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Existence of global attractors and gradient property for a class of non local evolution equations

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Abstract. In this work we prove the existence of a compact global attractor for the flow of the equation

$$\frac{\partial m(r,t)}{\partial t} = -m(r,t) + g(\beta J * m(r,t) + \beta h), \;\; h, \, \beta \geq 0,$$

in $L^2(S^1)$. We also give uniform estimates on the size of the attractor and show that the flow is gradient.

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1. Introduction

We consider here the non local evolution equation

$$\frac{\partial m(r,t)}{\partial t} = -m(r,t) + g\left(\beta J * m(r,t) + \beta h\right), \tag{1.1}$$

where $m(r,t)$ is a real function on $\mathbb{R} \times \mathbb{R}_+$, h and β are non negative constants and $J \in C^1(\mathbb{R})$ is a non negative even function supported in the

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interval $[-1, 1]$ and integral equal to 1. The $*$ above denotes convolution product, namely:

$$(J * m)(x) = \int_{\mathbb{R}} J(x-y)m(y)dy.$$

An equilibrium of (1.1) is a solution for (1.1) that is constant with respect to t, that is, m satisfies

$$m(r) = g(\beta J * m(r) + \beta h).$$

There are several works in the literature dedicated to the analysis of the particular case of (1.1) where $g \equiv \tanh$, that is

$$\frac{\partial m(r,t)}{\partial t} = -m(r,t) + \tanh(\beta J * m(r,t) + \beta h). \tag{1.2}$$

In this case, if $\beta \leq 1$, equation (1.2) has only one (stable) equilibrium, (see [9] and [11]). If $\beta > 1$ there is h^*, implicitly defined by equation (1.3) below, such that, for $0 \leq h < h^*$, equation (1.2) has three spatially homogeneous equilibria, m_β^-, m_β^0, m_β^+, each of which is identically equal to one of the roots of the equation

$$m_\beta = \tanh(\beta m_\beta + \beta h). \tag{1.3}$$

In [7], the existence and uniqueness (modulo translation) of a travelling front connecting the equilibria m_β^- and m_β^+ is proved. In [9], the existence of a non-homogeneous stationary solution referred to as the "bump" is proved for h "sufficiently close" to 1. In [11], the existence of a such solution is established for $0 < h < h^*$.

The existence and uniqueness (modulo translation) of an equilibrium for (1.2) which tends asymptotically to $\pm m_\beta^+$, referred to as the *"instanton"* is proved in [8] and [10] for the case $h = 0$.

In [1], the existence of a global attractor for (1.2) is proved for the case of bounded domain and $h = 0$.

We now collect the conditions on g which will be used used as hypotheses along the paper and indicate the points where each one is needed.

(H1) The function $g : \mathbb{R} \to \mathbb{R}$, is globally Lipschitz, that is, there exists a positive constant k_1 such that

$$|g(x) - g(y)| \leq k_1|x - y|, \ \ \forall\, x, y \in \mathbb{R}.$$

In particular, there exist non negative constants k_2 and k_3 such that

$$|g(x)| \leq k_2|x| + k_3, \ \ \forall\, x \in \mathbb{R}. \tag{1.4}$$

(H2) The function $g \in C^1(\mathbb{R})$ and g' is locally Lipschitz.

(H3) There exist non negative constants k_4 and k_5, such that

$$|g'(x)| \leq k_4|x| + k_5, \ \ \forall x \in \mathbb{R}.$$

(Observe that if **(H1)** and **(H2)** hold then **(H3)** also holds with $k_4 = 0$ and $k_5 = k_1$.

(H4) The function g has positive derivative. In particular it is increasing.

(H5) There exists $a > 0$ such that, for all $x \in \mathbb{R}$, $|g(x)| < a$. In particular, when $a < \infty$ (1.4) holds with $k_2 = 0$ and $k_3 = a$.

(H6) The function g^{-1} is continuous in $(-a, a)$ and the function

$$f(m) = -\frac{1}{2}m^2 - hm - \beta^{-1}i(m), \ \ m \in [-a, a],$$

where i is defined by

$$i(m) = -\int_0^m g^{-1}(s)ds, \ \ m \in [-a, a],$$

has a global minimum $\overline{m}$ in $(-a, a)$.

This paper is organized as follows. In Section 2 we prove that, under hypothesis **(H1)**, (1.1) (restricted to $P_{2\tau}$)) generates a flow in $L^2(S^1)$, which is of class C^1 if one also assumes **(H2)**. Section 3 is dedicated to the proof of existence of the global attractor, generalizing some results of [1], where the case $h = 0$ was considered. This is done using hypotheses **(H1)** and **(H3)** (we don't need to assume **(H2)** at this point). In Section 4, we prove a comparison result under the hypotheses **(H1)** and **(H4)**, generalizing Theorem 2.7 of [8]. Assuming also **(H3)** and **(H5)** (with $a < 0$) we prove an uniform estimate for the attractor. Finally, in Section 5 assuming **(H6)**, we exhibit a continuous Lyapunov functional for the flow of (1.1), and as used it to prove that, under hypotheses **(H1)**, **(H3)**, **(H4)**, **(H5)** and **(H6)**, the flow is gradient in the sense of [5]. As consequence, the global attractor coincides with the unstable manifolds of the equilibria.

2. Well posedness in $L^2(S^1)$

The Cauchy problem for equation (1.1) in the space of continuous bounded functions, $C_b(\mathbb{R})$, with the sup norm is well posed, since the function given by the right hand side of (1.1) is uniformly Lipschitz in this space, (see [2] and [3]).

It is an easy consequence of the uniquennes theorem that the subspace $\mathbb{P}_{2\tau}$ of 2τ periodic functions is invariant, We considerer here the equation (1.1) restricted to the $\mathbb{P}_{2\tau}$, $\tau > 1$. As we will see below, this leads naturally to the consideration of a flow in $L^2(S^1)$, where S^1 denotes the unit sphere.

