An Efficient Projection-type Method for Monotone Variational Inequalities in Hilbert Spaces

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Abstract

We consider the monotone variational inequality problem in a Hilbert space and describe a projection-type method with inertial terms under the following properties: (a) The method generates a strongly convergent iteration sequence; (b) The method requires, at each iteration, only one projection onto the feasible set and two evaluations of the operator; (c) The method is designed for variational inequality for which the underline operator is monotone and uniformly continuous; (d) The method includes an inertial term. The latter is also shown to speed up the convergence in our numerical results. A comparison with some related methods is given and indicates that the new method is promising.

1 Introduction

Let H be a real Hilbert space with scalar product $\langle ., . \rangle$ and the norm $\|.\|$. Suppose C is a nonempty, closed and convex subset of H and $A : C \to H$ be a continuous mapping. In this paper, we consider the following variational inequality (for short, VI(A, C)): find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \forall y \in C. \tag{1}$$

Let SOL denote the solution set of VI(A, C) (1). It is well known that x solves the VI(A, C) (1) if and only if x solves the fixed point equation (see [20] for the details)

$$x = P_C(x - \gamma Ax), \gamma > 0 \quad \text{and} \quad r_\gamma(x) := x - P_C(x - \gamma Ax) = 0.$$

Therefore, the knowledge of fixed-point algorithms (see, for example, [19, 45]) can be used to solve VI(A, C) (1).

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Variational inequality theory is an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework (see, for example, [6, 7, 20, 29, 31]) and several numerical methods have been developed for solving it (see, e.g., [8, 19, 31] and the references therein).
The extragradient method, introduced in 1976 by Korpelevich [30], which is given by

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \gamma A x_n) \\ x_{n+1} = P_C(x_n - \gamma A y_n), \quad n \geq 1, \end{cases} \tag{2}$$

where $\gamma \in (0, \frac{1}{L})$ for a finite-dimensional space, provides an iterative process converging to a solution of VI(A, C) (1) by assuming that $A : C \to \mathbb{R}^n$ is monotone and L-Lipschitz continuous. The extragradient method was further extended to infinite dimensional spaces by many authors; see for instance, [2, 15, 16, 23, 25, 26, 44, 51, 49, 50, 53]. In the setting of Hilbert spaces, this method obtains only weak convergence. Furthermore, it is easy to see that the extragradient method of Korpelevich needs two projections onto the set C and two values of A per iteration. A crucial feature regarding the design of numerical methods related to extragradient method is to minimize the number of evaluation of P_C per iteration. So the extragradient method needs to be improved in situations, where a projection onto C is hard to evaluate or computationally expensive. Several alternatives to the extragradient method or its modifications have also been proposed in the literature by several authors (see, for example, [17, 33, 42, 48, 53]).

Recently, Malitsky and Semenov [41] obtained strong convergence result when there is only one projection onto the feasible set C per iteration using the method of Haugazeau when A is monotone and L-Lipschitz continuous with constant step size. Similarly, Kraikaew and Saejung [35] obtained strong convergence result using a combination of Halpern iterative scheme and subgradient extragradient method in real infinite dimensional Hilbert spaces. More recently, Mainge and Gobinddass [36] (see also Mainge [37]) obtained weak convergence result for solving the VI(A, C) (1) in real Hilbert spaces with monotone and L-Lipschitz continuous mapping A, by means of a projected reflected gradient-type method [40] and inertial terms.

It is well known that one the main features of the extragradient method (2) and other related methods mentioned above is that they are explicit methods, hence easily implementable. As such, it is quite important to pay attention to computational issues, e.g., stepsizes. The extragradient method is an extension of the projected gradient method, with an additional step which makes it convergent under plain monotonicity of the operator, rather than strong monotonicity. Now, even for the finite dimensional unconstrained optimization case ($C = \mathbb{R}^n, A = \nabla f$ for a convex $f : \mathbb{R}^n \to \mathbb{R}$) it is well known that the use of exogenous stepsizes tending to zero and with sizes bounded by the inverse of the Lipschitz constant gives rise to a sublinear convergence rate and is quite inefficient, among other things because in most cases a global Lipschitz constant (if it indeed exists) cannot be accurately estimated, and estimates usually overestimate it, resulting in too small stepsizes. In the case of the gradient method, this obstacle was removed through the introduction of a linesearch allowing for larger stepsizes, e.g. in [34] and [26]. These linesearches were later incorporated to more general variants of the method, like the algorithm in [25]. The method proposed in [35] improves over [25] in the addition of the Halpern's regularization step which allows for strong convergence, but on the other hand it sticks to the inefficient stepsizes bounded in terms of the Lipschitz constant.

Recently, there have been increasing interests in studying inertial type algorithms. For example, inertial forward-backward splitting methods [5, 32, 46], inertial Douglas-Rachford splitting method [11], inertial ADMM [12, 18], and inertial forward-backward-forward method [13]. The inertial term is based upon a discrete version of a second order dissipative dynamical system [3, 4] and can be regarded as a procedure of speeding up the convergence properties. The results in [1, 10, 12, 32, 38, 39, 46, 47] and other related ones analyzed the convergence properties of inertial type algorithms and demonstrated their performance numerically on some imaging and data analysis problems.

