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Inertial Projection-Type Methods for Solving Quasi-Variational Inequalities in Real Hilbert Spaces

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Abstract In this paper, we introduce an inertial projection-type method with different updating strategies for solving quasi-variational inequalities with strongly monotone and Lipschitz continuous operators in real Hilbert spaces. Under standard assumptions, we establish different strong convergence results for the proposed algorithm. Primary numerical experiments demonstrate the potential applicability of our scheme compared with some related methods in the literature.

Keywords Quasi-variational inequalities · inertial extrapolation step · strongly monotone · Hilbert spaces

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1 Introduction

In this paper, we study the *Quasi-Variational Inequality* (QVI) problem, which generalizes the classical *Variational Inequality* (VI) problem of Fichera [1,2] and Stampacchia [3] (see also Kinderlehrer and Stampacchia [4]). In a QVI, the associated feasible set of the problem is not fixed, but varies according to some explicit or implicit rule. For example, in many applications, the feasible set is defined as a " moving set ", i.e., there is a closed and convex set, also known as the core set, that is shifted by another single-valued mapping; see for example [5–9] and the references therein. In such a setting, the problem is often called " moving set " QVI.

There exist many techniques for solving QVIs; for example, very recently, Antipin et al. [5] presented gradient projection and extragradient methods for solving QVIs under the assumptions of strong monotonicity and Lipschitz continuity of the associated mappings. The main disadvantage of the extragradient method, with respect to the classical gradient projection algorithm, is that the number of orthogonal projections and mapping evaluations is doubled per each iteration. Moreover, while for VIs the extra projection and evaluation per iteration of the extragradient method guarantee convergence under weaker assumptions than strong monotonicity of the associated mapping, for QVIs this is not the case and thus the extragradient does not have any advantage over the gradient projection method.

A more general gradient projection method, with strong convergence for solving QVIs in real Hilbert spaces, is introduced by Mijajlović et al. in [10]. This method holds great potential since it works well on different practical applications. Other efficient solution methods for solving QVIs can be found in [11–14].

In the field of continuous optimization, inertial type algorithms attracted much interest in recent years mainly due to their convergence properties. The idea is derived from the field of second-order dissipative dynamical systems [15,16]. It is shown that such inertial terms speed up the convergence rate of the existing algorithms, see, e.g., the inertial proximal point algorithm [17–21], the inertial forward-backward splitting method [22–24], the inertial Douglas-Rachford splitting method [25,26], the inertial ADMM [27,28], and the inertial forward-backward-forward method [29].

Motivated by the above results, we wish to present a new inertial type algorithm for solving QVIs with the following three major advantages.

1. We consider general QVIs, in contrast to what is done e.g. in [5] where only the ”moving set” case, described above, is analyzed.
2. Our new proposed scheme requires only one operator evaluation and one orthogonal projection per each iteration, in contrast to other methods, such as those provided in [5, 10], just to name a few.
3. The convergence speed of our proposed method is better than other projection methods for solving QVIs, see e.g. [10].

The outline of the paper is as follows. In Section 2 we list some basic facts, concepts, and lemmas, which are needed in the sequel. In Section 3 the new method with two different inertial and relaxation parameters is presented and analyzed. In Section 4, numerical examples illustrate the behaviour of the proposed schemes and, finally, in Section 5 conclusion is given.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let K be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator and $K : \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued mapping which associates for any element $u \in \mathcal{H}$ a closed and convex set $K(u) \subset \mathcal{H}$. With the above data, we are concerned with the following *Quasi-Variational Inequality* (QVI), which consists of finding a point $u^* \in \mathcal{H}$ such that $u^* \in K(u^*)$ and

$$\langle \mathcal{F}(u^*), v - u^* \rangle \geq 0 \text{ for all } v \in K(u^*). \tag{1}$$

Clearly, if $K(u) \equiv K$ for all $u \in \mathcal{H}$, then the problem is reduced to the classical variational inequality, that is, find a point $u^* \in K$ such that

$$\langle \mathcal{F}(u^*), v - u^* \rangle \geq 0 \text{ for all } v \in K. \tag{2}$$

Definition 2.1 Let $T : \mathcal{H} \to \mathcal{H}$ be a given mapping.

– The mapping T is called *L-Lipschitz continuous* $(L > 0)$, if

$$\|F(x) - F(y)\| \leq L\|x - y\| \text{ for all } x, y \in \mathcal{H}. \tag{3}$$

– The mapping T is called *μ-strongly monotone* ($\mu > 0$), if

$$\langle T(x) - T(y), x - y \rangle \geq \mu \|x - y\|^2 \text{ for all } x, y \in \mathcal{H}. \tag{4}$$

– The mapping T is called *monotone*, if

$$\langle F(x) - F(y), x - y \rangle \geq 0 \text{ for all } x, y \in \mathcal{H}. \tag{5}$$

For each point $x \in \mathcal{H}$, there exists a unique nearest point in K, denoted by $P_K(x)$, such that

$$\|x - P_K(x)\| \leq \|x - y\| \text{ for all } y \in K. \tag{6}$$

The mapping $P_K : \mathcal{H} \to K$ is called the *metric projection* of $\mathcal{H}$ onto K and is characterized [30, Section 3] by the following two properties:

$$P_K(x) \in K \tag{7}$$

and

$$\langle x - P_K(x), P_K(x) - y \rangle \geq 0 \text{ for all } x \in \mathcal{H},\ y \in K, \tag{8}$$

and if K is a hyper-plane, then (8) becomes an equality.

The following result, which is proved in [31], gives sufficient conditions for the existence of solutions of QVIs (1).

