

Liapunov Stability and the ring of P -adic integers

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Abstract. It is well-known that stable Cantor sets are topologically conjugate to adding machines. In this work we show are also conjugate to an algebraic object, the ring of P -adic integers with respect to group translation. This ring is closely related to the field of p -adic numbers; connections and distinctions are explored. The inverse limit construction provides a purely dynamical proof of an algebraic result: the classification of adding machines, or P -adic integers, up to group isomorphism.

1. Introduction

Although well-known from other contexts, namely harmonic analysis on compact groups [6], adding machines occur naturally in dynamical systems. In symbolic dynamics, they arise as factors of subshifts of finite type. In ‘real-world’ dynamical systems, it is well-known that the dynamics of the logistic map at the limit point of period-doubling (known as *Feigenbaum map*) is topologically conjugate to a 2-adic adding machine on the corresponding invariant Cantor set. In fact the universality of the logistic map within the family of \mathcal{S} -unimodal maps ensures that this dynamical adding machine is, in this sense, typical. Adding machines also appear naturally in dynamics via sections of (continuous- or discrete-time) solenoidal attractors [1].

More recent and possibly less well-known is the connection between dynamical (Liapunov) stability and adding machines in discrete maps [3, 4]. Under appropriate but extremely general assumptions about the phase space X (which includes the case of manifolds of any dimension, or even infinite-dimensional) invariant transitive sets, quotiented out by connected

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components, are either finite and cyclically permuted or Cantor sets on which the induced map is topologically transitive (see 3.1). If in addition the original invariant set was Liapunov stable, then this map is an adding machine. So one sees that the occurrence of adding machines in the Feigenbaum map or in solenoidal attractors is not at all accidental; in either case the corresponding adding machine is forced by stability.

An interesting question of a somewhat algebraic flavor naturally arises. It is a kind of ‘folklore’ result that the 2 – *adic* adding machine is topologically conjugate to the operation of ‘addition with carry’ on the group of 2–adic integers; see e.g. [7]. One can generalize this construction to p -adic adding machines and p -adic (for prime p) numbers in the obvious way. The group of 2-adic, or p -adic, integers extends without difficulty to the field of p -adic numbers, and no problems arise. However, adding machines arising from dynamics do not in general have the same base p at each level. What should then be the general algebraic object associated to a dynamical adding machine?

This paper is devoted to this construction, and is to be considered an announcement of results. It is expository by its very nature; formal proofs are omitted and shall be given elsewhere. The results we describe were obtained in joint work with H. Lopes.

2. p -adic numbers and the ring of P -adic integers

In this section we shall very quickly review the properties of p -adic numbers and see how some of their properties may be used to construct a generalization which we call P -adic integers (where P will be a sequence of primes). Excellent references for this material are Gouvêa [5] and Koblitz [8]. For divisibility reasons, in what follows p must be a prime number; the construction would collapse otherwise.

Given an integer $x \in \mathbb{Z}$ and a prime number p , there exists a unique expansion

$$x = \sum_{i \geq 0} a_i p^i, \quad 0 \leq a_i \leq p - 1. \quad (2.1)$$

This expansion may be constructed by the usual congruence method, and addition is defined by the operation of “carry to the right”. If we allow the index i to run through the negative integers we have the corresponding expansions for $x \in \mathbb{Q}$. That these (p -adic) expansions are well-defined for $x \in \mathbb{Q}$ is a consequence of the following facts.

Definition 2.1. Let \mathbb{K} be an ordered group. A valuation [10] in \mathbb{K} is a function $v : \mathbb{K} \rightarrow \overline{\mathbb{R}}$ such that

$$(1) \quad v(0) = \infty;$$

- (2) $v(ab) = v(a) + v(b)$;
 (3) $v(a + b) \geq \min\{v(a), v(b)\}$.

The p -adic valuation on \mathbb{Z} is defined as follows. Given a nonzero integer $n \in \mathbb{Z}$, $n \neq 0$, $v_p(n)$ is the only positive integer satisfying

$$n = p^{v_p(n)} n' \text{ with } p \nmid n'.$$

If $n = 0$, we define $v_p(0) = \infty$.

The p -adic valuation extends uniquely to \mathbb{Q} . Moreover, it induces an absolute value on \mathbb{Q} via

$$|x|_p = p^{-v_p(x)}$$

which in turn induces in the natural way a metric on \mathbb{Q} on which the expansion (2.1) is convergent for every $x \in \mathbb{Q}$. The absolute value and metric thus defined on \mathbb{Q} are, however, non-archimedean.

As happens with the usual metric, the field \mathbb{Q} is not complete with respect to the p -adic metric. The standard procedure of completion leads to a field which is not algebraically closed; algebraic closure of this bigger field leads to another field which now fails to be complete. However, a new topological completion preserves the algebraic closure, and the field thus constructed, sometimes referred to as \mathbb{C}_p [8], is what is referred as the field of p -adic numbers, where the p -adic analog of complex analysis takes place.