The aim of this paper is to present an projection-type method for the solution of a monotone and uniformly continuous variational inequality with the following properties:

(a) The iterates converge strongly to a solution of the $\mathrm{VI}(A, C)$ (1);

(b) The method requires, at each iteration, only one projection onto C and two evaluations of A.

(c) The method includes an inertial term.

To the best of our knowledge, it is the first method which has these three properties in an infinite-dimensional Hilbert space setting. In order to get properties (a) and (b), most existing methods require two or more projections onto C (see, for example, [25, 28, 43]). As we have observed earlier, the inertial term is generally believed to speed up the convergence of an iterative scheme, though a formal proof seems to be known only for optimization problems, but numerical evidence indicates that a suitable choice of this inertial term indeed improves the computational behaviour of the underlying method. Hence we believe that property (c) is important. It complicates some of the proofs, and most papers dealing with inertial terms prove weak convergence only. The only exception seems to be the recent paper [38] for certain fixed point problems, whose specification to variational inequalities, however, needs either stronger assumptions regarding A or two projections onto C.

The paper is therefore organized as follows: We first recall some basic definitions and results in Section 2. Some discussions about our projection-type method used in this paper are given in Section 3. The strong convergence of our Algorithm 3.3 is then investigated in Section 4. Some numerical experiments can be found in Section 5. We conclude with some final remarks in Section 6.

2 Preliminaries

This section contains some definitions and basic results that will be used in our subsequent analysis. Some elementary properties of real Hilbert spaces are summarized in the following result.

Lemma 2.1. *The following statements hold in any real Hilbert space H:*

(a) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$.

(b) $2\langle x-y, x-z\rangle = \|x-y\|^2 + \|x-z\|^2 - \|y-z\|^2$ *for all* $x, y, z \in H$.

Definition 2.2. *A mapping $A : C \to H$ is called*

(a) monotone *on X if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$;*

(b) η-strongly monotone *on C if there exists a constant $\eta > 0$ such that*

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x-y\|^2, \ \forall x, y \in C;$$

(c) Lipschitz continuous *on C if there exists a constant $L > 0$ such that*

$$\|Ax - Ay\| \leq L\|x-y\|, \ \forall x, y \in C.$$

A variational inequality defined by a monotone and continuous operator has the nice property that its solution set is closed and convex (see, for example, Theorem 1 of [52]).

Lemma 2.3. *Let $C \subseteq H$ be a nonempty, closed, and convex subset of a real Hilbert space H, and let $A : H \to H$ be continuous and monotone on C. Then the solution set of the variational inequality VI(A, C) is closed and convex (possibly empty).*

We next recall some properties of the projection. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \ \ \forall x, y, \in H. \tag{3}$$

In particular, we get from (3) that

$$\langle x - y, x - P_C y\rangle \geq \|x - P_C y\|^2, \quad \forall x \in C, y \in H. \tag{4}$$

Furthermore, $P_C x$ is characterized by the properties

$$P_C x \in C \quad \text{and} \quad \langle x - P_C x, P_C x - y\rangle \geq 0, \ \forall y \in C. \tag{5}$$

This characterization implies that

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \ \ \forall x \in H, \forall y \in C. \tag{6}$$

Recall that the solution set SOL of a variational inequality is closed and convex under the assumptions of Lemma 2.3. Therefore, if we assume, in addition, that SOL is nonempty, the projection onto SOL is well-defined. Hence, we can formulate the following result that will be used to prove our strong convergence theorem.

Lemma 2.4. *Let $S \subseteq H$ be a nonempty, closed, and convex subset of a real Hilbert space H. Let $u \in H$ be arbitrarily given, $z := P_S u$, and $\Omega := \{x \in H : \langle x-u, x-z \rangle \leq 0\}$. Then $\Omega \cap S = \{z\}$.*

Proof. By definition, it follows immediately that $z \in \Omega \cap S$. Conversely, take an arbitrary $y \in \Omega \cap S$. Then, in particular, we have $y \in \Omega$, and it therefore follows that

$$\begin{aligned} \|y-z\|^2 &= \langle y-z, y-z \rangle \\ &= \langle y-z, y-u \rangle + \langle y-z, u-z \rangle \\ &\leq \langle y-z, u-z \rangle. \end{aligned} \tag{7}$$

Using $z = P_S u$ together with the characterization (5), we also have

$$\langle u-z, z-x \rangle \geq 0 \quad \forall x \in S.$$

In particular, since $y \in S$, we therefore have $\langle u-z, z-y \rangle \geq 0$. Hence (7) implies $\|y-z\|^2 \leq 0$, so that $y = z$. This completes the proof. □

The following lemma was stated in [25, Prop. 2.11], see also [27, Prop. 4].

Lemma 2.5. *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then $A(M)$ is bounded.*

Lemma 2.6. *([22]) Let C be a nonempty closed and convex subset of H. Let h be a real-valued function on H and define $K := \{x \in H : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then*

$$\operatorname{dist}(x, K) \geq \theta^{-1} \max\{h(x), 0\}, \ \forall x \in C,$$

where $\operatorname{dist}(x, K)$ denotes the distance function from x to K.