Lemma 2.1 *Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be L-Lipschitz continuous and μ-strongly monotone on $\mathcal{H}$ and $K(\cdot)$ be a set-valued mapping with nonempty, closed and convex values such that there exists $\lambda \geq 0$ such that*

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \leq \lambda \|x - y\|, \quad x, y, z \in \mathcal{H}, \quad \lambda + \sqrt{1 - \frac{\mu^2}{L^2}} < 1. \tag{9}$$

Then the QVI (1) has a unique solution.

The next result is a fixed point formulation characterizing the solutions of the QVI (1).

Lemma 2.2 *Let $K(\cdot)$ be a set-valued mapping with nonempty, closed and convex values in $\mathcal{H}$. Then $x^* \in K(x^*)$ is a solution of the QVI (1) if and only if for any $\gamma > 0$ it holds that*

$$x^* = P_{K(x^*)}(x^* - \gamma \mathcal{F}(x^*)).$$

A technical result, which is useful for our analysis, is given next; see [32].

Lemma 2.3 *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{k+1} \leq (1-\alpha_k)a_k + \alpha_k \sigma_k + \gamma_k, \quad k \geq 1,$$

where

(a) $\{\alpha_k\} \subset [0,1]$, $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(b) $\limsup \sigma_k \leq 0$;

(c) $\gamma_k \geq 0$ $(n \geq 1)$, $\sum_{k=1}^{\infty} \gamma_k < \infty$.

Then, $a_k \to 0$ *as* $k \to \infty$.

3 The Inertial Method

In this section, we introduce the inertial-type method and establish its strong convergence theorems.

Algorithm 3.1

Initialization: *Select arbitrary starting points* $x^0, x^1 \in \mathcal{H}$.

Iterative step: *Given the iterates* x^k *and* x^{k-1}*, compute the next iterate* x^{k+1} *as follows*

$$\begin{cases} y^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} = (1-\alpha_k)y^k + \alpha_k P_{K(y^k)}(y^k - \gamma \mathcal{F}(y^k)) \end{cases} \tag{10}$$

Set $k \leftarrow k+1$ *and go to* ***Iterative step****.*

In Algorithm 3.1, $\{\theta_k\}$ and $\{\alpha_k\}$ are sequences satisfying several conditions that are specified in the convergence theorems below.

Remark 3.1 1. If $\theta_k = 0$ for all $k \geq 1$, in Algorithm 3.1, then [10, Algorithm 1] is obtained. Thus, Algorithm 3.1 is actually [10, Algorithm 1] with an inertial extrapolation step y^k.

2. If both $\theta_k = 0$ and $\alpha_k = 1$, for all $k \geq 1$ in Algorithm 3.1, we obtain the procedure [5, Eq. (5)] (also studied in [7, 33–35]).

We start the strong convergence analysis of Algorithm 3.1 with the special choice of parameters:

$$0 \leq \theta_k \leq \bar{\theta}_k, \quad \bar{\theta}_k := \begin{cases} \min\left\{\frac{k-1}{k+\eta-1}, \frac{\epsilon_k}{\|x^k - x^{k-1}\|}\right\}, & \text{if } x^k \neq x^{k-1}, \\ \\ \frac{k-1}{k+\eta-1}, & \text{if } x^k = x^{k-1}, \end{cases} \tag{11}$$

for some $\eta \geq 3$ and $\epsilon_k \in]0, \infty[$.

We observe that in this case Algorithm 3.1 generates a sequence such that $\sum_{k=1}^{\infty} \theta_k \|x^k - x^{k-1}\| < \infty$, because for every $k \geq 1$ we get $\theta_k \|x^k - x^{k-1}\| \leq \epsilon_k$ when $x^k \neq x^{k-1}$ and $\theta_k \|x^k - x^{k-1}\| = 0$ when $x^k = x^{k-1}$.

Theorem 3.1 *Consider the QVI (1) with $\mathcal{F}$ being μ-strongly monotone and L-Lipschitz continuous and assume there exists $\lambda \geq 0$ such that (9) holds. Let $\{x^k\}$ be any sequence generated by Algorithm 3.1 with the updating rule given by (11). Assume in addition that $\gamma \geq 0$ satisfies*

$$\left|\gamma - \frac{\mu}{L^2}\right| < \frac{\sqrt{\mu^2 - L^2 \lambda (2 - \lambda)}}{L^2}, \tag{12}$$

the sequence $\{\alpha_k\} \subseteq]0, 1]$ is such that $\sum_{k=1}^{\infty} \alpha_k = \infty$, and the sequence $\{\epsilon_k\}$ satisfies $\sum_{k=1}^{\infty} \epsilon_k < \infty$, then $\{x^k\}$ converges strongly to the unique solution $x^* \in K(x^*)$ of the QVI (1).

Proof We know that

$$x^* = (1 - \alpha_k) x^* + \alpha_k P_{K(x^*)}(x^* - \gamma \mathcal{F}(x^*)).$$

Now,

$$\begin{aligned}
\|x^{k+1} - x^*\| &= \|(1 - \alpha_k) y^k + \alpha_k P_{K(y^k)}(y^k - \gamma \mathcal{F}(y^k)) \\
&\quad - (1 - \alpha_k) x^* + \alpha_k P_{K(x^*)}(x^* - \gamma \mathcal{F}(x^*))\| \\
&\leq \|(1 - \alpha_k)(y^k - x^*)\| + \alpha_k \|P_{K(y^k)}(y^k - \gamma \mathcal{F}(y^k)) - P_{K(x^*)}(x^* - \gamma \mathcal{F}(x^*))\| \\
&\leq (1 - \alpha_k)\|y^k - x^*\| + \alpha_k \|P_{K(y^k)}(y^k - \gamma \mathcal{F}(y^k)) - P_{K(x^*)}(y^k - \gamma \mathcal{F}(y^k))\| \\
&\quad + \alpha_k \|P_{K(x^*)}(y^k - \gamma \mathcal{F}(y^k)) - P_{K(x^*)}(x^* - \gamma \mathcal{F}(x^*))\| \\
&\leq (1 - \alpha_k)\|y^k - x^*\| + \alpha_k \lambda \|y^k - x^*\| \\
&\quad + \alpha_k \|(y^k - \gamma \mathcal{F}(y^k)) - (x^* - \gamma \mathcal{F}(x^*))\|.
\end{aligned} \tag{13}$$

