx}. \tag{9.3}$$

This problem led to the following studies ([2]*):*

(i) *- the derivation and analysis of Helmholtz conditions as necessary and (locally) sufficient conditions for a differential operator to coincide with the Euler-Lagrange operator for some Lagrangian;*
ii) *- the characterization of the obstructions to the existence of global variational principles for different operators defined on manifolds;*
iii) *- the invariant inverse problem for different operators with symmetry; and*
(iv) *- the variational multiplier problem wherein variational principles are found, not for a given differential operator, but rather for the differential equations determined by that operator.*

That is: find a matrix $B = [B^j_i]$ *such that* $B^j_i P_j = E_i(L)$ *for some* L *with* B *being non-singular.*

Let $E \to M$ *be a fibered manifold.* $J^\infty(E)$ *is the infinite jet of* E.

Let

$$\theta^i = du^i - u^i_x dx \tag{9.4}$$

$$\theta^i_x = du^i_x - u^i_{xx} dx \tag{9.5}$$

and

$$\Omega_P = P_i \theta^i \wedge dx + 1/2[\partial P_i/\partial u^j_x - D_x \partial P_i/\partial u^j_{xx}]\theta^i \wedge \theta^j \\ +1/2[\partial P_i/\partial u^j_{xx} + \partial P_j/\partial u^i_{xx}]\theta^i \wedge \theta^j_x. \tag{9.6}$$

If P *satisfies the Helmholtz conditions, then* $d\Omega_P = 0$. *If the* $H^{n+1}(E)$*−n+1 de Rham cohomology group of* E *is trivial. then* Ω_P *is exact. This fact implies that* P_i *is globally variational. If* $\theta_L = Ldx + \partial L/\partial u^i_x \theta^i$,

then $d\theta_L = \Omega_P$. *In 1913, Volterra showed that if*
$L = \int_N u^i P_i(x, tu^j, tu^j_x, tu^j_{xx})dt$ *where* $N = [0, 1]$, *then*

$$E_i(L) = P_i. \tag{9.7}$$

Thus, we have a global solution for the inverse problem in the case of one independent variable and to equations $P_i = 0$ *of second order.*

Vaingberg [1969] generalized this result; however his Lagrangian is usually of a much higher order than necessary.

In [2] *we can find the following theorem.*

Theorem 9.1. *Let* P_i *be a differential operator of order* $2k$

$$P_i = P_i(x, u^j, u^j_1, ..., u^j_{2k}). \tag{9.8}$$

Then P_i *is the Euler-Lagrange operator of a* $k-th$ *order Lagrangian* $L = L(x, u^j, u^j_1, ..., u^j_k)$ *if and only if the functions* P_i *satisfy the higher order Helmholtz conditions, and the functions*

$$p_i(t) = P_i(x, u^j, u^j_1, ..., u^j_k, tu^j_{k+1}, ..., t^k u^j_{2k}) \tag{9.9}$$

are polynomials in t *of degree less or equal to* k.

Example 5. *Let us now look to another example where we have a function of three independent variables* x, y *and* z, *with a single dependent variable* u. *Let* $T = T(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, , ..., u_{zz})$ *be a second order operator.*

$$E[L] = \partial L/\partial u - D_x \partial L/\partial u_x - D_y \partial L/\partial u_y - D_z \partial L/\partial u_z \tag{9.10}$$

Let v *be a lift to the momentum space of an infinitesimal variation* $F_*(\partial/\partial t)$ *of* $f = \pi \circ g$, *where* g *is a solution of* $(C(\Psi), \pi^*\omega)$. *The Lie-derivative of* $\psi = \pi^* L\omega + (\pi^j \circ i')^*[i^*(\chi)] \wedge \pi^*\omega_j$ *by* v *is*

$$v \lrcorner d\psi + d(v \lrcorner \psi) = E[L](u)v^1\pi^*(dx \wedge dy \wedge dz)$$
$$+d(\partial L/\partial u_x v^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y v^1 \pi^*(dx \wedge dz) + \partial L/\partial u_z v^1 \pi^*(dx \wedge dy)). \tag{9.11}$$