Lemma 2.7. *Let C be a nonempty closed and convex subset of H, $y := P_C(x)$ and $x^* \in C$. Then*

$$\|y - x^*\|^2 \leq \|x - x^*\|^2 - \|x - y\|^2. \tag{8}$$

We finally restate a result which essentially states the equivalence between a primal and a weak form of variational inequality for continuous, monotone operators as given in [54, Lem. 7.1.7].

Lemma 2.8. *Let C be a nonempty, closed, and convex subset of H. Let $A : C \to H$ be a continuous, monotone mapping and $z \in C$. Then*

$$z \in \mathrm{SOL} \iff \langle Ax, x-z \rangle \geq 0 \quad \textit{for all } x \in C.$$

3 Projection-type Method with Inertial

In this section, we give a precise statement of our projection-type method with inertial terms and discuss some of its elementary properties. Its convergence analysis is postponed to the next section. We first state the assumptions that we will assume to hold through the rest of this paper.

Assumption 3.1. Suppose that the following hold:

(a) The feasible set C is a nonempty, closed, and affine subset of the real Hilbert space H.

(b) $A : C \to H$ is monotone and uniformly continuous on bounded subsets of H.

(c) The solution set SOL of VI(A, C) (1) is nonempty.

We next give the conditions which must be satisfied by our sequence of parameters in our proposed method.

Assumption 3.2. The sequences $\{\alpha_n\}$ and $\{\theta_n\}$ satisfy the following conditions:

(a) $\{\alpha_n\} \subset (0, 1]$ is non-increasing with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(b) $\{\theta_n\}$ is non-decreasing with $\theta_n \in [0, \theta]$ for all $n \in \mathbb{N}$ for some $\theta \in [0, 1/3)$.

Throughout this paper, we use the abbreviation

$$r(x) := x - P_C(x - Ax), x \in H$$

for the residual. Observe that if we take $y = x - Ax$ in (4), then we have

$$\langle Ax, r(x)\rangle \geq \|r(x)\|^2, \ \forall x \in C. \tag{9}$$

We next give a precise statement of our projection-type method.

Algorithm 3.3. *(Projection-type Method with Inertial)*

(S.0) *Choose sequences* $\{\alpha_n\}$ *and* $\{\theta_n\}$ *such that the conditions from Assumption 3.2 hold,* $\sigma \in (0,1)$, $\gamma \in (0,1)$. *Let* $x_0, x_1 \in H$ *be given starting points, and set* $n := 1$.

(S.1) *Compute*

$$\begin{aligned} w_n &:= \alpha_n x_0 + (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}), \\ z_n &:= P_C(w_n - Aw_n). \end{aligned}$$

(S.2) *If* $r(w_n) = w_n - z_n = 0$*: STOP. Otherwise*

(S.3) *Compute* $y_n = w_n - \gamma^{k_n} r(w_n)$, *where* k_n *is the smallest nonnegative integer satisfying*

$$\langle Ay_n, r(w_n)\rangle \geq \frac{\sigma}{2}\|r(w_n)\|^2. \tag{10}$$

Set $\eta_n := \gamma^{k_n}$.

(S.4) *Compute*

$$x_{n+1} = P_{C_n}(w_n), \tag{11}$$

where $C_n = \{x \in H : h_n(x) \leq 0\}$ *and*

$$h_n(x) := \langle Ay_n, x - y_n\rangle. \tag{12}$$

(S.5) *Set* $n \leftarrow n+1$, *and go to (S.1).*

Before we investigate the convergence properties of Algorithm 3.3, we first summarize a number of simple observations.

Remark 3.4. (a) Throughout our convergence analysis, we always assume implicitly that $w_n \neq z_n$ so that Algorithm 3.3 does not terminate after finitely many iterations.

(b) The termination test in (S.2) is justified by the following observation: If $w_n = z_n$, we have $w_n = P_C(w_n - \lambda_n Aw_n)$, hence the fixed-point characterization of a solution of VI(A,C) (1) implies that w_n is already a solution of the variational inequality. Furthermore, our subsequent convergence analysis will show that $\|w_n - z_n\| \to 0$ for $n \to \infty$, which justifies our stopping criterion. On the other hand, it is easy to see that the test from (S.2) can be replaced by a number of other suitable criteria.

(c) In general, Algorithm 3.3 requires two starting points $x_0, x_1 \in H$. This comes from the particular recursion for the vector w_n for $n = 1$. On the other hand, if we take $\theta_1 = 0$ (this choice is explicitly allowed), then only one starting point $x_1 \in H$ is needed.