Now, if $\tau > 1$ is a given positive number, we define J^τ as the 2τ periodic extension of the restriction of J to interval $[-\tau, \tau]$. It is then easy to show that, if $u \in \mathbb{P}_{2\tau}$, then

$$(J * u)(x) = \int_{-\tau}^{\tau} J^\tau(x-y)u(y)dy. \tag{2.5}$$

In view of the (2.5), the equation (1.1), restricted to $\mathbb{P}_{2\tau}$, with $\tau > 1$, can be written as

$$\frac{\partial m(x,t)}{\partial t} = -m(x,t) + g\left(\beta \int_{-\tau}^{\tau} J^\tau(x-y)m(y,t)dy + \beta h\right).$$

Define $\varphi : \mathbb{R} \to S^1$ by

$$\varphi(x) = e^{i\frac{\pi}{\tau}x}$$

and, for $u \in \mathbb{P}_{2\tau}$, $v : S^1 \to \mathbb{R}$ by

$$v(\varphi(x)) = u(x).$$

In particular, we write $\widetilde{J}(\varphi(x)) = J^\tau(x)$. Then we have the following result, whose simple proof is omitted.

Proposition 2.1. *The function $u(x,t)$ is a 2τ periodic solution of (1.1) if and only if $v(w,t) = u(\varphi^{-1}(w),t)$ is a solution of*

$$\frac{\partial m(w,t)}{\partial t} = -m(w,t) + g\left(\beta \widetilde{J} * m(w,t) + \beta h\right), \tag{2.6}$$

where, now, $(*)$ denote convolution in S^1, that is

$$(\widetilde{J} * m)(w) = \int_{S^1} \widetilde{J}(w \cdot z^{-1})m(z)dz$$

and $dz = \frac{\tau}{\pi}d\theta$, where $d\theta$ denote integration with respect to arc length.

From now on we will write J instead of $\widetilde{J}$ for simplicity.

Proposition 2.2. *Suppose that the hypothesis (H1) holds. Then the function*

$$F(u) = -u + g(\beta J * u + \beta h)$$

is uniformly Lipschitz in $L^2(S^1)$.

Proof From (H1) and the triangle inequality, we obtain

$$\begin{aligned}
\|F(m) - F(u)\|_{L^2} &= \\
&= \| -(m-u) + g(\beta J * m + \beta h) - g(\beta J * u + \beta h)\|_{L^2} \\
&\le \|m-u\|_{L^2} + k_1 \|\beta(J*m) - \beta(J*u)\|_{L^2} \\
&= \|m-u\|_{L^2} + k_1\beta\|J*(m-u)\|_{L^2}.
\end{aligned}$$

But, from Young's inequality, (see [4]),

$$\begin{aligned}
\|J*(m-u)\|_{L^2} &\le \|J\|_{L^1}\|m-u\|_{L^2} \\
&= \|m-u\|_{L^2}.
\end{aligned}$$

Thus

$$\|F(m) - F(u)\|_{L^2} \le (1 + k_1\beta)\,\|m-u\|_{L^2},$$

which concludes the proof. □

From Proposition 2.2, it follows that the Cauchy problem for (2.6) is well posed in $L^2(S^1)$ with a unique global solution, (see [2] and [3]). More precisely, we have

Corollary 2.3. *Equation (2.6) has a unique solution for any initial condition in $L^2(S^1)$, which is globally defined.*

The following result has been proven in [12].

Proposition 2.4. *Let X and Y be normed linear spaces, $F : X \to Y$ a map and suppose that the Gateaux derivative of F, $DF : X \to \mathcal{L}(X,Y)$ exists and is continuous at $x \in X$. Then the Frechet derivative F' of F exists and is continuous at x.*

Remark 2.5. *If $u \in L^2(S^1)$, then*

$$|(J*u)(w)| \le \sqrt{2\tau}\|J\|_\infty \|u\|_{L^2}, \ \forall\, w \in S^1. \tag{2.7}$$

In fact,

$$\begin{aligned}
|(J*u)(w)| &\le \int_{S^1} |J(wz^{-1})||u(z)|dz \\
&\le \int_{S^1} \|J\|_\infty |u(z)|dz,
\end{aligned}$$

and the estimate follows from Hölder's inequality, (see [2]).

Proposition 2.6. *Assume that the hypotheses (H1) and (H2) hold. Then the function*

$$F(u) = -u + g(\beta J * u + \beta h)$$

is continuously Frechet differentiable in $L^2(S^1)$ with derivative given by

$$F'(u)v = -v + g'(\beta J * u + \beta h)\beta(J * v).$$

Proof By a simple computation, using the hypothesis (H1), it follows that the Gateaux's derivative of F is given by

$$DF(u)v = -v + g'(\beta J * u + \beta h)\beta(J * v).$$

Now, note that for each $u \in L^2(S^1)$, due to linearity of the convolution, $DF(u)$ is a linear operator. Furthermore

$$\|DF(u)v\|_{L^2} \quad \leq \quad \|v\|_{L^2} + \|g'(\beta J * u + \beta h)\beta(J * v)\|_{L^2}.$$

But, from (2.7), we have

$$|\beta(J * v)(w)| \leq \sqrt{2\tau}\beta\|J\|_\infty\|v\|_{L^2}, \ \forall\, w \in S^1$$

and, from (H2)

$$\int_{S^1} |g'(\beta(J * u)(w) + \beta h)|^2 dw = L < \infty.$$

Hence

$$\begin{aligned}
&\|g'(\beta J * u + \beta h)\beta(J * v)\|_{L^2}^2 = \\
&\qquad = \int_{S^1} |g'(\beta(J * u)(w) + \beta h)|^2\beta^2|(J * v)(w)|^2 dw \\
&\qquad \leq \int_{S^1} |g'(\beta(J * u)(w) + \beta h)|^2\beta^2 2\tau\|J\|_\infty^2\|v\|_{L^2}^2 dw \\
&\qquad = \beta^2 2\tau\|J\|_\infty^2\|v\|_{L^2}^2 \int_{S^1} |g'(\beta(J * u)(w) + \beta h)|^2 dw \\
&\qquad = L\beta^2 2\tau\|J\|_\infty^2\|v\|_{L^2}^2.
\end{aligned}$$