Suppose that for some vector w *with* $\pi_* w \in T_f V(I^*, L^*, \varphi, [h])$ *(i.e.* $w \lrcorner d\theta + d(w \lrcorner \theta)$ *for* $\theta = du - u_x dx - u_y dy - u_z dz$ *and* $w \lrcorner \theta|_{\partial N} = 0$*) we have* $v \lrcorner d\psi + d(v \lrcorner \psi) =$

$$T[u]v^1\pi^*(dx \wedge dy \wedge dz) + d(\partial L/\partial u_x w^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y w^1 \pi^*(dx \wedge dz)$$
$$+\partial L/\partial u_z w^1 \pi^*(dx \wedge dy)). \tag{9.12}$$

Then we have $T[u] = E[L](u)$

If we identify e_1 *with* $dy \wedge dz$, e_2 *with* $dz \wedge dx$ *and* e_3 *with* $dx \wedge dy$ *at each point of the integral manifold of* $(C(\Psi), \pi^*\omega)$, *we can write*

$$d(\partial L/\partial u_x v^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y v^1 \pi^*(dx \wedge dz) \tag{9.13}$$

$$+\partial L/\partial u_z v^1 \pi^*(dx \wedge dy)) = DivV[u]\pi^*(dx \wedge dy \wedge dz), \tag{9.14}$$

where

$$V[u] = \partial L/\partial u_x v^1 e_1 + \partial L/\partial u_y v^1 e_2 + \partial L/\partial u_z v^1 e_3. \tag{9.15}$$

The divergence operator is defined in terms of the total derivatives D_x, D_y and D_z.

We can conclude that $v \lrcorner d\psi + d(v \lrcorner \psi) = (E[L](u)v + DivV[u])\pi^*(dx \wedge dy \wedge dz)$.

We have

$$E[L](u) = 0 \quad \textit{whenever} \quad L[u] = DivW[u]. \tag{9.16}$$

Suppose $T[u] = E[L](u)$. Then the first variation formula is

$$v \lrcorner d\psi + d(v \lrcorner \psi) = (T[u]v^1 + DivW[u])\pi^*(dx \wedge dy \wedge dz). \tag{9.17}$$

By applying the Euler-Lagrange operator (i.e. $E[\alpha[u]\pi^*(dx \wedge dy \wedge dz)] \doteq E[\alpha[u]]\pi^*(dx \wedge dy \wedge dz)]$), we obtain

$$E[v \lrcorner d\psi + d(v \lrcorner \psi)] = E[T[u]v]\pi^*(dx \wedge dy \wedge dz), \ \textit{since } E(DivW)(u) = 0. \tag{9.18}$$

We have

$$E[v \lrcorner d\psi + d(v \lrcorner \psi)] = (v \lrcorner dE[L](u) + d(v \lrcorner dE[L](u)))\pi^*(dx \wedge dy \wedge dz) \tag{9.19}$$

$$= (v \lrcorner dT + d(v \lrcorner dT))\pi^*(dx \wedge dy \wedge dz). \tag{9.20}$$

Therefore

$$E[T[u]v]\pi^*(dx \wedge dy \wedge dz) = (v \lrcorner dT + d(v \lrcorner dT))\pi^*(dx \wedge dy \wedge dz). \tag{9.21}$$

Let

$$\psi' = \pi^* T\omega + (\pi^j o i')^*[i^*(\chi)]\pi^*\omega_j, \tag{9.22}$$

and

$$v \lrcorner d\psi' + d(v \lrcorner \psi') = E[T[u]v]\pi^*(dx \wedge dy \wedge dz). \tag{9.23}$$