Using the fact that $\mathcal{F}$ is μ-strongly monotone and L-Lipschitz continuous, we obtain

$$\begin{aligned}
\|(y^k - \gamma \mathcal{F}(y^k)) - (x^* - \gamma \mathcal{F}(x^*))\|^2 &= \|y^k - x^*\|^2 - 2\gamma \langle \mathcal{F}(y^k) - \mathcal{F}(x^*), y^k - x^* \rangle \\
&\quad + \gamma^2 \|\mathcal{F}(y^k) - \mathcal{F}(x^*)\|^2 \\
&\leq (1 - 2\mu\gamma + \gamma^2 L^2)\|y^k - x^*\|^2.
\end{aligned} \tag{14}$$

Combining (13) and (14), we get

$$\begin{aligned}\|x^{k+1}-x^*\| &\leq (1-\alpha_k)\|y^k-x^*\|+\alpha_k\lambda\|y^k-x^*\| \\ &+\alpha_k\sqrt{1-2\mu\gamma+\gamma^2L^2}\|y^k-x^*\| \\ &= (1-\alpha_k)\|y^k-x^*\|+\alpha_k\beta\|y^k-x^*\|, \end{aligned} \tag{15}$$

where $\beta := \sqrt{1-2\mu\gamma+\gamma^2L^2}+\lambda$. Now,

$$\begin{aligned}\|y^k-x^*\| &= \|x^k-x^*+\theta_k(x^k-x^{k-1})\| \\ &\leq \|x^k-x^*\|+\theta_k\|x^k-x^{k-1}\|. \end{aligned} \tag{16}$$

Plugging (16) into (15), we get

$$\begin{aligned}\|x^{k+1}-x^*\| &\leq (1-\alpha_k)\|y^k-x^*\|+\alpha_k\beta\|y^k-x^*\| \\ &= (1-\alpha_k(1-\beta))\|y^k-x^*\| \\ &\leq (1-\alpha_k(1-\beta))(\|x^k-x^*\|+\theta_k\|x^k-x^{k-1}\|) \\ &\leq (1-\alpha_k(1-\beta))\|x^k-x^*\|+\theta_k\|x^k-x^{k-1}\|. \end{aligned} \tag{17}$$

Observe that by (12), we have $0<\beta<1$. Since $\sum_{k=1}^{\infty}\theta_k\|x^k-x^{k-1}\|<\infty$, using Lemma 2.3, we get that $x^k\to x^*, k\to\infty$, and the proof is complete. □

Remark 3.2 Inequality (condition) (12) can always be satisfied by setting γ sufficiently close to the ratio $\frac{\mu}{L^2}$.

Moreover, Theorem 3.1 still holds if in (11) the term $\frac{k-1}{k+\eta-1}$ is replaced with some constant in $[0,1[$. The idea of using it with $\eta\geq 3$ derives from the recent inertial extrapolated step introduced in [19, 36].

Complexity bound for Algorithm 3.1 with the updating rule (11) is presented next.

Theorem 3.2 *Consider the QVI* (1) *with the same assumptions as in Theorem 3.1. Let* $\{x^k\}$ *be any sequence generated by Algorithm 3.1 with the updating rule* (11) *and let* $x^*\in K(x^*)$ *be the unique solution of the QVI* (1)*. Let* $\alpha_k=\alpha$ *and* $\epsilon_k=\epsilon$ *be constant. Then, given* $\rho\in]0,\alpha(1-\beta)[$, *for any*

$$k\geq\bar{k}:=\Big\lfloor \log_{(1-\rho)}\left(\left(\frac{\epsilon}{\|x^0-x^*\|}\right)\left(\frac{1-\alpha(1-\beta)}{\alpha(1-\beta)-\rho}\right)\right)\Big\rfloor,$$

assuming $\bar{k} \geq 0$, *it holds that*

$$\|x^k - x^*\| \leq \epsilon \left(\frac{1-\alpha(1-\beta)}{\alpha(1-\beta)-\rho} + 1 \right). \tag{18}$$

Proof From the proof of Theorem 3.1, for any $k \geq 1$ we get

$$\begin{aligned} \|x^{k+1} - x^*\| &\leq (1-\alpha(1-\beta))(\|x^k - x^*\| + \theta_k \|x^k - x^{k-1}\|) \\ &\leq (1-\alpha(1-\beta))(\|x^k - x^*\| + \epsilon), \end{aligned} \tag{19}$$

because $(1-\alpha(1-\beta)) \geq 0$. Without the loss of generality, assume that for every $k < \bar{k}$ we get

$$\|x^k - x^*\| \geq \epsilon \frac{1-\alpha(1-\beta)}{\alpha(1-\beta)-\rho}. \tag{20}$$

Concatenating (19) and (20) we obtain, for every $k < \bar{k}$,

$$\begin{aligned} \|x^{k+1} - x^*\| &\leq (1-\alpha(1-\beta)) \left(1 + \frac{\alpha(1-\beta)-\rho}{1-\alpha(1-\beta)} \right) \|x^k - x^*\| \\ &= (1-\rho)\|x^k - x^*\|. \end{aligned} \tag{21}$$

Therefore, by the definition of $\bar{k}$, it holds that

$$\|x^{\bar{k}} - x^*\| \leq (1-\rho)^{\bar{k}} \|x^0 - x^*\| \leq \epsilon \frac{1-\alpha(1-\beta)}{\alpha(1-\beta)-\rho}.$$