Thus

$$\|g'(\beta J * u + \beta h)\beta(J * v)\|_{L^2} \leq \sqrt{L}\beta\sqrt{2\tau}\|J\|_\infty\|v\|_{L^2}.$$

Therefore

$$\|DF(u)v\|_{L^2} \leq (1 + \sqrt{L2\tau}\beta\|J\|_\infty)\|v\|_{L^2}.$$

Furthermore, DF is a continuous operator. In fact

$$\begin{aligned}
&\|DF(u_1)v - DF(u_2)v\|_{L^2} = \\
&\qquad = \|[g'(\beta J * u_1 + \beta h) - g'(\beta J * u_2 + \beta h)]\beta(J * v)\|_{L^2}.
\end{aligned}$$

Keeping $u_1 \in L^2(S^1)$ fixed and letting $u_2 \to u_1$ in $L^2(S^1)$ it follows, from (2.7), that $(\beta J * u_2 + \beta h)$ is in a ball of $L^\infty(S^1)$ centered in $(\beta J * u_1 + \beta h)$. Thus, using hypothesis (H2), there exists a constant $M > 0$ such that

$$|g'(\beta J * u_1 + \beta h)(w) - g'(\beta J * u_2 + \beta h)(w)| \leq M\beta |J * (u_1 - u_2)(w)|.$$

Using this last estimative and (2.7), we obtain

$$\begin{aligned}\|DF(u_1)v - DF(u_2)v\|_{L^2} = \Bigg(\int_{S^1} &|g'(\beta J * u_1 + \beta h)(w) - \\ &- g'(\beta J * u_2 + \beta h)(w)|^2 \beta^2 |(J * v)(w)| dw \Bigg)^{\frac{1}{2}} \\ &\leq \left(\int_{S^1} M^2 \beta^2 |J * (u_1 - u_2)(w)|^2 \beta^2 |(J * v)(w)|^2 dw \right)^{\frac{1}{2}} \\ &\leq M\beta^2 2\tau\sqrt{2\tau}\|J\|_\infty^2 \|u_1 - u_2\| \|v\|_{L^2}.\end{aligned}$$

It follows from Proposition 2.4 that F is Frechet differentiable with continuous derivative in $L^2(S^1)$. □

Remark 2.7. *Since the right-hand side of (2.6) is a C^1 function, the flow generated by (2.6) is C^1 with respect to initial conditions, (see* [6]*).*

3. Existence of a global attractor

We prove, in this section, the existence of a global maximal invariant compact set $\mathcal{A} \subset L^2(S^1)$ for the flow of (2.6), which attracts each bounded set of $L^2(S^1)$ (the global attractor, see [5] and [13]).

We recall that a set $\mathcal{B} \subset L^2(S^1)$ is an absorbing set for the flow $T(t)$ if, for any bounded set $C \subset L^2(S^1)$, there is a $t_1 > 0$ such that $T(t)C \subset \mathcal{B}$ for any $t \geq t_1$.

The following result was proven in [13]

Theorem 3.1. *Let X be a Banach space and $T(t)$ a semigroup on X. Assume that, for every t, $T(t) = T_1(t) + T_2(t)$ where the operators $T_1(\cdot)$ are uniformly compact for t large, that is, for every bounded set B there exists t_0, which may depend on B, such that*

$$\bigcup_{t \geq t_0} T_1(t)B$$

is relatively compact in X and $T_2(t)$ is a continuous mapping from X into itself such that the following holds: For every bounded set $C \subset X$,

$$r_c(t) = \sup_{\varphi \in C} \|T_2(t)\varphi\|_X \to 0 \quad as \quad t \to \infty.$$

Assume also that there exists an open set $\mathcal{U}$ and bounded subset $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}$ is absorbing in $\mathcal{U}$. Then the ω-limit set of $\mathcal{B}$, $\mathcal{A} = \omega(\mathcal{B})$, is a compact attractor which attracts the bounded sets of $\mathcal{U}$. It is the maximal bounded attractor in $\mathcal{U}$ (for the inclusion relation). Furthermore, if $\mathcal{U}$ is convex and connected, then $\mathcal{A}$ is connected.

Lemma 3.2. *Assume that the hypothesis (H1) holds and $k_2\beta < 1$. Then the ball of radius $\frac{2\sqrt{2\tau}(k_2\beta h + k_3)}{1-k_2\beta}$ is an absorbing set for the flow $T(t)$ generated by (2.6).*

Proof If $u(w,t)$ is a solution of (2.6) with initial condition $u(w,0)$ then, by the variation of constants formula

$$u(w,t) = e^{-t}u(w,0) + \int_0^t e^{-(t-s)} g(\beta(J * u(w,s) + h))ds.$$

Thus

$$\frac{d}{dt}\int_{S^1} |u(w,t)|^2 dw = \\ = -2\int_{S^1} u^2(w,t)dw + 2\int_{S^1} u(w,t)g(\beta J * u(w,t) + \beta h)dw.$$

But, by Hölder inequality

$$\int_{S^1} u(w,t)g(\beta J * u(w,t) + \beta h)dw \leq \\ \leq \|u(\cdot,t)\|_{L^2}\left(\int_{S^1}(g(\beta J * u(w,t) + \beta h))^2 dw\right)^{\frac{1}{2}}.$$

Using (1.4) and Young's inequality in the right-hand side of the inequality above, we obtain

$$\begin{aligned}\int_{S^1} u(w,t)g(\beta J * u(w,t) + \beta h)dw &\leq \|u(\cdot,t)\|_{L^2} * \\ &* \left[k_2\beta\left(\int_{S^1}(J * u(w,t))^2 dw\right)^{\frac{1}{2}} + \left(\int_{S^1}(k_2\beta h + k_3)^2 dw\right)^{\frac{1}{2}}\right] \\ &\leq \|u(\cdot,t)\|_{L^2}\left[k_2\beta\|J\|_{L^1}\|u(\cdot,t)\|_{L^2} + \sqrt{2\tau}(k_2\beta h + k_3)\right] \\ &= \|u(\cdot,t)\|_{L^2}\left[k_2\beta\|u(\cdot,t)\|_{L^2} + \sqrt{2\tau}(k_2\beta h + k_3)\right].\end{aligned}$$