If we define

$$H(T)[v]\pi^*(dx \wedge dy \wedge dz) = v \lrcorner d\psi' + d(v \lrcorner \psi') - E[T(u)v]\pi^*(dx \wedge dy \wedge dz), \tag{9.24}$$

then $H(T) = 0$ if $T[u]$ is Euler-Lagrange. Helmholtz equations are:

(i)

$$\partial T/\partial u_x = D_x \partial T/\partial u_{xx} + 1/2 D_y \partial T/\partial u_{xy} + 1/2 D_z \partial T/\partial u_{xz}, \tag{9.25}$$

(ii)

$$\partial T/\partial u_y = D_y \partial T/\partial u_{yy} + 1/2 D_x \partial T/\partial u_{yx} + 1/2 D_z \partial T/\partial u_{yz}, \tag{9.26}$$

(iii)

$$\partial T/\partial u_z = D_z \partial T/\partial u_{zz} + 1/2 D_x \partial T/\partial u_{zx} + 1/2 D_y \partial T/\partial u_{zy}. \tag{9.27}$$

We have a sequences of spaces

$$0 \rightarrow R \rightarrow F[u] \xrightarrow{Grad} V(u) \xrightarrow{Curl} V(u) \xrightarrow{Div} F(u) \xrightarrow{E} F(u) \xrightarrow{H} V(u) \tag{9.28}$$

that is cochain complex, the Euler-Lagrange complex. Since this complex is exact, the inverse problem is globally solved in this second example.

9.1. **Variational Bicomplex.** Let us introduce now a very important tool for a globalization of the inverse problem.

Definition 9.1. *A p form ω on $J^\infty(E)$ is said to be of type (r,s), where $r+s=p$, if at each point x of $J^\infty(E)$*

$$\omega(X_1, X_2, ..., X_p) = 0, \tag{9.29}$$

whenever either

(i) *more than s of the vectors $X_1, X_2, ..., X_p$ are π_M^∞ vertical, or*
(ii) *more than r of the vectors $X_1, X_2, ..., X_p$ annihilate all contact one forms.*

Note: $\Omega^{r,s}$ denotes the space of type (r,s) forms on $J^\infty(E)$.

(i) *$\pi : E \rightarrow M$ be a fiber bundle.*
(ii) *Let us assume that there exists a transformation group G acting on E, and*
(iii) *that there exists a set of differential equations on sections of E.*

$$d = d_H + d_V,$$

$$d_H : \Omega^{r,s}(J^\infty(E)) \rightarrow \Omega^{r+1,s}(J^\infty(E)), \tag{9.30}$$

$$d_V : \Omega^{r,s}(J^\infty(E)) \rightarrow \Omega^{r,s+1}(J^\infty(E)), \tag{9.31}$$

$$d_H^2 = 0, \quad d_H d_V = -d_V d_H, \quad d_V^2 = 0. \tag{9.32}$$

In local coordinates

$$d_H f = [\partial f/\partial x^i + u\alpha_i \partial f/\partial u^\alpha + u_{ij}^\alpha \partial f/\partial u_j^\alpha + ...]dx^i \tag{9.33}$$

$$d_V f = \partial f/\partial u^\alpha \theta^\alpha + \partial f/\partial u_i^\alpha \theta_i^\alpha + ... \tag{9.34}$$