For any $k > \bar{k}$ there are two possibilities. If

$$\|x^{k-1} - x^*\| \leq \epsilon \frac{1-\alpha(1-\beta)}{\alpha(1-\beta)-\rho},$$

then, by (19) and recalling that $(1-\alpha(1-\beta)) \leq 1$, we obtain that x^k satisfies (18). Otherwise, if

$$\epsilon \frac{1-\alpha(1-\beta)}{\alpha(1-\beta)-\rho} \leq \|x^{k-1} - x^*\| \leq \epsilon \left(\frac{1-\alpha(1-\beta)}{\alpha(1-\beta)-\rho} + 1 \right),$$

then

$$\|x^k - x^*\| \leq (1-\rho)\|x^{k-1} - x^*\| \leq \|x^{k-1} - x^*\|,$$

and the desired result holds. □

Remark 3.3 We observe that, in contradiction with the assumptions of Theorem 3.1, in Theorem 3.2 the summability of $\{\epsilon_k\}$ is not required. However if one wants a good bound in (18) then a small value of ϵ must be set, but, in this case, small values of θ_k are allowed.

Next, we present another convergence theorem for Algorithm 3.1 under different conditions on the inertial terms. **Theorem 3.3** *Consider the QVI* (1) *with $\mathcal{F}$ being μ-strongly monotone and L-Lipschitz continuous and assume there exists $\lambda \geq 0$ such that* (9) *holds. Let $\{x^k\}$ be any sequence generated by Algorithm 3.1 with $\gamma \geq 0$ satisfying* (12)*, and $\{\alpha_k\}$ and $\{\theta_k\}$ satisfying the following conditions for some $\epsilon \in]0,1[$:*

1. $0 < \frac{1}{2(1-\beta)} \leq \alpha_k < \frac{1}{1+\epsilon}$, *with* $\beta := \sqrt{1 - 2\mu\gamma + \gamma^2 L^2} + \lambda$, *and* $\sum_{k=1}^{\infty} \alpha_k = \infty$;
2. $0 \leq \theta_k \leq \theta_{k+1} \leq \theta < \frac{\sqrt{1+8\epsilon}-1-2\epsilon}{2(1-\epsilon)} < 1$.

Then $\{x^k\}$ converges strongly to the unique solution $x^* \in K(x^*)$ of the QVI (1).

Proof Define $T(y^k) := P_{K(y^k)}(y^k - \gamma\mathcal{F}(y^k))$. Then for the unique solution x^* of (1), we obtain

$$
\begin{aligned}
\|T(y^k) - T(x^*)\| &= \|P_{K(y^k)}(y^k - \gamma\mathcal{F}(y^k)) - P_{K(x^*)}(x^* - \gamma\mathcal{F}(x^*))\| \\
&\leq \|P_{K(y^k)}(y^k - \gamma\mathcal{F}(y^k)) - P_{K(x^*)}(y^k - \gamma\mathcal{F}(y^k))\| \\
&\quad + \|P_{K(x^*)}(y^k - \gamma\mathcal{F}(y^k)) - P_{K(x^*)}(x^* - \gamma\mathcal{F}(x^*))\| \\
&\leq \lambda\|y^k - x^*\| + \|y^k - x^* + \gamma(\mathcal{F}(x^*) - \mathcal{F}(y^k))\|. \qquad (22)
\end{aligned}
$$

Since $\mathcal{F}$ is μ-strongly monotone and $L-$Lipschitz continuous, we get

$$
\begin{aligned}
\|y^k - x^* - \gamma(\mathcal{F}(x^*) - \mathcal{F}(y^k))\|^2 &= \|y^k - x^*\|^2 - 2\gamma\langle \mathcal{F}(y^k) - \mathcal{F}(x^*), y^k - x^*\rangle \\
&\quad + \gamma^2\|\mathcal{F}(y^k) - \mathcal{F}(x^*)\|^2 \\
&\leq (1 - 2\mu\gamma + \gamma^2 L^2)\|y^k - x^*\|^2. \qquad (23)
\end{aligned}
$$

Combining (22) and (23), we get

$$
\begin{aligned}
\|T(y^k) - T(x^*)\| &\leq \lambda\|y^k - x^*\| + \sqrt{1 - 2\mu\gamma + \gamma^2 L^2}\|y^k - x^*\| \\
&= \beta\|y^k - x^*\| \\
&\leq \|y^k - x^*\|. \qquad (24)
\end{aligned}
$$

From the definition of Algorithm 3.1, we get

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &= (1 - \alpha_k)\|y^k - x^*\|^2 + \alpha_k\|T(y^k) - x^*\|^2 \\
&\quad - \alpha_k(1 - \alpha_k)\|y^k - T(y^k)\|^2 \\
&\leq \|y^k - x^*\|^2 - \alpha_k(1 - \alpha_k)\|y^k - T(y^k)\|^2. \qquad (25)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\|y^k - x^*\|^2 &= \|x^k + \theta_k(x^k - x^{k-1}) - x^*\|^2 \\
&= \|(1+\theta_k)(x^k - x^*) - \theta_k(x^{k-1} - x^*)\|^2 \\
&= (1+\theta_k)\|x^k - x^*\|^2 - \theta_k\|x^{k-1} - x^*\|^2 \\
&\quad +\theta_k(1+\theta_k)\|x^k - x^{k-1}\|^2. \qquad (26)
\end{aligned}
$$