(d) Geometrically, the set C_n is describes a half-space and there is a simple analytic expression for the projection onto C_n, meaning that x_{n+1} can easily be computed by

$$x_{n+1} := \begin{cases} w_n - \frac{\langle Ay_n, w_n - y_n\rangle}{\|Ay_n\|^2} Ay_n, & \text{if } \langle Ay_n, w_n - y_n\rangle > 0, \\ w_n, & \text{if } \langle Ay_n, w_n - y_n\rangle \leq 0, \end{cases}$$

see, e.g., [14]. Hence the main effort at each iteration of Algorithm 3.3 is one projection onto C and two evaluations of the operator A to get Aw_n and Ay_n. Therefore, the effort per iteration is even less than for the original (and only weakly convergent) extragradient method which requires two projections onto C and two evaluations of A.

(e) Our Algorithm 3.3 is much more applicable than the proposed methods in [15, 16, 24, 36, 40, 41, 44, 53] because the Lipschitz constant of A or an estimate of it is needed in order to implement the proposed methods in these papers. Neither the Lipschitz constant of A nor its estimate is needed during implementation of our Algorithm 3.3 and A is not even required to be Lipschitz continuous. Hence, our Algorithm 3.3 is applicable for a much more general class of *monotone and uniformly continuous mapping A.* $\Diamond$

Remark 3.5. Using the fact that A is continuous and (9), we can see that Step (S.3) in Algorithm 3.3 is well-defined. Furthermore, if SOL $\neq \emptyset$, the Step (S.4) is well-defined since SOL $\subset C_n$ by the lemma below and hence $C_n \neq \emptyset$ for all $n \in \mathbb{N}$. $\Diamond$

Lemma 3.6. *Let $x^* \in SOL$ and the function h_n be defined by* (12). *Then*

$$h_n(w_n) \geq \frac{\sigma\eta_n}{2}\|w_n - z_n\|^2$$

and $h_n(x^*) \leq 0$. In particular, if $w_n \neq z_n$, then $h_n(w_n) > 0$.

Proof. Since $y_n = w_n - \eta_n(w_n - z_n)$, using (10) we have

$$\begin{aligned} h_n(w_n) &= \langle Ay_n, w_n - y_n\rangle \\ &= \eta_n\langle Ay_n, w_n - z_n\rangle \geq \eta_n\frac{\sigma}{2}\|w_n - z_n\|^2 \geq 0. \end{aligned}$$

If $w_n \neq z_n$, then $h_n(w_n) > 0$. Since $x^* \in$ SOL, we have

$$\langle Ax^*, y - x^*\rangle \geq 0,\ \forall y \in C,$$

and thus implies by Lemma 2.8 that $h_n(x^*) = \langle Ay_n, x^* - y_n\rangle \leq 0$. $\square$

4 Convergence Analysis

Here using the idea of proof in [38], we show that Algorithm 3.3 generates a sequence $\{x_n\}$ which converges strongly to a solution of the underlying variational inequality VI(A, C) (1) under the Assumptions 3.1 and 3.2. To this end we begin with a technical lemma that will be used in our subsequent analysis. For the rest of this paper, let $z := P_{\mathrm{SOL}}x_0$.

Lemma 4.1. *Let Assumptions 3.1 and 3.2 hold. Then for all $n \in \mathbb{N}$ the inequality*

$$\begin{aligned} &-2\alpha_n\langle x_n - z, x_n - x_0\rangle \\ &\geq\ \|x_{n+1} - z\|^2 - \|x_n - z\|^2 + 2\theta_{n+1}\|x_{n+1} - x_n\|^2 - 2\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad +\alpha_{n+1}\|x_0 - x_{n+1}\|^2 - \alpha_n\|x_n - x_0\|^2 - \theta_n\|x_n - z\|^2 + \theta_{n-1}\|x_{n-1} - z\|^2 \\ &\quad +(1 - 3\theta_{n+1} - \alpha_n)\|x_n - x_{n+1}\|^2 \end{aligned} \tag{13}$$

holds for the sequences generated by Algorithm 3.3.

Proof. By Lemma 2.7 we get (since $z \in C_n$) that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|P_{C_n}(w_n) - z\|^2 \le \|w_n - z\|^2 - \|x_{n+1} - w_n\|^2 \\ &= \|w_n - z\|^2 - \text{dist}^2(w_n, C_n). \end{aligned} \tag{14}$$

Moreover, from the definition of w_n, we obtain using Lemma 2.1 (a) that

$$\begin{aligned} \|w_n - z\|^2 &= \|(x_n - z) + \theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2 \\ &= \|x_n - z\|^2 + \|\theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2 \\ &\quad +2\big\langle x_n - z, \theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\big\rangle \\ &= \|x_n - z\|^2 + 2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle - 2\alpha_n\langle x_n - z, x_n - x_0\rangle \\ &\quad +\|\theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2, \end{aligned} \tag{15}$$

and, similarly, with z replaced by x_{n+1} in the previous formula,

$$\begin{aligned} &\|w_n - x_{n+1}\|^2 \\ &\quad = \|x_n - x_{n+1}\|^2 + 2\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle \\ &\qquad -2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle + \|\theta_n(x_n - x_{n-1}) - \alpha_n(x_n - x_0)\|^2. \end{aligned} \tag{16}$$