The sequences of spaces

$$\begin{array}{ccccccccc}
 & & & & & & \uparrow d_V & I \quad \uparrow \delta_V & \\
 & 0 & \rightarrow \Omega^{0,3} & ... & & & \rightarrow \Omega^{n,3} & \rightarrow F^3 & \rightarrow 0 \\
 & & \uparrow d_V & d_H \uparrow d_V & ... & d_H \uparrow d_V & d_H \uparrow d_V & I \quad \uparrow \delta_V & \\
 & 0 & \rightarrow \Omega^{0,2} & \rightarrow \Omega^{1,2} & ... & \rightarrow \Omega^{n-1,2} & \rightarrow \Omega^{n,2} & \rightarrow F^2 & \rightarrow 0 \\
 & & \uparrow d_V & d_H \uparrow d_V & ... & d_H \uparrow d_V & d_H \uparrow d_V & I \quad \uparrow \delta_V & \\
 & 0 & \rightarrow \Omega^{0,1} & \rightarrow \Omega^{1,1} & ... & \rightarrow \Omega^{n-1,1} & \rightarrow \Omega^{n,1} & \rightarrow F^1 & \rightarrow 0 \\
 & & \uparrow d_V & d_H \uparrow d_V & ... & d_H \uparrow d_V & d_H \uparrow d_V & & \\
0 & \rightarrow R & \rightarrow \Omega^{0,0} & \rightarrow \Omega^{1,0} & ... & \rightarrow \Omega^{n-1,0} & \rightarrow \Omega^{n,0} & &
\end{array}$$

is the Variational Bicomplex.

Therefore the generalization of (9.28) is:

$$0 \;\to R \;\to \Omega^{0,0} \overset{d_H}{\to} \Omega^{1,0} \overset{d_H}{\to} \Omega^{2,0} \;\ldots\; \overset{d_H}{\to} \Omega^{n-1,0} \overset{d_H}{\to}$$

$$\overset{d_H}{\to} \Omega^{n,0} \overset{E}{\to} F^1 \overset{\delta_V}{\to} F^2 \overset{\delta_V}{\to} F^3.$$

9.2. **Lagrange problem with non-holonomic constraints.** Let us recall from [26] the Lagrange problem with non-holonomic constraints. We showed that a well-posed variational problem with mixed endpoint conditions for $n = 1$ is locally a Lagrange problem with non-holonomic constraints.

Proposition 9.1. *Let us assume that a Lagrange problem with non-holonomic constraints $g^\rho(x, u^j, u^j_x) = 0$, with $rank[\partial g^\rho/\partial u^j_x] = m - l$ with $1 \leq j \leq m$ and $1 \leq \rho \leq m-l, l \geq 0$ is given. If $det[L_{\mu\nu}] \neq 0$ and $Ldet[A_{\mu\nu}] \neq 0$ for all $(\lambda_1, ..., \lambda_{m-l}) \in R^{m-l}$, then $(I^*, L^*, \varphi, I^*, L^*)$ is a well-posed valued differential system, where $I^* = span\ \{\theta^\alpha | 1 \leq \alpha \leq m\}$, and $L^* = span\ \{\theta^\alpha, dx | 1 \leq \alpha \leq m\}$*

$$\theta^\rho = g^\rho_{u^\sigma_x}(du^\sigma - u^\sigma_x dx) + g^\rho_{u^\mu_x}(du^\mu - u^\mu_x dx) \quad 1 \leq \sigma \leq m-l, \tag{9.35}$$

$$\theta^\mu = du^\mu - u^\mu_x dx \quad m - l + 1 \leq \mu, \nu \leq m. \tag{9.36}$$

In this setting we have

$$\theta^\mu = -du^\mu_x \wedge dx, \tag{9.37}$$

$$d\theta^\rho \equiv -A^\rho_{\mu\alpha} du^\mu_x \wedge \theta^\alpha - B^\rho_\alpha dx \wedge \theta^\alpha \ \mathrm{mod}\{\theta^\alpha \wedge \theta^{\alpha'} | 1 \leq \alpha, \alpha' \leq m\}, \tag{9.38}$$

$$A^\rho_{\mu\rho'} = g^\rho_{u^\sigma_x u^{\sigma'}_x} a^\sigma_{\rho'} a^{\sigma'}_{\rho''} g^{\rho''}_{u^\mu_x} + g^\rho_{u^\sigma_x u^\mu_x} a^\sigma_{\rho'}, \tag{9.39}$$