Observe that

$$
\|x^{k+1} - y^k\|^2 = \alpha_k^2\|y^k - T(y^k)\|^2
$$

and so

$$
\|y^k - T(y^k)\|^2 = \frac{1}{\alpha_k^2}\|x^{k+1} - y^k\|^2. \qquad (27)
$$

Combining (27) with (25) yields

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|y^k - x^*\|^2 - \frac{\alpha_k(1-\alpha_k)}{\alpha_k^2}\|x^{k+1} - y^k\|^2 \\
&= \|y^k - x^*\|^2 - \frac{(1-\alpha_k)}{\alpha_k}\|x^{k+1} - y^k\|^2. \qquad (28)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\|x^{k+1} - y^k\|^2 &= \|x^{k+1} - x^k - \theta_k(x^k - x^{k-1})\|^2 \\
&= \|x^{k+1} - x^k\|^2 + \theta_k^2\|x^k - x^{k-1}\|^2 - 2\theta_k\langle x^{k+1} - x^k, x^k - x^{k-1}\rangle \\
&\geq \|x^{k+1} - x^k\|^2 + \theta_k^2\|x^k - x^{k-1}\|^2 - 2\theta_k\|x^{k+1} - x^k\|\|x^k - x^{k-1}\| \\
&\geq (1-\theta_k)\|x^{k+1} - x^k\|^2 + (\theta_k^2 - \theta_k)\|x^k - x^{k-1}\|^2. \qquad (29)
\end{aligned}
$$

Using (26) and (29) with (28), we get

$$\begin{aligned}
\|x^{k+1}-x^*\|^2 &\le (1+\theta_k)\|x^k-x^*\|^2-\theta_k\|x^{k-1}-x^*\|^2\\
&\quad+\theta_k(1+\theta_k)\|x^k-x^{k-1}\|^2-\frac{(1-\alpha_k)}{\alpha_k}\Big[(1-\theta_k)\|x^{k+1}-x^k\|^2\\
&\quad+(\theta_k^2-\theta_k)\|x^k-x^{k-1}\|^2\Big]\\
&=(1+\theta_k)\|x^k-x^*\|^2-\theta_k\|x^{k-1}-x^*\|^2\\
&\quad-\frac{(1-\alpha_k)}{\alpha_k}(1-\theta_k)\|x^{k+1}-x^k\|^2\\
&\quad+\Big[\theta_k(1+\theta_k)-\frac{(1-\alpha_k)}{\alpha_k}(\theta_k^2-\theta_k)\Big]\|x^k-x^{k-1}\|^2\\
&=(1+\theta_k)\|x^k-x^*\|^2-\theta_k\|x^{k-1}-x^*\|^2\\
&\quad-\rho_k\|x^{k+1}-x^k\|^2+\sigma_k\|x^k-x^{k-1}\|^2, \qquad (30)
\end{aligned}$$

where $\rho_k := \frac{(1-\alpha_k)}{\alpha_k}(1-\theta_k)$ and $\sigma_k := \theta_k(1+\theta_k)-\frac{(1-\alpha_k)}{\alpha_k}(\theta_k^2-\theta_k)$.

Let $\Gamma_k := \|x^k-x^*\|^2-\theta_k\|x^{k-1}-x^*\|^2+\sigma_k\|x^k-x^{k-1}\|^2$. Then we obtain from (30) that

$$\begin{aligned}
\Gamma_{k+1}-\Gamma_k &= \|x^{k+1}-x^*\|^2-(1+\theta_{k+1})\|x^k-x^*\|^2\\
&\quad+\theta_k\|x^{k-1}-x^*\|^2+\sigma_{k+1}\|x^{k+1}-x^k\|^2-\sigma_k\|x^k-x^{k-1}\|^2\\
&\le \|x^{k+1}-x^*\|^2-(1+\theta_k)\|x^k-x^*\|^2\\
&\quad+\theta_k\|x^{k-1}-x^*\|^2+\sigma_{k+1}\|x^{k+1}-x^k\|^2-\sigma_k\|x^k-x^{k-1}\|^2\\
&\le -(\rho_k-\sigma_{k+1})\|x^{k+1}-x^k\|^2. \qquad (31)
\end{aligned}$$

Since $0\le\theta_k\le\theta_{k+1}<\theta$, we have

$$\begin{aligned}
\rho_k-\sigma_{k+1} &= \frac{(1-\alpha_k)}{\alpha_k}(1-\theta_k)-\theta_{k+1}(1+\theta_{k+1})\\
&\quad+\frac{(1-\alpha_{k+1})}{\alpha_{k+1}}(\theta_{k+1}^2-\theta_{k+1})\\
&\ge \frac{(1-\alpha_k)}{\alpha_k}(1-\theta_{k+1})-\theta_{k+1}(1+\theta_{k+1})\\
&\quad+\frac{(1-\alpha_{k+1})}{\alpha_{k+1}}(\theta_{k+1}^2-\theta_{k+1})\\
&\ge \epsilon(1-\theta)-\theta(1+\theta)+\epsilon(\theta^2-\theta)\\
&= \epsilon-2\epsilon\theta-\theta-\theta^2+\epsilon\theta^2\\
&= -(1-\epsilon)\theta^2-(1+2\epsilon)\theta+\epsilon. \qquad (32)
\end{aligned}$$

Combining (31) and (32), we get

$$\Gamma_{k+1} - \Gamma_k \leq -\delta\|x^{k+1} - x^k\|^2, \tag{33}$$

where $\delta := -(1-\epsilon)\theta^2 - (1+2\epsilon)\theta + \epsilon$. Therefore, $\Gamma_{k+1} \leq \Gamma_k$. Hence $\{\Gamma_k\}$ is nonincreasing. Furthermore,

$$\begin{aligned}\Gamma_k &= \|x^k - x^*\|^2 - \theta_k\|x^{k-1} - x^*\|^2 + \sigma_n\|x^k - x^{k-1}\|^2 \\ &\geq \|x^k - x^*\|^2 - \theta_k\|x^{k-1} - x^*\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x^k - x^*\|^2 &\leq \theta_k\|x^{k-1} - x^*\|^2 + \Gamma_k \\ &\leq \theta\|x^{k-1} - x^*\|^2 + \Gamma_1 \\ &\vdots \\ &\leq \theta^k\|x^0 - x^*\|^2 + \Gamma_1(\theta^{k-1} + \theta^{k-2} + \ldots + 1) \\ &\leq \theta^k\|x^0 - x^*\|^2 + \frac{\Gamma_1}{1-\theta} \qquad (34)\end{aligned}$$

and it can also be seen that

$$\begin{aligned}-\theta\|x^{k-1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - \theta\|x^{k-1} - x^*\|^2 \\ &\leq \Gamma_k \leq \Gamma_1. \qquad (35)\end{aligned}$$