Substituting (15) and (16) into (14) and eliminating identical terms, we get

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\quad \le \|x_n - z\|^2 + 2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle \\ &\qquad -2\alpha_n\langle x_n - z, x_n - x_0\rangle - \|x_n - x_{n+1}\|^2 \\ &\qquad -2\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle + 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle \\ &\quad = \|x_n - z\|^2 + 2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle \\ &\qquad -2\alpha_n\langle x_n - z, x_n - x_0\rangle - \|x_n - x_{n+1}\|^2 + \theta_n\|x_n - x_{n+1}\|^2 + \theta_n\|x_n - x_{n-1}\|^2 \\ &\qquad -\theta_n\|x_n - x_{n+1} + (x_n - x_{n-1})\|^2 + 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle. \end{aligned} \tag{17}$$

Therefore, we obtain

$$\begin{aligned} &\|x_{n+1} - z\|^2 - \|x_n - z\|^2 - \theta_n\|x_n - x_{n-1}\|^2 + (1 - \theta_n)\|x_n - x_{n+1}\|^2 \\ &\quad \le -2\alpha_n\langle x_n - z, x_n - x_0\rangle + 2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle + 2\alpha_n\langle x_n - x_{n+1}, x_n - x_0\rangle \\ &\quad = -2\alpha_n\langle x_n - z, x_n - x_0\rangle - \theta_n\|x_{n-1} - z\|^2 + \theta_n\|x_n - z\|^2 + \theta_n\|x_n - x_{n-1}\|^2 \\ &\qquad -\alpha_n\|x_0 - x_{n+1}\|^2 + \alpha_n\|x_{n+1} - x_n\|^2 + \alpha_n\|x_n - x_0\|^2, \end{aligned} \tag{18}$$

where the last identity exploits Lemma 2.1 (a) twice. We therefore have

$$-2\alpha_n\langle x_n - z, x_n - x_0\rangle \tag{19}$$

$$\begin{aligned} &\ge \|x_{n+1} - z\|^2 - \|x_n - z\|^2 + 2\theta_{n+1}\|x_{n+1} - x_n\|^2 - 2\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad +\theta_n\big(\|x_{n-1} - z\|^2 - \|x_n - z\|^2\big) + \alpha_n\big(\|x_0 - x_{n+1}\|^2 - \|x_n - x_0\|^2\big) \\ &\quad +(1 - \theta_n - 2\theta_{n+1} - \alpha_n)\|x_{n+1} - x_n\|^2. \end{aligned} \tag{20}$$

Using the fact that $\{\theta_n\}$ is non-decreasing and $\{\alpha_n\}$ is non-increasing, we then obtain

$$-2\alpha_n\langle x_n - z, x_n - x_0\rangle$$

$$\begin{aligned}
&\geq \|x_{n+1}-z\|^2 - \|x_n - z\|^2 + 2\theta_{n+1}\|x_{n+1}-x_n\|^2 - 2\theta_n\|x_n - x_{n-1}\|^2 \\
&\quad +\alpha_{n+1}\|x_0 - x_{n+1}\|^2 - \alpha_n\|x_n - x_0\|^2 - \theta_n\|x_n - z\|^2 + \theta_{n-1}\|x_{n-1}-z\|^2 \\
&\quad +(1-3\theta_{n+1}-\alpha_n)\|x_n - x_{n+1}\|^2,
\end{aligned}$$

which is the desired inequality. □

Our first central result below shows that the sequence $\{x_n\}$ generated by Algorithm 3.3 is bounded under the given assumptions.

Theorem 4.2. *Let Assumptions 3.1 and 3.2 hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.3 is bounded.*

Proof. A simple re-ordering of (13) implies that

$$\begin{aligned}
&\|x_{n+1}-z\|^2 - \|x_n - z\|^2 \\
&\leq \theta_n\|x_n - z\|^2 - \theta_{n-1}\|x_{n-1}-z\|^2 - (1-3\theta_{n+1}-\alpha_n)\|x_n - x_{n+1}\|^2 \\
&\quad -2\theta_{n+1}\|x_{n+1}-x_n\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 - \alpha_{n+1}\|x_0 - x_{n+1}\|^2 \\
&\quad +\alpha_n\|x_n - x_0\|^2 - 2\alpha_n\langle x_n - x_0, x_n - z\rangle \\
&= \theta_n\|x_n - z\|^2 - \theta_{n-1}\|x_{n-1}-z\|^2 - (1-3\theta_{n+1}-\alpha_n)\|x_n - x_{n+1}\|^2 \\
&\quad -2\theta_{n+1}\|x_{n+1}-x_n\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 - \alpha_{n+1}\|x_0 - x_{n+1}\|^2 \\
&\quad +\alpha_n\|x_n - x_0\|^2 + \alpha_n\|x_0 - z\|^2 - \alpha_n\|x_n - x_0\|^2 - \alpha_n\|x_n - z\|^2, \qquad (21)
\end{aligned}$$

where the equality uses once again Lemma 2.1 (a). Hence, by cancellation, re-ordering, and neglecting a non-positive term on the right-hand side, we obtain

$$\begin{aligned}
&\|x_{n+1}-z\|^2 - \|x_n - z\|^2 + \alpha_n\|x_n - z\|^2 \\
&\leq \theta_n\|x_n - z\|^2 - \theta_{n-1}\|x_{n-1}-z\|^2 - (1-3\theta_{n+1}-\alpha_n)\|x_n - x_{n+1}\|^2 \\
&\quad -2\theta_{n+1}\|x_{n+1}-x_n\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 + \alpha_n\|x_0 - z\|^2. \qquad (22)
\end{aligned}$$