$$A^\rho_{\mu\nu} = g^\rho_{u^\sigma_x u^{\sigma'}_x} a^\sigma_{\rho'} g^{\rho'}_{u^\nu_x} a^{\sigma'}_{\rho''} g^{\rho''}_{u^\mu_x} - g^\rho_{u^\sigma_x u^\mu_x} a^\sigma_{\rho'} g^{\rho'}_{u^\nu_x} - g^\rho_{u^\nu_x u^{\sigma'}_x} a^{\sigma'}_{\rho'} g^{\rho'}_{u^\mu_x} + g^\rho_{u^\nu_x u^\mu_x}, \tag{9.40}$$

$$\begin{aligned} B^\rho_\sigma = g^\rho_{u^{\sigma'}_x u^{\sigma''}_x} a^{\sigma'}_\sigma a^{\sigma''}_{\rho''} (g^{\rho''}_x - g^{\rho''}_{u^\alpha_x} u^\alpha_x) + g^\rho_{u^\sigma_x u^\alpha_x} a^{\sigma'}_\sigma u^\alpha_x \\ - g^\rho_{u^{\sigma'}_x x} a^{\sigma'}_\sigma + g^\rho_{u^{\sigma'}_x} a^{\sigma'}_\sigma, \end{aligned} \tag{9.41}$$

$$B^\rho_\mu = -g^\rho_{u^\sigma_x u^{\sigma'}_x} a^\sigma_{\rho'} g^{\rho'}_{u^\mu_x} a^{\sigma'}_{\rho''} (g^{\rho''}_x - g^{\rho''}_{u^\alpha_x} u^\alpha_x) - g^\sigma_{u^{\sigma'}_x u^\alpha_x} a^{\sigma'}_\sigma g^\rho_{u^\mu_x} u^\alpha_x + g^\rho_{u^{\sigma'}_x x} a^{\sigma'}_\sigma g^\sigma_{u^\mu_x} + g^\rho_{u^\mu}$$

$$-g^{\rho}_{u^{\sigma'}} a^{\sigma'}_{\sigma} g^{\sigma}_{u^{\mu}_x} + g^{\rho}_{u^{\mu}_x u^{\sigma}_x} a^{\sigma}_{\rho'} (g^{\rho'}_x - g^{\rho'}_{u^{\alpha}_x} u^{\alpha}_x) + g^{\rho}_{u^{\mu}_x u^{\sigma}_x} a^{\sigma}_{\rho'} u^{\rho'}_x + g^{\rho}_{u^{\mu}_x u^{\nu}_x} u^{\nu}_x. \tag{9.42}$$

$$L_{\mu} = (\partial/\partial u^{\mu}_x - a^{\sigma}_{\rho} g^{\rho}_{u^{\mu}_x} \partial/\partial u^{\sigma}_x) L, \tag{9.43}$$

$$L_{\mu\nu} = (\partial/\partial u^{\mu}_x - a^{\sigma}_{\rho} g^{\rho}_{u^{\mu}_x} \partial/\partial u^{\sigma}_x) L_{\mu}, \tag{9.44}$$

and

$$A_{\mu\nu}(\lambda_1, ..., \lambda_{m-l})$$
$$= L_{\mu\nu} + \lambda_{\rho} (g^{\rho}_{u^{\sigma}_x u^{\sigma'}_x} a^{\sigma}_{\rho'} g^{\rho'}_{u^{\nu}_x} a^{\sigma'}_{\rho''} g^{\rho''}_{u^{\mu}_x} - g^{\rho}_{u^{\sigma}_x u^{\mu}_x} a^{\sigma}_{\rho'} g^{\rho'}_{u^{\nu}_x} - g^{\rho}_{u^{\nu}_x u^{\sigma}_x} a^{\sigma}_{\rho'} g^{\rho'}_{u^{\mu}_x} + g^{\rho}_{u^{\nu}_x} u^{\mu}_x), \tag{9.45}$$