Note that

$$\begin{aligned}\Gamma_{k+1} &= \|x^{k+1} - x^*\|^2 - \theta_{k+1}\|x^k - x^*\|^2 + \sigma_{k+1}\|x^{k+1} - x^k\|^2 \\ &\geq -\theta_{k+1}\|x^k - x^*\|^2. \qquad (36)\end{aligned}$$

Using (34) and (36), we get

$$\begin{aligned}-\Gamma_{k+1} &\leq \theta_{k+1}\|x^k - x^*\|^2 \leq \theta\|x^k - x^*\|^2 \\ &\leq \theta^{k+1}\|x^0 - x^*\|^2 + \frac{\theta\Gamma_1}{1-\theta}.\end{aligned}$$

By (33), we have

$$\delta\|x^{k+1}-x^k\|^2 \le \Gamma_k - \Gamma_{k+1},$$

and so by (34) and (35), we get

$$\begin{aligned}\delta\sum_{j=1}^{k}\|x^{j+1}-x^j\|^2 &\le \Gamma_1 - \Gamma_{k+1}\\ &\le \Gamma_1 + \theta\|x^k-x^*\|^2\\ &\le \Gamma_1 + \theta^{k+1}\|x^0-x^*\|^2 + \frac{\theta\Gamma_1}{1-\theta}\\ &= \theta^{k+1}\|x^0-x^*\|^2 + \frac{\Gamma_1}{1-\theta}.\end{aligned}$$

This shows that

$$\sum_{k=1}^{\infty}\|x^{k+1}-x^k\|^2 < \infty$$

and hence also,

$$\sum_{k=1}^{\infty}\theta_k\|x^{k+1}-x^k\|^2 < \infty.$$

From the above, we deduce that $\lim_{k\to\infty}\|x^{k+1}-x^k\| = 0$. Following the same arguments that derived (17) in the proof of Theorem 3.1, we obtain

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\le (1-\alpha_k(1-\beta))\|y^k-x^*\|^2\\ &\le (1-\alpha_k(1-\beta))(\|x^k-x^*\| + \theta_k\|x^k-x^{k-1}\|)^2\\ &\le (1-\alpha_k(1-\beta))(\|x^k-x^*\|^2 + 2\theta_k\|x^k-x^*\|\|x^k-x^{k-1}\|\\ &\quad +\theta_k\|x^k-x^{k-1}\|^2).\end{aligned} \tag{37}$$

Since $2\alpha_k(1-\beta)\ge 1$, it implies that

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\le (1-\alpha_k(1-\beta))\|x^k-x^*\|^2\\ &\quad +2(1-\alpha_k(1-\beta))\theta_k\|x^k-x^*\|\|x^k-x^{k-1}\| + \theta_k\|x^k-x^{k-1}\|^2\\ &\le (1-\alpha_k(1-\beta))\|x^k-x^*\|^2\\ &\quad +2\alpha_k(1-\beta)\theta_k\|x^k-x^*\|\|x^k-x^{k-1}\| + \theta_k\|x^k-x^{k-1}\|^2.\end{aligned} \tag{38}$$

Observe that

$$\begin{aligned}\limsup_{k\to\infty}(2\alpha_k(1-\beta)\theta_k\|x^k-x^*\|\|x^k-x^{k-1}\|) &= \lim_{k\to\infty}(2\alpha_k(1-\beta)\theta_k\|x^k-x^*\|\|x^k-x^{k-1}\|)\\ &= 0,\end{aligned}$$

since $\{x^k\}$ is bounded and $\lim_{n\to\infty} \|x^{k+1} - x^k\| = 0$. Applying Lemma 2.3 to (38), we get that $x^k \to x^*, k \to \infty$ and the desired result is obtained. □

We discuss some relationships between the sets of assumptions of $\{\theta_k\}$ given in Theorem 3.1 and Theorem 3.3 in the following remark.

Remark 3.4 1. One can see from the choices of $\{\theta_k\}$ in Theorem 3.1 (see (11)) and Theorem 3.3 that the two choices of $\{\theta_k\}$ are independent of each other. For example, when $x^k = x^{k-1}$, one can see from (11) that

$$\theta_k \leq \theta_{k+1} \quad \text{but} \quad \theta_{k+1} \nleq \frac{\sqrt{1+8\epsilon} - 1 - 2\epsilon}{2(1-\epsilon)}$$

for all $k \geq 1$ and $\epsilon \in]0,1[$. This negates the second assumption in Theorem 3.3.

2. Also, from the proof of Theorem 3.3, we can see that $\sum_{k=1}^{\infty} \theta_k \|x^{k+1} - x^k\|^2 < \infty$ while in Theorem 3.1, we have $\sum_{k=1}^{\infty} \theta_k \|x^{k+1} - x^k\| < \infty$.

4 Numerical Experiments

In this section, we compare the performances of our proposed scheme (Algorithm 3.1) with those of Algorithms 1 and 2 proposed in [10], and the extragradient method studied in [5].