Let $\mu_j := e^{\sum_{i=1}^{j}\alpha_i}, j \geq 1$. Since $e^x \geq x+1$ for all $x \in \mathbb{R}$, we also have

$$\begin{aligned}
&\frac{1}{\mu_{n+1}}\left(\mu_{n+1}\|x_{n+1}-z\|^2 - \mu_n\|x_n - z\|^2\right) \\
&= \|x_{n+1}-z\|^2 - \|x_n - z\|^2 + \frac{1}{\mu_{n+1}}(\mu_{n+1}-\mu_n)\|x_n - z\|^2 \\
&\leq \|x_{n+1}-z\|^2 - \|x_n - z\|^2 + \alpha_{n+1}\|x_n - z\|^2.
\end{aligned}$$

Since $\{\alpha_n\}$ is non-increasing in (0,1], this implies

$$\begin{aligned}
&\frac{1}{\mu_{n+1}}\left(\mu_{n+1}\|x_{n+1}-z\|^2 - \mu_n\|x_n - z\|^2\right) \\
&\leq \|x_{n+1}-z\|^2 - \|x_n - z\|^2 + \alpha_n\|x_n - z\|^2. \qquad (23)
\end{aligned}$$

It then follows from (22) and (23) that

$$\begin{aligned}
&\frac{1}{\mu_{n+1}}\left(\mu_{n+1}\|x_{n+1}-z\|^2 - \mu_n\|x_n - z\|^2\right) \\
&\leq \theta_n\|x_n - z\|^2 - \theta_{n-1}\|x_{n-1}-z\|^2
\end{aligned}$$

$$
\begin{aligned}
&-(1-3\theta_{n+1}-\alpha_n)\|x_n-x_{n+1}\|^2-2\theta_{n+1}\|x_{n+1}-x_n\|^2\\
&+2\theta_n\|x_n-x_{n-1}\|^2+\alpha_n\|x_0-z\|^2.
\end{aligned}
$$

Since $\mu_n \le \mu_{n+1}$, $\mu_{n+1}=\mu_n e^{\alpha_{n+1}}$ and $\{\alpha_n\}$ is non-increasing in (0,1], we therefore get

$$
\begin{aligned}
&\mu_{n+1}\|x_{n+1}-z\|^2-\mu_n\|x_n-z\|^2\\
&\quad\le \mu_{n+1}\theta_n\|x_n-z\|^2-\mu_n\theta_{n-1}\|x_{n-1}-z\|^2-\mu_{n+1}(1-3\theta_{n+1}-\alpha_n)\|x_{n+1}-x_n\|^2\\
&\qquad -2\mu_{n+1}\theta_{n+1}\|x_{n+1}-x_n\|^2+2\mu_n\theta_n e^{\alpha_{n+1}}\|x_n-x_{n-1}\|^2+\mu_{n+1}\alpha_n\|x_0-z\|^2,
\end{aligned}
$$

which can be rewritten as (since $\{\alpha_n\}$ is non-increasing in (0,1])

$$
\begin{aligned}
&\mu_{n+1}\|x_{n+1}-z\|^2-\mu_n\|x_n-z\|^2\\
&\quad\le \mu_{n+1}\theta_n\|x_n-z\|^2-\mu_n\theta_{n-1}\|x_{n-1}-z\|^2\\
&\qquad -\mu_{n+1}\left[1-\theta_{n+1}\big(3+2(e^{\alpha_{n+1}}-1)\big)-\alpha_n\right]\|x_{n+1}-x_n\|^2\\
&\qquad -2\mu_{n+1}\theta_{n+1}e^{\alpha_{n+1}}\|x_{n+1}-x_n\|^2+2\mu_n\theta_n e^{\alpha_n}\|x_n-x_{n-1}\|^2+\mu_{n+1}\alpha_n\|x_0-z\|^2.
\end{aligned}
$$

Since the sequence $\{\theta_n\}$ belongs to the interval $[0,\theta]$ by Assumption 3.2, we have

$$
1-\theta_{n+1}\big(3+2(e^{\alpha_{n+1}}-1)\big)-\alpha_n \ge 1-\theta\big(3+2(e^{\alpha_{n+1}}-1)\big)-\alpha_n, \quad \forall n\in\mathbb{N}.
$$