Thus

$$\frac{d}{dt}\|u(\cdot,t)\|^2 \leq$$
$$\leq -2\|u(\cdot,t)\|_{L^2}^2 + 2k_2\beta\|u(\cdot,t)\|_{L^2}^2 + 2\sqrt{2\tau}(k_2\beta h + k_3)\|u(\cdot,t)\|_{L^2}$$
$$= 2\|u(\cdot,t)\|_{L^2}^2 \left[-1 + k_2\beta + \frac{\sqrt{2\tau}\,(k_2\beta h + k_3)}{\|u(\cdot,t)\|_{L^2}}\right].$$

Since $k_2\beta < 1$, let $\varepsilon = 1 - k_2\beta > 0$. Then, while $\|u(\cdot,t)\|_{L^2} \geq \frac{2\sqrt{2\tau}(k_2\beta h+k_3)}{\varepsilon}$, we have

$$\begin{aligned}\frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2 &\leq 2\|u(\cdot,t)\|_{L^2}^2(-\varepsilon + \frac{\varepsilon}{2}) \\ &= -\varepsilon\|u(\cdot,t)\|_{L^2}^2.\end{aligned}$$

Therefore

$$\begin{aligned}\|u(\cdot,t)\|_{L^2} &\leq e^{-\varepsilon t}\|u(\cdot,0)\|_{L^2} \\ &= e^{-(1-k_2\beta)t}\|u(\cdot,0)\|_{L^2},\end{aligned}$$

which concludes the proof. □

The next result generalizes Theorem 3.3 of [1].

Theorem 3.3. *Suppose that (H1), (H3) hold and* $k_2\beta < 1$. *Then there exists a global attractor* $\mathcal{A}$ *for the flow* $T(t)$ *generated by (2.6) in* $L^2(S^1)$, *which is contained in the ball of radius* $\frac{2\sqrt{2\tau}(k_2\beta h+k_3)}{1-k_2\beta}$.

Proof If $u(w,t)$ is the solution of (2.6) with initial condition $u(w,0)$ we have, by the variation of constants formula

$$u(w,t) = e^{-t}u(w,0) + \int_0^t e^{s-t}g(\beta(J * u(w,s) + h))ds. \tag{3.8}$$

Write

$$T_1(t)u(w) = e^{-t}u(w,0)$$

and

$$T_2(t)u(w) = \int_0^t e^{s-t}g(\beta(J * u(w,s) + h))ds$$

and suppose $u(\cdot,0) \in C$, where C is a bounded set in $L^2(S^1)$. Then

$$\|T_1(t)u\|_{L^2} \longrightarrow 0, \text{ as } t \longrightarrow \infty, \text{ uniformly in } u.$$

Also, using (3.8), we have that $\|u(\cdot,t)\|_{L^2} \leq K$, for $t \geq 0$, where $K = \max\left\{R, \frac{2\sqrt{2\tau}(k_2\beta h+k_3)}{1-k_2\beta}\right\}$. Therefore, for $t \geq 0$ we have

$$\begin{aligned}\frac{\partial T_2(t)u(w)}{\partial w} &= \int_0^t e^{s-t}\frac{\partial}{\partial w}g(\beta(J*u(w,s)+h))ds\\ &= \beta\int_0^t e^{s-t}g'(\beta(J*u(w,s)+h))(J'*u)(w,s)ds.\end{aligned}$$

Thus

$$\left|\frac{\partial T_2(t)u(w)}{\partial w}\right| \leq \beta\int_0^t e^{s-t}|g'(\beta J*u(w,s)+\beta h)||(J'*u)(w,s)|ds.$$

Using (H3) and (2.7), we obtain

$$\begin{aligned}&|g'(\beta J*u(w,s)+\beta h)||(J'*u)(w,s)| \leq\\ &\quad\leq [k_4|\beta J*u(w,s)+\beta h|+k_5]|J'*u(w,s)|\\ &\quad\leq [k_4|\beta J*u(w,s)|+k_4\beta h+k_5]|J'*u(w,s)|\\ &\quad\leq \left[k_4\beta\sqrt{2\tau}\|J\|_\infty\|u(\cdot,s)\|_{L^2}+k_4\beta h+k_5\right]\sqrt{2\tau}\|J'\|_\infty\|u(\cdot,s)\|_{L^2}\\ &\quad\leq k_4\beta 2\tau\|J\|_\infty\|J'\|_\infty K^2+(k_4\beta h+k_5)\sqrt{2\tau}\|J'\|_\infty K.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\frac{\partial T_2(t)u(w)}{\partial w}\right| \leq\\ &\leq \beta\int_0^t e^{s-t}\left[k_4\beta 2\tau\|J\|_\infty\|J'\|_\infty K^2+(k_4\beta h+k_5)\sqrt{2\tau}\|J'\|_\infty K\right]ds\\ &= \left[k_4\beta^2 2\tau\|J\|_\infty\|J'\|_\infty K^2+(k_4\beta^2 h+k_5\beta)\sqrt{2\tau}\|J'\|_\infty K\right]\int_0^t e^{s-t}ds\\ &\leq \left[k_4\beta^2 2\tau\|J\|_\infty\|J'\|_\infty K^2+(k_4\beta^2 h+k_5\beta)\sqrt{2\tau}\|J'\|_\infty K\right].\end{aligned}$$

It follows that, for $t > 0$ and any $u \in C$, the value of $\|\frac{\partial T_2(t)u}{\partial w}\|_{L^2}$ is bounded by a constant (independent of t and u). Thus, for all $u \in C$, we have that $T_2(t)u$ belongs to a ball of $W^{1,2}(S^1)$. From Sobolev's Imbedding Theorem, it follows that

$$\bigcup_{t\geq 0} T_2(t)C$$

is relatively compact. Therefore, the result follows from Theorem 3.1, the attractor $\mathcal{A}$ being the set ω-limit of the ball $B\left(0, \frac{2\sqrt{2\tau}(k_2\beta h+k_3)}{1-k_2\beta}\right)$ in $L^2(S^1)$. □

4. Comparison and boundedness results

In this section we prove a comparison result that generalizes Theorem 2.7 of [8], where the case $g \equiv \tanh$ and $h = 0$ was considered.