We mention these facts to stress the parallels but also the differences with adding machines. It is more or less clear that, in the case of the 2-*adic* adding machine, an isomorphism may be constructed with the 2-adic integers, the dynamics being given by “adding one with carry” (that is, 2-adic addition). However, it is also clear from § 1 that p -adic numbers cannot describe all adding machines generated from dynamics: in the expansion in (2.1) the base p is by construction the same at each level, whereas in dynamical adding machines this is not true in general. In fact, the adding machine of the Feigenbaum map is isomorphic to the 2-*adic* numbers precisely because it corresponds to the limit of period doubling.

On the other hand, there are definite limitations on possible constructions, the most striking of which is Ostrowski’s theorem [5, 8], which states that any non-trivial absolute value on \mathbb{Q} is either equivalent to the standard absolute value or to some p -adic absolute value. So whatever the adequate algebraic construction, Ostrowski’s theorem forbids some norm structure to extend to \mathbb{Q} .

The crux of the matter is to allow the analog expansions of (2.1) to have a *variable* base. To this end, we fix a sequence of primes $P = (p_1, p_2, \dots)$, and define $P_k = \prod_{i=1}^k p_i$. An integer $x \in \mathbb{Z}$ may be uniquely expanded in

base P as

$$x = a_0 + a_1P_1 + a_2P_2 + \dots + a_kP_k + \dots = a_0 + \sum_{i=1}^{\infty} a_iP_i, \quad (2.2)$$

with $a_i \in \mathbb{Z}$, $0 \leq a_i < p_{i+1}$, $i = 0, 1, \dots$. Again no difficulties arise since this expansion only entails congruence relationships. We define the group operation, as previously, as “addition with carry to the right”.

It is easy to show that the relevant algebraic structure of the integers with this operation is that of a ring, and the corresponding additive group is profinite, since it is the inverse limit group the inverse limit system corresponding to the sequence of (compatible) finite groups $\{\mathbb{Z}_{P_i}\}$. It then follows from the general theory [9] that this ring of P -adic integers is compact, Hausdorff and totally disconnected in the induced topology.

In general, no valuation function exists in the ring of P -adic integers since this would contradict Ostrowski's theorem. However, one may prove that the following weaker structure does exist.

Definition 2.2. Consider the function $q_P : \mathbb{Z} - \{0\} \rightarrow \mathbb{R}$ defined as follows: for each $n \in \mathbb{Z} - \{0\}$, $q_P(n)$ is the only positive integer such that

$$n = P_{q_P(n)}n' \text{ where } P_{q_P(n)} \nmid n'.$$

The function $q_P(\cdot)$ is called a quasivaluation. It is not hard to show it satisfies the following properties:

Lemma 2.3. For all $x, y \in \mathbb{Z}$,

- (1) $q_P(x) = +\infty$ if and only if $x = 0$;
- (2) $q_P(x) = q_P(-x)$;
- (3) $q_P(x + y) \geq \min\{q_P(x), q_P(y)\}$.

Observe that properties (i) and (iii) are identical to the corresponding properties of a valuation; however, property (ii) of a quasivaluation is strictly weaker than the corresponding one for a valuation. In particular, a valuation is a quasivaluation.

It now follows that the P -adic analog of the p -adic distance may be naturally defined, even though no absolute value exists:

Definition 2.4. Let $x, y \in \mathbb{Z}$. Define the P -adic distance, $d_P(x, y)$ as

$$d_P(x, y) = P_{q_P(x-y)}^{-1}. \quad (2.3)$$

It is understood that if $q_P(x - y) = 0$ then $P_0 = 1$.

It is not hard to show that indeed d_P is a metric in the ring \mathcal{Z}_P of P -adic integers compatible with its topology. It follows that \mathcal{Z}_P is perfect in this

metric. Thus, \mathcal{Z}_P is a compact, totally disconnected, perfect metric space, and is therefore a Cantor set.

Two remarks are in order. In the first place, the construction of p -adic numbers may be considered as a special case of the construction just described. In the case of p -adic numbers, $P = (p, p, p, \dots)$ and the P -adic metric defined in 2.3 is precisely the old p -adic metric. In the second place, observe that for p -adic integers it is the existence of a valuation that allows the extension of the metric to the field \mathbb{Q} , and thus the construction of p -adic numbers. For the ring of P -adic integers no such construction is possible: the weaker properties of the quasivaluation do not allow for an extension of the P -adic metric to \mathbb{Q} .

Fortunately, from the point of view of dynamics this fact is irrelevant: we will show below that general stable adding machines are topologically conjugate to a ring \mathcal{Z}_P with respect to group addition (in fact only the additive group structure is required).