Using $\lim_{n\to\infty}\alpha_n=0$ and $\theta\in[0,1/3)$ from Assumption 3.2, it follows that the right-hand side is eventually bounded from below by a positive number, i.e., there is a constant $\gamma>0$ such that $1-\theta_{n+1}\big(3+2(e^{\alpha_{n+1}}-1)\big)-\alpha_n\ge\gamma$ for all $n\in\mathbb{N}$ sufficiently large, say, for all $n\ge n_0$. Hence, we have

$$
\begin{aligned}
&\mu_{n+1}\|x_{n+1}-z\|^2-\mu_n\|x_n-z\|^2\\
&\quad\le \mu_{n+1}\theta_n\|x_n-z\|^2-\mu_n\theta_{n-1}\|x_{n-1}-z\|^2-2\mu_{n+1}\theta_{n+1}e^{\alpha_{n+1}}\|x_{n+1}-x_n\|^2\\
&\qquad -\gamma\mu_{n+1}\|x_{n+1}-x_n\|^2+2\mu_n\theta_n e^{\alpha_n}\|x_n-x_{n-1}\|^2+\mu_{n+1}\alpha_n\|x_0-z\|^2.
\end{aligned}
$$

This implies that for $n\ge n_0$,

$$
\begin{aligned}
&\|x_0-z\|^2\sum_{k=n_0+1}^{n}\mu_{k+1}\alpha_k\\
&\quad\ge \mu_{n+1}\|x_{n+1}-z\|^2+2\mu_{n+1}\theta_{n+1}e^{\alpha_{n+1}}\|x_{n+1}-x_n\|^2-\mu_{n+1}\theta_n\|x_n-z\|^2\\
&\qquad -\mu_{n_0+1}\|x_{n_0+1}-z\|^2-2\mu_{n_0+1}\theta_{n_0+1}e^{\alpha_{n_0+1}}\|x_{n_0+1}-x_{n_0}\|^2\\
&\qquad +\mu_{n_0+1}\theta_{n_0}\|x_{n_0}-z\|^2. \qquad (24)
\end{aligned}
$$

Thus, dividing by μ_{n+1} and omitting a non-positive term, we get

$$
\begin{aligned}
&\|x_{n+1}-z\|^2-\theta_n\|x_n-z\|^2\\
&\quad\le e^{-t_{n+1}}\left[\mu_{n_0+1}\|x_{n_0+1}-z\|^2+2\mu_{n_0+1}\theta_{n_0+1}e^{\alpha_{n_0+1}}\|x_{n_0+1}-x_{n_0}\|^2\right.\\
&\qquad \left.-\mu_{n_0+1}\theta_{n_0}\|x_{n_0}-z\|^2\right]+\|x_0-z\|^2e^{-t_{n+1}}\sum_{k=n_0+1}^{n}\alpha_k e^{t_k+1}, \qquad (25)
\end{aligned}
$$

where $t_n:=\sum_{i=1}^{n}\alpha_i$. Since $\alpha_k\in(0,1]$ for all $k\in\mathbb{N}$, it is easy to see that $\alpha_k e^{t_{k+1}}\le e^2(e^{t_k}-e^{t_{k-1}})$ for all $k\ge 2$, so that

$$
\sum_{k=n_0+1}^{n}\mu_{k+1}\alpha_k=\sum_{k=n_0+1}^{n}\alpha_k e^{t_{k+1}}\le e^2\big(e^{t_n}-e^{t_{n_0}}\big)\le e^2e^{t_n},
$$

which, by (25), $e^{-t_{n+1}} \leq 1$, and the fact that $\{\theta_n\}$ belongs to the interval $[0, \theta] \subset [0, \frac{1}{3})$, yields

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq \ \theta\|x_n - z\|^2 + \mu_{n_0+1}\|x_{n_0+1} - z\|^2 + 2\mu_{n_0+1}\theta_{n_0+1}e^{\alpha_{n_0+1}}\|x_{n_0+1} - x_{n_0}\|^2 \\
&\quad +e^2\|x_0 - z\|^2. \qquad (26)
\end{aligned}
$$

Using (26), $\theta \in [0, 1)$, and the convergence of the geometric series, a simple calculation gives

$$
\begin{aligned}
\|x_{n+1} - z\|^2 \ \leq \ & \theta^{n-n_0}\|x_{n_0+1} - z\|^2 + \frac{1}{1-\theta}\big[\mu_{n_0+1}\|x_{n_0+1} - z\|^2 \\
& +2\mu_{n_0+1}\theta_{n_0+1}e^{\alpha_{n_0+1}}\|x_{n_0+1} - x_{n_0}\|^2 + e^2\|x_0 - z\|^2\big].
\end{aligned}
$$

Using once again that $\theta < 1$, this shows that $\{x_n\}$ is bounded. □

In the next lemma, we show that certain sequences obtained in Algorithm 3.3 are null subsequences. These two lemmas are necessary in order to show that the weak limit of $\{x_n\}$ is an element of SOL.

Lemma 4.3. *Let $\{x_n\}$ generated by Algorithm 3.3 above and Assumptions 3.1 and 3.2 hold. If $\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0$, then*

(a) $\lim\limits_{n\to\infty} \eta_n\|w_n - z_n\|^2 = 0$;

(b) $\lim\limits_{n\to\infty} \|w_n - z_n\| = 0$.