$$[a^{\sigma}_{\rho}] = [g^{\sigma}_{u^{\rho}_x}]^{-1} \text{ with } 1 \le \rho, \rho', \rho'', \sigma, \sigma' \le m - l \text{ and } m - l + 1 \le \mu, \nu \le m. \tag{9.46}$$

$$\begin{aligned} \psi \equiv (L_{\mu} - \lambda_{\mu}) \pi^*(du^{\mu}_x \wedge dx) + (d\lambda_{\mu} - (A_{\mu} + \lambda_{\rho} B^{\rho}_{\mu}) \pi^* dx + \lambda_{\rho} A^{\rho}_{\mu\nu} \pi^* du^{\nu}_x) \wedge \pi^* \theta^{\mu} \\ + (d\lambda_{\sigma} - (A_{\sigma} + \lambda_{\rho} B^{\rho}_{\sigma}) \pi^* dx + \lambda_{\rho} A^{\rho}_{\mu\sigma} \pi^* du^{\mu}_x) \wedge \pi^* \theta^{\sigma} \\ \mathrm{mod}\{\pi^*(\theta^{\alpha} \wedge \theta^{\alpha'}) | 1 \le \alpha, \alpha' \le m\}, \end{aligned} \tag{9.47}$$

with

$$A_{\mu} = L_{u^{\mu}} - L_{u^{\sigma'}_x} a^{\sigma'}_{\rho'} g^{\rho'}_{u^{\mu}_x} + L_{u^{\sigma'}_x} a^{\sigma'}_{\rho'} g^{\rho'}_{u^{\sigma}} a^{\sigma}_{\rho''} g^{\rho''}_{u^{\mu}_x} - L_{u^{\rho}} a^{\rho}_{\sigma} g^{\sigma}_{u^{\mu}_x} \tag{9.48}$$

$$A_{\sigma} = L_{u^{\rho}} a^{\rho}_{\sigma} - L_{u^{\sigma'}_x} a^{\sigma'}_{\rho'} g^{\rho'}_{u^{\rho}} a^{\rho}_{\sigma}. \tag{9.49}$$

The Cartan system is

$$\pi^* \theta^{\alpha} \quad (1 \le \alpha \le m), \tag{9.50}$$

$$(L_{\mu} - \lambda_{\mu}) \pi^* dx \quad (m - l + 1 \le \mu \le m), \tag{9.51}$$

$$(d\lambda_{\mu} - (A_{\mu} + \lambda_{\rho} B^{\rho}_{\mu}) \pi^* dx + \lambda_{\rho} A^{\rho}_{\mu\nu} \pi^* du^{\nu}_x) \quad (m - l + 1 \le \mu \le m), \tag{9.52}$$

$$(d\lambda_{\sigma} - (A_{\sigma} + \lambda_{\rho} B^{\rho}_{\sigma}) \pi^* dx + \lambda_{\rho} A^{\rho}_{\mu\sigma} \pi^* du^{\mu}_x) \quad (1 \le \sigma \le m - l). \tag{9.53}$$

Proposition 9.2. *Let* (I^*, L^*) *be a locally embeddable differential system defined in* $X = J^1(R, R^m)|g^\rho(x, u^j, u^j_x) = 0$, $rank[\partial g^\rho/\partial u^j_x] = m - l, 1 \leq j \leq m$ *and* $1 \leq \rho \leq m - l, l \geq 0$, *where* $I^* = span\ \{\theta^\alpha | 1 \leq \alpha \leq m\}$ *and* $L^* = span\ \{\theta^\alpha, dx | 1 \leq \alpha \leq m\}$,

$$\theta^\rho = g^\rho_{u^\sigma_x}(du^\sigma - u^\sigma_x dx) + g^\rho_{u^\mu_x}(du^\mu - u^\mu_x dx) \quad 1 \leq \sigma, \rho \leq m - l, \tag{9.54}$$

$$\theta^\mu = du^\mu - u^\mu_x dx \quad m - l + 1 \leq \mu, \nu \leq m. \tag{9.55}$$