3. Liapunov stability, adding machines and P -adic integers

In this section we shall describe our main results. We must first recall some previous results and definitions.

Let X be a locally compact, locally connected metric space and $f : X \rightarrow X$ a continuous map. Let A be a compact forward-invariant set. The connected components of A induce an equivalence relation such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \pi \downarrow & & \downarrow \pi \\ K & \xrightarrow{\tilde{f}} & K \end{array}$$

commutes (π is the projection on K and \tilde{f} a the induced map). We then have (see [3]):

Theorem 3.1. *In the above conditions, either*

- (1) K is finite and \tilde{f} is a cyclic permutation;
- (2) K is a Cantor set and \tilde{f} is topologically transitive on K .

If A is further required to be (Liapunov) stable, then the map \tilde{f} is much more restricted: it must be an adding machine [3, 4]. This theorem is proved constructively; we shall not formulate it explicitly since it will be a special case of our main result, Theorem 3.2. Let us for the moment define symbolic adding machines formally.

Let $P = (p_1, p_2, \dots)$ be a sequence of primes. Denote by

$$\Sigma_P = \prod_{n=1}^{\infty} \{0, 1, \dots, p_n - 1\}$$

the space of one-sided sequences $\underline{i} = (i_n)_{n \geq 1}$ such that $0 \leq i_n < p_n$ endowed with the product topology. Σ_P is homeomorphic to the Cantor set. We define the *base P adding machine* to be the map $\alpha_P : \Sigma_P \rightarrow \Sigma_P$ given by

$$\alpha_P(i_1, i_2, \dots) = \begin{cases} (0, \dots, 0, \underbrace{i_l + 1}_{l-1}, i_{l+1}, \dots), & \text{if } i_l < p_l - 1 \text{ and } i_j = p_j - 1 \text{ for } j < l \\ (0, \dots, 0, \dots) & \text{if } i_j = p_j - 1 \text{ for all } j. \end{cases}$$

We should note that, at this point, restriction of P to be a sequence of primes is really unnecessary. However, it will be useful in the sequel to consider adding machines in which the sequence base P is composed by primes; we shall refer to it as a *prime adding machine*. The adding machine map is easily shown to be topologically transitive (indeed minimal – every orbit is dense in Σ_P).

We can then prove constructively the following result:

Theorem 3.2. *Let X be a locally compact, locally connected metric space, $f : X \rightarrow X$ be a continuous map and A be a compact, invariant transitive set. Suppose A is Liapunov stable and has infinitely many connected components. Then $\tilde{f} : K \rightarrow K$ is topologically conjugate to a prime adding machine $\alpha_P : \Sigma_P \rightarrow \Sigma_P$.*

The proof of theorem 3.2 is constructive and hinges critically on the condition of Liapunov stability. The stability condition allows us to construct a cover \mathcal{C} of K by clopen sets whose inverse images are forward-invariant and such that the diagram

$$\begin{array}{ccccccc} \mathcal{C}_1 & \xleftarrow{\delta_1} & \mathcal{C}_2 & \xleftarrow{\dots} & \mathcal{C}_{n-1} & \xleftarrow{\delta_{n-1}} & \mathcal{C}_n & \xleftarrow{\delta_n} & \mathcal{C}_{n+1} & \xleftarrow{\dots} \\ \psi_1 \downarrow & & \psi_2 \downarrow & & \psi_{n-1} \downarrow & & \psi_n \downarrow & & \psi_{n+1} \downarrow & \\ \mathcal{S}_1 & \xleftarrow{\theta_1} & \mathcal{S}_2 & \xleftarrow{\dots} & \mathcal{S}_{n-1} & \xleftarrow{\theta_{n-1}} & \mathcal{S}_n & \xleftarrow{\theta_n} & \mathcal{S}_{n+1} & \xleftarrow{\dots} \end{array}$$

is commutative, and so are the inverse limit maps. The connection with P -adic numbers is that base P adding machines and P -adic integers are

also related by a commutative diagram:

$$\begin{array}{ccccccc}
 \mathcal{S}_1 & \xleftarrow{\theta_1} & \mathcal{S}_2 & \xleftarrow{\quad} & \cdots & \xleftarrow{\quad} & \mathcal{S}_{n-1} & \xleftarrow{\theta_{n-1}} & \mathcal{S}_n & \xleftarrow{\theta_n} & \mathcal{S}_{n+1} & \xleftarrow{\quad} & \cdots \\
 \phi_1 \downarrow & & \phi_2 \downarrow & & & & \phi_{n-1} \downarrow & & \phi_n \downarrow & & \phi_{n+1} \downarrow & & \\
 \mathbb{Z}_{P_1} & \xleftarrow{\gamma_1} & \mathbb{Z}_{P_2} & \xleftarrow{\quad} & \cdots & \xleftarrow{\quad} & \mathbb{Z}_{P_{n-1}} & \xleftarrow{\gamma_{n-1}} & \mathbb{Z}_{P_n} & \xleftarrow{\gamma_n} & \mathbb{Z}_{P_{n+1}} & \xleftarrow{\quad} & \cdots
 \end{array}$$