Definition 4.1. *A function $v(w,t)$ is a subsolution of the Cauchy problem for (2.6) with initial condition $u(\cdot,0)$ if $v(w,0) \leq u(w,0)$ for almost all $w \in S^1$, v is continuously differentiable with respect to t and satisfies*

$$\frac{\partial v(w,t)}{\partial t} \leq -v(w,t) + g(\beta(J * v(w,t) + h)), \tag{4.9}$$

almost everywhere.

Analogously, the function $V(w,t)$ is a super solution if has the same regularity properties as above, satisfies (4.9) with reversed inequality and $V(w,0) \geq u(w,0)$ for almost all $w \in S^1$.

Theorem 4.2. *(Comparison Theorem) Assume hypotheses (H1) and (H4) hold and let $v(w,t)$, $[V(w,t)]$ be a sub solution [super solution] of the Cauchy problem of (2.6) with initial condition $u(\cdot,0)$. Then*

$$v(w,t) \leq u(w,t) \leq V(w,t),$$

almost everywhere.

Proof

Define the operator G on $L^\infty(S^1 \times [0,T])$ by

$$G(f)(w,t) = e^{-t} f(w,0) + \int_0^t e^{-(t-s)} g(\beta(J * f(w,s) + h))ds.$$

Then $(G(f))(w,0) = f(w,0)$. Also, from (H4), it follows that G is monotonic, that is, for any $f_1, f_2 \in L^\infty(S^1 \times [0,T])$ with $f_1 \geq f_2$ (*a.e.* in $S^1 \times [0,T]$), $G(f_1) \geq G(f_2)$ (*a.e.* in $S^1 \times [0,T]$).

From (1.4), we obtain

$$\begin{aligned}
|G(f)(w,t)| &\leq e^{-t}|f(w,0)| + \int_0^t e^{-(t-s)} |g(\beta(J * f)(w,s) + \beta h)|ds \\
&\leq e^{-t}|f(w,0)| + \int_0^t e^{-(t-s)} [k_2|\beta(J * f)(w,s) + \beta h| + k_3]ds \\
&\leq e^{-t}|f(w,0)| + \int_0^t e^{-(t-s)} k_2\beta|(J * f)(w,s)|ds + \\
&+ \int_0^t e^{-(t-s)}(k_2\beta h + k_3)ds.
\end{aligned}$$

Since $|(J * f)(w,s)| \leq \|f\|_\infty$ almost everywhere in $S^1 \times [0,T]$, we obtain

$$\begin{aligned} &\|G(f)\|_\infty \leq \\ &\leq e^{-t}\|f\|_\infty + k_2\beta\|f\|_\infty \int_0^t e^{-(t-s)}ds + (k_2\beta h + k_3)\int_0^t e^{-(t-s)}ds \\ &\leq \|f\|_\infty + k_2\beta\|f\|_\infty + k_2\beta h + k_3. \end{aligned}$$

Therefore $G : L^\infty(S^1 \times [0,T]) \to L^\infty(S^1 \times [0,T])$.

Furthermore, if $k_1\beta T < 1$, G is a contraction in any subset of functions of $L^\infty(S^1 \times [0,T])$ with the same values at $t = 0$. In fact

$$\begin{aligned} &|G(f_1)(w,t) - G(f_2)(w,t)| = \\ &= \left| \int_0^t e^{-(t-s)}[g(\beta(J * f_1)(w,s) + \beta h) - g(\beta(J * f_2)(w,s) + \beta h)]ds \right| \\ &\leq \int_0^t e^{-(t-s)} k_1\beta|(J * f_1)(w,s) - (J * f_2)(w,s)|ds \\ &\leq \int_0^t e^{-(t-s)} k_1\beta(J * |f_1 - f_2|(w,s))ds \\ &\leq \int_0^t e^{-(t-s)} k_1\beta J * \|f_1 - f_2\|_\infty ds \\ &= k_1\beta T\|f_1 - f_2\|_\infty \int_0^t e^{-(t-s)}ds \\ &\leq k_1\beta T\|f_1 - f_2\|_\infty, \end{aligned}$$

almost everywhere in $S^1 \times [0,T]$. Hence $\|G(f_1) - G(f_2)\|_\infty \leq k_1\beta T\|f_1 - f_2\|_\infty$. Therefore, if $k_1\beta T < 1$, G is a contraction. Thus, if $u(w,t)$ is a solution of (2.6) with $u^0 = u(w,0)$, we have

$$u = \lim_{n \longrightarrow \infty} G^n(u^0)$$

on $L^\infty(S^1 \times [0,T])$. The same holds for a solution $\widetilde{u}$ with $\widetilde{u}^0 = \widetilde{u}(w,0)$. If $\widetilde{u}^0 \leq u^0$ *a.e.*, with g monotonic, it follows that

$$G^n(\widetilde{u}^0) \leq G^n(u^0), \ a.e.$$

Now, if v is a sub solution of (2.6) we have

$$\frac{d}{dt}v(w,t) + v(w,t) \leq g(\beta(J * v(w,t) + h)), \ a.e.$$

Multiplying both sides of the inequality above by e^t, we have

$$\frac{d}{dt}\left(e^t v(w,t)\right) \leq e^t g(\beta(J * v(w,t) + h)), \ a.e.$$