We choose to use the test problem library QVILIB taken from [37]; the feasible map K is assumed to be given by $K(x) := \{z \in \mathbb{R}^n \, : \, g(z,x) \leq 0\}$. We implemented Algorithm 3.1 in Matlab/Octave. We implemented the projection over a convex set as the solution of a convex program. We considered the following performance measures for optimality and feasibility

$$\text{opt}(x) := -\min_z \{\mathcal{F}(x)^T(z-x) \, : \, z \in K(x)\}, \;\; \text{feas}(x) := \|\max\{0, g(x,x)\}\|_\infty.$$

A point x^* is considered as a solution of the QVI if opt(x^*) $\leq$1e-3 and feas(x^*) $\leq$1e-3. As a nonlinear programming solver, we used the built-in function `sqp` with `maxiter` = 1000.

All the experiments were carried out on an Intel Core i7-4702MQ CPU @ 2.20GHz x 8 with Ubuntu 14.04 LTS 64-bit and by using GNU Octave version 3.8.1.

In Table 1 the results of Algorithm 3.1 with $\gamma = 0.5$ and constant sequences $\{\alpha_k = \alpha\}$ and $\{\theta_k = \theta\}$ are presented. Different values for α and θ are considered, and we also report in Table 1 the number of iterations the algorithm requires to reach the stopping criterion. Failure is reported in case that the algorithm does not converge within 5000 iterations. We remark that when $\alpha = 1$ and $\theta = 0$ we obtain the classical gradient projection algorithm, and in general when $\theta = 0$ [10, Algorithm 1] is obtained.

It can be seen in Table 1 that small values of α affects the robustness of the algorithm. Specifically, we observe only 5 successes when $\alpha = 1$, 20 successes when $\alpha = 0.2$, and 22-24 successes when $\alpha = 0.05$. On the other hand, these results also show that the inertial step can significantly improve the performances of the projected gradient method. In Table 1 we report the performance measure iter./success that gives the average number of iterations needed to get a run successfully solved. These results clearly show that the algorithm with the inertial parameter $\theta = 0.75$ outperforms the case with $\alpha = 0$, i.e. Algorithm 1 in [10].

Table 1 Numerical results of Algorithm 3.1 with $\gamma = 0.5$: number of iterations needed for satisfying the stopping criterion.

α	1	0.2	0.2	0.2	0.2	0.2	0.05	0.05	0.05	0.05	0.05
θ	0	0	0.25	0.5	0.75	1	0	0.25	0.5	0.75	1
OutZ40-1	fail.	460	343	223	88	320	1854	1389	919	438	1116
OutZ40-2	fail.	454	340	222	90	348	1825	1368	908	439	1080
OutZ40-3	fail.	429	321	213	89	398	1718	1288	857	420	660
OutZ41-1	123	411	310	207	89	269	1638	1229	822	410	1508
OutZ41-2	115	417	312	205	82	184	1675	1256	834	403	1147
OutZ41-3	122	442	331	217	87	245	1776	1331	884	427	1066
OutZ42-1	fail.	50	36	25	30	33	206	154	98	46	135
OutZ42-2	fail.	44	32	22	29	60	184	137	88	41	205
OutZ42-3	fail.	37	27	19	14	53	153	114	73	37	150
OutZ42-4	fail.	37	26	15	25	45	159	118	73	45	216
OutZ43-1	fail.	32	23	17	24	34	134	100	63	45	142
OutZ43-2	fail.	27	19	14	22	33	112	83	52	40	115
OutZ43-3	fail.	19	13	6	16	21	79	59	36	25	66
OutZ44-1	fail.	33	23	18	24	39	137	102	64	46	106
OutZ44-2	fail.	27	19	15	23	38	115	86	53	29	56
OutZ44-3	fail.	21	15	13	11	27	89	66	41	25	77
MovSet1A-1	fail.	46	33	18	21	61	192	143	90	35	194
MovSet1A-2	fail.	63	45	21	41	56	267	198	125	50	496
MovSet2A-1	33	42	30	16	20	25	178	132	83	32	163
MovSet2A-2	42	59	42	28	39	76	252	187	117	61	418
Box1A-1	fail.	fail.	fail.	fail.	fail.	fail.	150	112	71	75	fail.
Box1A-2	fail.	fail.	fail.	fail.	fail.	fail.	244	181	114	81	fail.
BiLin1A-1	fail.	fail.	fail.	fail.	fail.	fail.	131	96	63	36	123
BiLin1A-2	fail.	fail.	fail.	fail.	fail.	fail.	218	160	100	44	246
#success	5	20	20	20	20	20	24	24	24	24	22
iter./success	87	157.5	117	76.7	43.2	118.25	561.92	420.38	276.17	138.75	431.14

In Table 2 we compare our algorithm's performances with $\theta = 0.75$ and [10, Algorithm 2]. Recall that [10, Algorithm 2] requires an additional orthogonal projection onto a closed and convex set per each iteration compared to Algorithm 3.1, i.e., one additional convex program needs to be solved. Following this reason, we report in Table 2 the time in seconds needed for the algorithms to reach the stopping criterion. The number of iterations of [10, Algorithm 2] is however reported in brackets. Hence, clearly, the performance of Algorithm 3.1 is much better than [10, Algorithm 2] .

Table 2 Comparison between Algorithm 2 in [10] and Algorithm 3.1 with $\gamma = 0.5$: time in seconds to satisfy the stopping criterion. (For Alg2 [10] in brackets the number of iterations.)