Let $Q_i = Q_i(x, u^j, u^j_x, u^\mu_{xx}, \lambda_\rho \lambda_{\rho_x}), 1 \leq i \leq m$, *with* $Q_i(x, u^j, u^j_x, t u^\mu_{xx}, \lambda_\rho \lambda_{\rho_x})$ *being polynomial in* t *of degree less or equal to* 1, *and*

$$P_\mu = Q_\mu + \lambda_\rho B^\rho_\mu - \lambda_\rho A^\rho_{\mu\nu} \frac{du^\nu_x}{dx}, \tag{9.56}$$

$$R_\sigma = Q_\sigma - \lambda_{\sigma x} + \lambda_\rho B^\rho_\sigma - \lambda_\rho A^\rho_{\mu\sigma} \frac{du^\mu_x}{dx}, \tag{9.57}$$

and

$$R_\mu = P_\mu + D_x(\partial P_\mu/\partial u^\mu_{xx}). \tag{9.58}$$

Furthermore, let us assume that the functions P_μ *satisfy the Helmholtz conditions, that the functions* R_α *do not depend on* λ_ρ *and* $(\lambda_\rho)_x$ *coordinates, and the 1-form* $\Theta = R_\alpha(x, u^j, u^\mu_x, u^\mu_{xx})\theta^\alpha$ *is closed mod* R, *where* $R = C^\infty(Z, R^*), Z = J^2(R, R^m)|g^\rho(x, u^j, u^j_x) = 0$ *with coordinates* $\{x, u^j, u^\mu_x, u^\mu_{xx}\}$ *and* $R^* = span\ \{dx, du^\mu_x, du^\mu_{xx}\}$. *Then,* Q_i *is locally a Euler-Lagrange operator for a Lagrangian* $L(x, u^j, u^\mu_x)$.

Proof: From Theorem 9.1 we know that a function $F(x, u^j, u^j_x)$ can be found that does not depend on u^ν_{xx}, such that $E_\mu(F) = \partial F/\partial u^\mu - D_x \partial F/\partial u^\mu_x = P_\mu$ (note that if R_μ does not depend on λ_ρ, then neither does P_μ).

Therefore,

$$\partial P_\mu/\partial u^\nu_{xx} = F_{\mu\nu}, \tag{9.59}$$

where

$$F_{\mu\nu} = (\partial/\partial u^\mu_x - a^\sigma_\rho g^\rho_{u^\mu_x} \partial/\partial u^\sigma_x) F_\nu, \tag{9.60}$$

and

$$F_\mu = (\partial/\partial u^\mu_x - a^\sigma_\rho g^\rho_{u^\mu_x} \partial/\partial u^\sigma_x) F. \tag{9.61}$$

The R_μ functions satisfy

$$R_\mu = (\partial/\partial u^\mu - a^\sigma_\rho g^\rho_{u^\mu} \partial/\partial u^\sigma_x) F. \tag{9.62}$$

Hence, if the Θ-form is closed mod R, then locally

$$R_\sigma = (\partial/\partial u^\sigma - a^{\sigma'}_\rho g^\rho_{u^\sigma} \partial/\partial u^{\sigma'}_x) F. \tag{9.63}$$

Finally, we make $F = L$.

In addition, if the domain of the R_α functions is simply connected and

$$\Omega_P = P_\mu \theta^\mu \wedge dx + 1/2[\partial P_\mu/\partial u^j_x - D_x \partial P_\mu/\partial u^j_{xx}]\theta^\mu \wedge \theta^j$$
$$+1/2[\partial P_\mu/\partial u^j_{xx} + \partial P_j/\partial u^\mu_{xx}]\theta^\mu \wedge \theta^j_x. \tag{9.64}$$

is exact, then we have a global solution for the inverse problem.