Finally, since all connecting maps are homeomorphisms, it follows that the diagram for the inverse limit maps

$$\begin{array}{ccc}
 \mathcal{C}_\infty & \xrightarrow{\tilde{f}_\infty} & \mathcal{C}_\infty \\
 \Psi \downarrow & & \downarrow \Psi \\
 \mathcal{S}_\infty & \xrightarrow{\alpha_\infty} & \mathcal{S}_\infty \\
 \Phi \downarrow & & \downarrow \Phi \\
 \mathbb{Z}_P & \xrightarrow{\mathcal{F}} & \mathbb{Z}_P
 \end{array}$$

is commutative. The proof thus shows simultaneously that base P adding machines are topologically conjugate to (the additive group of) P -adic integers and that the dynamics on stable Cantor sets are conjugate to both.

The proof shows in fact a stronger result than stated in 3.2. To formulate it we need a few definitions.

Let \mathcal{P} be the set of primes. Given a prime sequence $P = (p_1, p_2, p_3, \dots)$, define the *multiplicity function of P* m_P as follows:

$$m_P(p) = \begin{cases} n & \text{if } p \text{ appears } n \text{ times in the sequence } P \\ \infty & \text{if } p \text{ appears infinitely often in the sequence } P. \end{cases}$$

The proof of theorem 3.2 is constructive and at each step is based on arbitrary choices. Different choices at each step lead us to different prime adding machines, so the prime adding machine whose existence is asserted in 3.2 far from unique. However, these distinct prime adding machines do have the same multiplicity function! Theorem 3.2 associates to a dynamical system not a unique adding machine but an equivalence class of adding machines: if (K, \tilde{f}) is topologically conjugate to two different prime adding machines, then obviously these are conjugate between themselves. Since they have the same multiplicity function, this means that m_P is a complete invariant for the conjugacy classes of adding machines (the other sense of the implication is trivial). We thus have the following classification theorem for adding machines.

Theorem 3.3. *Let P e P' be prime sequences. Then α_P e $\alpha_{P'}$ are topologically conjugate if and only if they have the same multiplicity function.*

It is interesting to note that the classification theorem appears recurrently in the literature [1, 2, 3] with several different proofs, ranging from the algebraic to the spectral. Note that from 3.2 the classification theorem is deduced, almost as an afterthought, from purely dynamical considerations.

4. Concluding remarks

First of all we should note that the construction of P -adic integers and P -adic adding machines is made with prime sequences for convenience reasons only. Nothing essential (not even in the proofs) would be lost in working with a non-prime sequence. This choice is motivated by two reasons: in the first place, it makes more transparent the parallel with p -adic numbers; in the second place, the classification theorem 3.3 becomes particularly simple to state. But everything would be essentially the same with non-prime sequences (see, e.g., [6]).

One of the most significant parts of the results just presented is the fact the classification theorem, which may be thought as a group isomorphism result (it is easy to see that in this case group isomorphism and topological conjugacy coincide), has been given a purely dynamical proof.

References

- [1] J. Aarts, R. Fokink, The classification of solenoids. *Proc. Am. Math. Soc.* **111** (1991) 4, 1161–1163.
- [2] V. Afraimovich, A. Maass, J. Urías, Symbolic dynamics for sticky sets in Hamiltonian systems. *Nonlinearity* **13** (2000) 1, 617–637.
- [3] J. Buescu, I. Stewart, Liapunov stability and adding machines. *Ergodic Theory & Dynamical Systems* **15** (1995), 271–290.
- [4] J. Buescu, M. Kulczycki, I. Stewart, Liapunov stability and adding machines revisited. *Dynamical Systems An International Journal* **21**, 3 (2006), 379–384.
- [5] F. Gouvêa, *p-adic Numbers: An Introduction*, Springer-Verlag, Universitext, 1991.
- [6] E. Hewitt, K. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, Mathematischen Wissenschaften **115**, 1963.
- [7] A. Katok, B. Hasselblatt, *Modern theory of dynamical systems*, C.U.P. Encyclopedia of Mathematics and its applications **54**, 1995.
- [8] N. Koblitz, *p-adic numbers, p-adic analysis and zeta functions*, Springer-Verlag, GTM 58, 2nd edition, 1984.
- [9] L. Ribes, P. Zalesskii, *Profinite Groups*, Springer-Verlag, Modern Surveys in Mathematics **40**, 2000.
- [10] B. van der Waerden *Algebra, vol II*, Springer-Verlag, 1991.