	$\alpha = 0.2$		$\alpha = 0.05$	
	Alg2 [10]	Alg. 3.1 with $\theta = 0.75$	Alg2 [10]	Alg. 3.1 with $\theta = 0.75$
OutZ40-1	152.1 (392)	17.3	652.6 (1777)	95.1
OutZ40-2	159.2 (384)	17.7	646.3 (1746)	97.9
OutZ40-3	139.9 (360)	18.2	606.2 (1640)	94.6
OutZ41-1	98.6 (340)	14.0	448.0 (1558)	72.3
OutZ41-2	103.0 (352)	13.0	464.7 (1602)	69.3
OutZ41-3	112.1 (374)	13.9	492.4 (1698)	75.5
OutZ42-1	5.7 (47)	1.5	25.2 (203)	4.3
OutZ42-2	4.4 (42)	1.4	19.2 (182)	3.4
OutZ42-3	2.7 (36)	0.7	11.6 (152)	2.3
OutZ42-4	1.2 (39)	0.9	5.4 (160)	1.2
OutZ43-1	3.0 (33)	0.8	12.7 (135)	2.5
OutZ43-2	1.5 (28)	0.7	7.1 (113)	1.8
OutZ43-3	0.3 (20)	0.2	1.7 (81)	0.2
OutZ44-1	3.7 (33)	1.0	16.4 (137)	3.0
OutZ44-2	2.4 (28)	0.8	10.3 (116)	2.2
OutZ44-3	0.5 (23)	0.1	2.5 (91)	0.5
MovSet1A-1	6.3 (45)	1.7	26.9 (191)	3.4
MovSet1A-2	9.1 (62)	3.3	40.2 (265)	5.6
MovSet2A-1	5.2 (42)	1.7	27.0 (178)	3.3
MovSet2A-2	9.5 (59)	3.3	40.6 (252)	5.9
Box1A-1	fail.	fail.	14.1 (149)	6.1
Box1A-2	fail.	fail.	17.0 (242)	5.9
BiLin1A-1	fail.	fail.	fail.	1.7
BiLin1A-2	fail.	fail.	fail.	2.2
#success	20	20	22	24
seconds/success	41.1	5.6	163.1	23.3

We also tested the extragradient method proposed in [5] on the same benchmark problems but we do not report it here since the algorithm does not work with the stepsizes considered for the other methods, and is slow with smaller stepsizes.

The experiments made here clearly show how the usage of α makes Algorithm 3.1 robust, while the inertial parameter θ can be used to speed up the method. Relying on these considerations, we tested Algorithm 3.1 with diminishing sequences $\{\alpha_k\}$ and $\{\theta_k\}$. Specifically, we set $\alpha_0 = 0.5$, $\theta_0 = 1$, and $\alpha_{k+1} = 0.99\alpha_k$, $\theta_{k+1} = 0.99\theta_k$ for $k \geq 0$. Performances of this version of the algorithm can be found in Table 3. Comparing these performances with those reported in Tables 1 and 2, implies that this variant is the fastest one among those that can solve all the test problems.

We also report in Table 3 the performances of Algorithm 3.1 with the updating rule from (11). For this algorithm we set $\theta_k = \bar{\theta}_k$, $\eta = 3$, and $\{\epsilon_k\}$ such that $\epsilon_0 = 1$ and $\epsilon_{k+1} = 0.99\epsilon_k$.

Table 3 Performances of Algorithm 3.1 with diminishing rule and $\gamma = 0.5$, and with updating rule (11), $\alpha_k = 0.2$, or 0.05, and $\gamma = 0.5$: time in seconds to satisfy the stopping criterion and, in brackets, the number of iterations.

	diminishing rule	rule (11) and $\alpha_k = 0.2$	rule (11) and $\alpha_k = 0.05$
OutZ40-1	39.6 (190)	29.1 (124)	38.9 (202)
OutZ40-2	21.1 (103)	24.8 (100)	57.1 (249)
OutZ40-3	7.3 (39)	24.0 (100)	58.6 (249)
OutZ41-1	20.9 (144)	9.5 (83)	36.5 (293)
OutZ41-2	12.6 (90)	12.3 (83)	19.7 (167)
OutZ41-3	15.1 (107)	11.5 (83)	37.6 (323)
OutZ42-1	1.2 (32)	1.8 (25)	4.9 (94)
OutZ42-2	1.2 (32)	2.2 (37)	4.8 (105)
OutZ42-3	1.5 (33)	1.8 (29)	3.9 (88)
OutZ42-4	1.6 (33)	0.3 (14)	1.6 (58)
OutZ43-1	0.7 (45)	1.2 (26)	4.1 (78)
OutZ43-2	0.6 (58)	0.6 (13)	2.5 (51)
OutZ43-3	0.4 (45)	0.1 (12)	0.5 (37)
OutZ44-1	2.1 (78)	1.3 (15)	4.2 (78)
OutZ44-2	1.4 (73)	1.1 (19)	2.5 (40)
OutZ44-3	1.5 (77)	0.2 (12)	0.6 (37)
MovSet1A-1	1.7 (23)	2.9 (23)	5.8 (62)
MovSet1A-2	2.5 (32)	4.4 (44)	14.7 (161)
MovSet2A-1	1.6 (21)	2.5 (22)	2.8 (31)
MovSet2A-2	2.4 (29)	5.4 (51)	15.5 (157)
Box1A-1	5.1 (148)	fail.	45.5 (457)
Box1A-2	5.3 (148)	fail.	64.7 (698)
BiLin1A-1	4.2 (219)	fail.	3.3 (44)
BiLin1A-2	4.2 (219)	fail.	5.3 (97)
#success	24	20	24
iter./success	84.08	45.75	160.67
seconds/success	6.5	6.8	18.2

5 Conclusions

In this paper, we propose a generalized gradient-type method with inertial extrapolation step for solving QVIs in real Hilbert spaces and obtained strong convergence results with different updating rules. Throughout this paper, we do not assume that the set-valued mapping $K(\cdot)$ is of the form $K(u) = K + m(u)$ for all $u \in \mathcal{H}$, as commonly assumed in most previously published papers in this subject area. Our numerical experiments show that our suggested method outperforms most of the recently proposed gradient-type methods for solving QVIs in real Hilbert spaces when $\mathcal{F}$ is strongly monotone and Lipschitz continuous.

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