Example 6. *Let X be the $J^1(R,R^3)|g(v,y,z,v_x,y_x,z_x)=0$, where*

$$g(v,y,z,v_x,y_x,z_x) = mvv_x - mgz_x + R\sqrt{1+(y_x)^2+(z_x)^2}. \tag{9.65}$$

Let

$$Q_1 = -\lambda_{\rho_x} - \frac{\sqrt{1+(y_x)^2+(z_x)^2}}{mv^3} = 0, \tag{9.66}$$

and

$$Q_2 = -\frac{Ry_x}{mv^3} - \frac{v(1+z_x^2)y_{xx} - y_x z_x z_{xx} - v_x y_x\sqrt{1+(y_x)^2+(z_x)^2}}{v^2(\sqrt{1+(y_x)^2+(z_x)^2})^3}$$
$$-\lambda_1\left(\frac{R(1+z_x^2)y_{xx}}{(\sqrt{1+(y_x)^2+(z_x)^2})^3} + \frac{Rz_x y_x z_{xx}}{(\sqrt{1+(y_x)^2+(z_x)^2})^3}\right) = 0, \tag{9.67}$$

$$Q_3 = -\frac{\sqrt{1+(y_x)^2+(z_x)^2}}{mv^3}\left(mg - \frac{Rz_x}{\sqrt{1+(y_x)^2+(z_x)^2}}\right)$$
$$-\frac{v(1+y_x^2)z_{xx} - y_x z_x y_{xx} - v_x z_x\sqrt{1+(y_x)^2+(z_x)^2}}{v^2(\sqrt{1+(y_x)^2+(z_x)^2})^3}$$
$$-\lambda_1\frac{R(1+y_x^2)z_{xx}}{(\sqrt{1+(y_x)^2+(z_x)^2})^3} + \frac{Rz_x y_x y_{xx}}{(\sqrt{1+(y_x)^2+(z_x)^2})^3} = 0. \tag{9.68}$$

Hence,

$$P_2 = -\frac{Ry_x}{mv^3} - \frac{v(1+z_x^2)y_{xx} - y_x z_x z_{xx} - v_x y_x\sqrt{1+(y_x)^2+(z_x)^2}}{v^2(\sqrt{1+(y_x)^2+(z_x)^2})^3}, \tag{9.69}$$

$$P_3 = -\frac{\sqrt{1+(y_x)^2+(z_x)^2}}{mv^3}\left(mg - \frac{Rz_x}{\sqrt{1+(y_x)^2+(z_x)^2}}\right)$$
$$+\frac{v(1+y_x^2)z_{xx} - y_x z_x y_{xx} - v_x z_x\sqrt{1+(y_x)^2+(z_x)^2}}{v^2(\sqrt{1+(y_x)^2+(z_x)^2})^3}, \tag{9.70}$$

and

$$R_1 = -\frac{\sqrt{1+(y_x)^2+(z_x)^2}}{mv^3}, \tag{9.71}$$

$$R_2 = -\frac{Ry_x}{mv^3\sqrt{1+(y_x)^2+(z_x)^2}}, \tag{9.72}$$

$$R_3 = -\frac{\sqrt{1+(y_x)^2+(z_x)^2}}{mv^3}(mg - \frac{Rz_x}{\sqrt{1+(y_x)^2+(z_x)^2}}). \tag{9.73}$$

It is easy to verify that P_2 *and* P_3 *satisfy Helmholtz conditions, and that the* 1*-form* $\Theta = R_1\theta^1 + R_2\theta^2 + R_3\theta^3$ *is closed mod* R*, with* $R^* =$ *span* $\{dx, dy_x, dz_x\}$ *and* $R = C^\infty(X, R^*)$*. The Lagrangian for this example is* $L = \frac{\sqrt{1+(y_x)^2+(z_x)^2}}{v}$.

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