Integrating from 0 to t, we obtain

$$v(w,t) \leq e^{-t}v(w,0) + \int_0^t e^{-(t-s)} g(\beta(J * v(w,s) + h))ds,$$

almost everywhere. Therefore $v(w,t) \leq G(v)(w,t)$, *a.e.*, and since g monotonic, it follows that $v(w,t) \leq G^n(v)(w,t)$ almost everywhere. Thus, $v(w,t) \leq z(w,t)$, *a.e.*, where

$$z = \lim_{n \longrightarrow \infty} G^{n+1}(v).$$

Now, from the continuity of G, it follows that

$$G(z) = G\left(\lim_{n \longrightarrow \infty} G^n(v)\right) = \lim_{n \longrightarrow \infty} G^{n+1}(v) = z.$$

Therefore z is a fixed point of G, that is, z is a solution of (2.6) in $S^1 \times [0,T]$ with initial condition $z(\cdot,0) = v(\cdot,0)$. Thus, if $z(\cdot,0) \leq u(\cdot,0)$, *a.e.*, then

$$v \leq z \leq u, \ a.e. \text{ in } S^1 \times [0,T],$$

where u is the solution of (2.6) with initial condition $u(\cdot,0)$. If $V(w,t)$ is a super solution we obtain, by the same arguments

$$u \leq \widetilde{z} \leq V, \ a.e. \text{ in } S^1 \times [0,T].$$

Therefore

$$v(w,t) \leq u(w,t) \leq V(w,t),$$

almost everywhere in $S^1 \times [0,T]$.

Since the estimates above do not depend on the initial condition, we may extend the result to $[T, 2T]$ and, by iteration, we can complete the proof of the theorem. □

Remark 4.3. *If we add the hypothesis (H5), with $a < \infty$, the comparison result holds in the ball $\mathbb{M} = \{L^\infty(S^1 \times [0,T]), \|\cdot\|_\infty \leq a\}$.*

In fact, it is enough to prove that $G|_{\mathbb{M}} : \mathbb{M} \to \mathbb{M}$. But, from (H4), it follows that

$$|(G|_{\mathbb{M}} f)(w,t)| \quad \leq \quad e^{-t}|f(w,0)| + a\int_0^t e^{-(t-s)}ds.$$

Hence

$$\begin{aligned} \|(G|_{\mathbb{M}} f)\|_\infty &\leq e^{-t}\|f\|_\infty + a\int_0^t e^{-(t-s)}ds \\ &\leq ae^{-t} + a\int_0^t e^{-(t-s)}ds \\ &= a. \end{aligned}$$

Therefore, $G|_{\mathbb{M}}(f) \in \mathbb{M}$.

Theorem 4.4. *Assume the hypotheses (H1) and (H5) with $a < \infty$. Then the attractor $\mathcal{A}$ belongs to the ball $\| \cdot \|_\infty \leq a$ in $L^\infty(S^1)$.*

Proof Since the hypothesis (H5) is a particular case of (1.4) with $k_2 = 0$ and $k_3 = a$, it follows from Theorem 3.3 that the attractor is contained in the ball $B[0, 2a\sqrt{2\tau}]$ in $L^2(S^1)$.

Let $u(w,t)$ be a solution of (2.6) in $\mathcal{A}$. Then, by the variation of constants formula

$$u(w,t) = e^{-(t-t_0)}u(w,t_0) + \int_{t_0}^{t} e^{-(t-s)}g(\beta(J*u)(w,s) + \beta h)ds.$$

Since $\|u\|_{L^2} \leq 2a\sqrt{2\tau}$ for all $u \in \mathcal{A}$, we obtain for all $(w,t) \in S^1 \times \mathbb{R}^+$ letting $t_0 \to -\infty$

$$u(w,t) = \int_{-\infty}^{t} e^{-(t-s)}g(\beta(J*u)(w,s) + \beta h)ds,$$

where the equality above is in the sense of $L^2(S^1)$. Thus, using (H5) again, we have

$$\begin{aligned} |u(w,t)| &\leq \int_{-\infty}^{t} e^{-(t-s)}|g(\beta(J*u)(w,s) + \beta h)|ds \\ &\leq \int_{-\infty}^{t} ae^{-(t-s)}ds \\ &\leq a. \end{aligned}$$

as claimed. □

5. Existence of a Lyapunov functional

In this section we exhibit a continuous " Lyapunov's functional" for the flow of (2.6), restricted to the ball of radius a in $L^\infty(S^1)$, concluding that this flow is gradient, in the sense of [5].

We claim that $\{L^2(S^1), \| \cdot \|_\infty \leq a\}$ is an invariant set for the flow generated by (2.6) . In fact, if $a = \infty$, there is nothing to prove. Otherwise, let

$$u(w,t) = e^{-t}u(w,0) + \int_{0}^{t} e^{-(t-s)}g(\beta J*u(w,s) + \beta h)ds$$

be the solution of (2.6) with initial condition $u(w,0) \in \{L^2(S^1), \|\cdot\|_\infty \leq a\}$. Then

$$\begin{aligned} |u(w,t)| &\leq e^{-t}|u(w,0)| + \int_0^t e^{-(t-s)}|g(\beta J * u(w,s) + \beta h)|ds \\ &\leq e^{-t}|u(w,0)| + a\int_0^t e^{-(t-s)}ds. \end{aligned}$$

Hence

$$\begin{aligned} \|u(\cdot,t)\|_\infty &\leq e^{-t}\|u(\cdot,0)\|_\infty + a\int_0^t e^{-(t-s)}ds \\ &\leq e^{-t}a + a\int_0^t e^{-(t-s)}ds \\ &= a. \end{aligned}$$

Define the functional $\mathbb{F}$: $(L^2(S^1), \|u\|_\infty \leq a) \to \mathbb{R}$ by

$$\mathbb{F}(u) = \int_{S^1}[f(u(w)) - f(\overline{m})]dw + \frac{1}{4}\int_{S^1}\int_{S^1} J(w\cdot z^{-1})[u(w)-u(z)]^2 dwdz, \tag{5.10}$$

where f is given in the hypothesis (H6).

Note that, if $a < \infty$, the functional in (5.10) is defined in the whole space $\{L^2(S^1),\ \|u\|_\infty \leq a\}$. This is not true for the similar functional

$$\widetilde{\mathbb{F}}(u) = \int_{\mathbb{R}}[f(u(w)) - f(m_\beta^+)]dw + \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} J(w-z)[u(w)-u(z)]^2 dwdz,$$

considered in [7], [8] and [11] with $g \equiv \tanh$.

It is proved in [8], in the case of unbounded domain, $g \equiv \tanh$ and $h = 0$, that the functional $\widetilde{\mathbb{F}}$ is lower semicontinuous in the weak L^2_{loc} topology. In our case, however, a stronger continuity property can be proved.

Theorem 5.1. *Assume (H6) holds with $a < \infty$. Then the functional given in (5.10) is continuous in the topology of $L^2(S^1)$.*

Proof Note that, if $\|u\|_\infty \leq a$, there exists a positive constant K such that

$$|f(u(w)) - f(\overline{m})| \leq |f(u(w))| + |f(\overline{m})| \leq K, \quad \text{for almost every } w \in S^1.$$

for any $u \in L^2(S^1)$, with $\|u\|_\infty \leq a$, let u_n a sequence converging to u in the norm of $L^2(S^1)$. We can extract a subsequence u_{n_k}, such that, $u_{n_k}(w) \longrightarrow u(w)$ *a.e.* in S^1. Since from (H6), it follows that f is continuous, $f(u_{n_k}(w)) \longrightarrow f(u(w))$ *a.e.* Thus

$$\lim_{k\longrightarrow\infty}[f(u_{n_k}(w)) - f(\overline{m})] = [f(u(w)) - f(\overline{m})],\ a.e.$$

and

$$\lim_{k\longrightarrow\infty}[u_{n_k}(w)-u_{n_k}(z)]^2=[u(w)-u(z)]^2,\ a.e.$$

Now, we write

$$\mathbb{F}(u)=\mathbb{F}_1(u)+\mathbb{F}_2(u),$$

where

$$\mathbb{F}_1=\int_{S^1}[f(u(w))-f(\overline{m})]dw$$

and

$$\mathbb{F}_2(u)=\frac{1}{4}\int_{S^1}\int_{S^1}J(w\cdot z^{-1})[u(w)-u(z)]^2dwdz.$$

Since

$$|f(u_{n_k}(w))-f(\overline{m})|\le K,$$

we can apply Lebesgue's Dominated Convergence Theorem to obtain

$$\lim_{k\longrightarrow\infty}\mathbb{F}_1(u_{n_k})=\mathbb{F}_1(u).$$

Similarly, as

$$|u_{n_k}(w)-u_{n_k}(z)|^2\le 4a^2\in L^1(S^1),$$

we have

$$\lim_{k\longrightarrow\infty}\mathbb{F}_2(u_{n_k})=\mathbb{F}_2(u).$$

Therefore

$$\lim_{k\longrightarrow\infty}\mathbb{F}(u_{n_k})=\mathbb{F}(u).$$

Thus $\mathbb{F}(u_n)$ is a sequence such that every subsequence has a subsequence that converges to $\mathbb{F}(u)$, and we obtain

$$\lim_{n\rightarrow\infty}\mathbb{F}(u_n)=\mathbb{F}(u).$$

□

Theorem 5.2. *Suppose that the hypotheses (H1), (H4) and (H5)-(H6), with $a<\infty$, hold. Let $u(\cdot,t)$ be a solution of (2.6) with $u(\cdot,t)\le a$. Then $\mathbb{F}(u(\cdot,t))$ is differentiable with respect to t for $t>0$ and*

$$\frac{d}{dt}\mathbb{F}(u(\cdot,t))=-I(u(\cdot,t))\le 0,$$

where, for any $u\in L^2(S^1)$ with $\|u\|_\infty\le a$,

$$I(u(\cdot))=\int_{S^1}[(J*u)(w)+h-\beta^{-1}g^{-1}(u(w))][g(\beta(J*u)(w)+\beta h)-u(w)]dw.$$

Furthermore, the integrand in $I(u(\cdot))$ is a non negative function and, u is a critical point of $\mathbb{F}$ if only if u is an equilibrium of (2.6).

Proof From (H1) and (H5), it follows that $\mathbb{F}(u(\cdot,t))$ is well defined for all $t \geq 0$. We assume first that, given $t > 0$, there exists $\varepsilon > 0$ such that $\|u(\cdot,s)\|_\infty \leq a - \varepsilon$, for $s \in \Delta$ where Δ is a closed finite interval containing t. For $s \in \Delta$ we write

$$\mathbb{F}(u(\cdot,s)) = \int_{S^1} \phi(w,s)dw, \text{ and } I(u(\cdot,s)) = \int_{S^1} \iota(w,s)dw.$$

As

$$\frac{\partial \phi}{\partial s}(w,s) =$$
$$[-u(w,s) - h + \beta^{-1}g^{-1}(u(w,s))][-u(w,s) + g(\beta((J*u)(w,s)+h))]$$
$$+ \frac{1}{2}\int_{S^1} J(w\cdot z^{-1})[u(w,s)-u(z,s)]\left[\frac{\partial u(w,s)}{\partial s} - \frac{\partial u(z,s)}{\partial s}\right]dz,$$

$\frac{\partial \phi(w,s)}{\partial s}$ is almost everywhere continuous and bounded in w for $s \in \Delta$, that is,

$$\sup_{s\in\Delta}\left\|\frac{\partial \phi(\cdot,s)}{\partial s}\right\|_{L^1} < \infty.$$

Therefore, we can derive under the integration sign obtaining

$$\begin{aligned}\frac{d}{ds}\mathbb{F}(u(\cdot,s)) &= \int_{S^1}[-u(w,s) - h + \beta^{-1}g^{-1}(u(w,s))]\frac{\partial u(w,s)}{\partial s}dw \\ &+ \frac{1}{2}\int_{S^1}\int_{S^1} J(w\cdot z^{-1})[u(w,s)-u(z,s)] \\ &* \left[\frac{\partial u(w,s)}{\partial s} - \frac{\partial u(z,s)}{\partial s}\right]dwdz,\end{aligned}$$

Since

$$\begin{aligned}
\int_{S^1}\int_{S^1} & J(w\cdot z^{-1})[u(w,s)-u(z,s)]\left[\frac{\partial u(w,s)}{\partial s}-\frac{\partial u(z,s)}{\partial s}\right]dwdz = \\
= & \int_{S^1}\int_{S^1} J(w\cdot z^{-1})u(w,s)\frac{\partial u(w,s)}{\partial s}dwdz \\
- & \int_{S^1}\int_{S^1} J(w\cdot z^{-1})u(w,s)\frac{\partial u(z,s)}{\partial s}dwdz \\
- & \int_{S^1}\int_{S^1} J(w\cdot z^{-1})u(z,s)\frac{\partial u(w,s)}{\partial s}dwdz \\
+ & \int_{S^1}\int_{S^1} J(w\cdot z^{-1})u(z,s)\frac{\partial u(z,s)}{\partial s}dwdz \\
= & \, 2\int_{S^1}\int_{S^1} J(w\cdot z^{-1})u(w,s)\frac{\partial u(w,s)}{\partial s}dwdz \\
- & \, 2\int_{S^1}\int_{S^1} J(w\cdot z^{-1})u(z,s)\frac{\partial u(w,s)}{\partial s}dwdz.
\end{aligned}$$

and

$$\int_{S^1} J(w\cdot z^{-1})dz = 1,$$

it follows that

$$\begin{aligned}
\frac{d}{ds}\mathbb{F}(u(\cdot,s)) &= \int_{S^1}\left[-u(w,s)-h+\beta^{-1}g^{-1}(u(w,s))\right]\frac{\partial u(w,s)}{\partial s}dw \\
&+ \int_{S^1}\left(\int_{S^1} J(w\cdot z^{-1})dz\right)u(w,s)\frac{\partial u(w,s)}{\partial s}dw \\
&- \int_{S^1}\left(\int_{S^1} J(w\cdot z^{-1})u(z,s)dz\right)\frac{\partial u(w,s)}{\partial s}dw \\
&= \int_{S^1}\left[-u(w,s)-h+\beta^{-1}g^{-1}(u(w,s))\right]\frac{\partial u(w,s)}{\partial s}dw \\
&+ \int_{S^1}[u(w,s)-(J*u)(w,s)]\frac{\partial u(w,s)}{\partial s}dw \\
&= \int_{S^1}\left[-(J*u)(w,s)-h+\beta^{-1}g^{-1}(u(w,s))\right] \\
&* \left[-u(w,s)+g(\beta(J*u(w,s)+h))\right]dw \\
&= -I(u(\cdot,s)).
\end{aligned}$$

This proves the first part of theorem with the additional hypothesis that $\|u(\cdot,s)\|_\infty \leq a-\varepsilon$, for $s \in \Delta$ and some $\varepsilon > 0$, where Δ is a closed finite interval containing t. We claim that this hypothesis actually holds for all $t > 0$.

Let $\lambda(w,t)$ be the solution of (2.6) such that $\lambda(w,0) = a$ for any $w \in S^1$. Then $\lambda(w,t) = \lambda(t)$ where

$$\frac{d\lambda}{dt} = -\lambda(t) + g(\beta(\lambda(t) + h)).$$

Since by hypothesis (H5), $|g(x)| < a, \forall\, x \in \mathbb{R}$, it follows easily that $\lambda(t) < a$ for any $t > 0$. Since $u(w,0) \leq a$, we obtain by the Comparison Theorem

$$u(w,t) \leq \lambda(t) < a,$$

for almost every $w \in S^1$ and $t > 0$. Repeating the same argument, starting from inequality $u(w,0) \geq -a$, for almost every $w \in S^1$, we obtain $u(w,t) \geq -\lambda(t) > -a$, and thus

$$\|u(\cdot,t)\|_\infty < \lambda(t) < a, \quad \text{for all} \quad t > 0$$

and the claim follows by continuity.

To conclude the proof, it is enough to show that u is a critical point of $\mathbb{F}$ if only if u is an equilibrium of (2.6). Let $u(w)$ be a critical point of the functional $\mathbb{F}$, then $I(u(\cdot)) = 0$. Since the integrand is non negative almost everywhere, it follows that

$$[(J * u(w)) + h - \beta^{-1} g^{-1}(u(w))][g(\beta(J * u(w) + h)) - u(w)] = 0$$

almost everywhere. But the annihilation of any of these factors implies

$$g(\beta(J * u(w) + h)) = u(w).$$

Reciprocally, if u is a equilibrium of (2.6), it is easy to see that $I(u(\cdot)) = 0$. □

As a immediate consequence of the existence of the functional $\mathbb{F}$ we obtain the following result.

Corollary 5.3. *There are no non trivial recurrent points under the flow of (2.6).*

Remark 5.4. *The integrand in the functional* $\mathbb{F}$ *above is always non negative since* J *is positive and* $\overline{m}$ *is a global minim of* f. *Thus* $\mathbb{F}$ *is lower bounded.*

We recall that a C^r-semigroup, $T(t)$, is *gradient* if each bounded positive orbit is precompact and there exists a Lyapunov Functional for $T(t)$, (see [5]).

Proposition 5.5. *Assume the hypotheses (H1), (H3), (H4) and (H5), (H6) with* $a < \infty$. *Then the flow generated by equation (2.6) is gradient.*

Proof The precompacity of the orbits follows from the existence of the global attractor. From Theorems 5.1 and 5.2, and Remark 5.4, we have existence of a continuous Lyapunov functional. □

As a consequence of the existence of the Lyapunov functional, we have the following characterization of the attractor (see [5] - Theorem 3.8.5).

Theorem 5.6. *Assume the same assumptions of Proposition 5.5. Then the attractor $\mathcal{A}$ is the unstable set of the equilibrium point set of $T(t)$, that is,*

$$\mathcal{A} = W^u(E).$$

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