Proof. Since A is uniformly continuous on bounded subsets of H, then $\{Ax_n\}, \{z_n\}, \{w_n\}$ and $\{Ay_n\}$ are bounded. In particular, there exists $M > 0$ such that $\|Ay_n\| \leq M$ for all $n \in \mathbb{N}$. Combining Lemma 2.6 and Lemma 3.6, we get

$$
\begin{aligned}
\|x_{n+1} - z\|^2 \ &= \ \|P_{C_n}(w_n) - z\|^2 \leq \|w_n - z\|^2 - \|x_{n+1} - w_n\|^2 \\
&= \ \|w_n - z\|^2 - \mathrm{dist}^2(w_n, C_n) \\
&\leq \ \|w_n - z\|^2 - \Big(\frac{1}{M}h_n(w_n)\Big)^2 \\
&\leq \ \|w_n - z\|^2 - \Big(\frac{1}{2M}\sigma\eta_n\|r(w_n)\|^2\Big)^2 \\
&= \ \|w_n - z\|^2 - \Big(\frac{1}{2M}\sigma\eta_n\|w_n - z_n\|^2\Big)^2. \qquad (27)
\end{aligned}
$$

Since $\{x_n\}$ and $\{w_n\}$ are bounded, we obtain from (27) that

$$
\begin{aligned}
\Big(\frac{1}{2M}\sigma\eta_n\|w_n - z_n\|^2\Big)^2 \ &\leq \ \|w_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&= \ \Big(\|w_n - z\| - \|x_{n+1} - z\|\Big)\Big(\|w_n - z\| + \|x_{n+1} - z\|\Big) \\
&\leq \ \Big(\|w_n - z\| - \|x_{n+1} - z\|\Big)M_1 \\
&\leq \ \|w_n - x_{n+1}\|M_1, \qquad (28)
\end{aligned}
$$

where $M_1 := \sup_{n\geq 1}\{\|w_n - z\| + \|x_{n+1} - z\|\}$. This establishes (a).

To establish (b), We distinguish two cases depending on the behaviour of (the bounded) sequence of stepsizes $\{\eta_n\}$.
Case 1: Suppose that $\liminf_{n\to\infty} \eta_n > 0$. Then

$$0 \leq \|r(w_n)\|^2 = \frac{\eta_n\|r(w_n)\|^2}{\eta_n}$$

and this implies that (using (a) above)

$$\begin{aligned}\limsup_{n\to\infty} \|r(w_n)\|^2 &\leq \limsup_{n\to\infty}\left(\eta_n\|r(w_n)\|^2\right)\left(\limsup_{n\to\infty}\frac{1}{\eta_n}\right)\\ &= \left(\limsup_{n\to\infty}\eta_n\|r(w_n)\|^2\right)\frac{1}{\liminf_{n\to\infty}\eta_n}\\ &= 0.\end{aligned}$$

Hence, $\limsup_{n\to\infty}\|r(w_n)\| = 0$. Therefore,

$$\lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}\|r(w_n)\| = 0.$$

Case 2: Suppose that $\liminf_{n\to\infty}\eta_n = 0$. It suffices to show that $\limsup_{n\to\infty}\|w_n - z_n\| = 0$. Subsequencing if necessary, we may assume without loss of generality that $\lim_{n\to\infty}\eta_n = 0$.
Define $\bar{y}_n := \frac{1}{\gamma}\eta_n z_n + \left(1 - \frac{1}{\gamma}\eta_n\right)w_n$ or, equivalently, $\bar{y}_n - w_n = \frac{1}{\gamma}\eta_n(z_n - w_n)$. Since $\{z_n - w_n\}$ is bounded and since $\lim_{n\to\infty}\eta_n = 0$ holds, it follows that

$$\lim_{n\to\infty}\|\bar{y}_n - w_n\| = 0. \tag{29}$$

From the stepsize rule and the definition of $\bar{y}_n$, we have

$$\langle A\bar{y}_n, w_n - z_n\rangle < \frac{\sigma}{2}\|w_n - z_n\|^2,\ \forall n \in \mathbb{N},$$

or equivalently

$$2\langle Aw_n, w_n - z_n\rangle + 2\langle A\bar{y}_n - Aw_n, w_n - z_n\rangle < \sigma\|w_n - z_n\|^2,\ \forall n \in \mathbb{N}.$$

Setting $t_n := w_n - Aw_n$, we obtain form the last inequality that

$$2\langle w_n - t_n, w_n - z_n\rangle + 2\langle A\bar{y}_n - Aw_n, w_n - z_n\rangle < \sigma\|w_n - z_n\|^2,\ \forall k \in \mathbb{N}.$$

Using Lemma 2.1 (iii) we get

$$2\langle w_n - t_n, w_n - z_n\rangle = \|w_n - z_n\|^2 + \|w_n - t_n\|^2 - \|z_n - t_n\|^2.$$

Therefore,

$$\|w_n - t_n\|^2 - \|z_n - t_n\|^2 < (\sigma - 1)\|w_n - z_n\|^2 - 2\langle A\bar{y}_n - Aw_n, w_n - z_n\rangle\ \forall n \in \mathbb{N}.$$

Since A is uniformly continuous on bounded subsets of H and (29), if $a > 0$ then the right hand side of the last inequality converges to $(\sigma - 1)a < 0$ as $n \to \infty$. From the last inequality we have

$$\limsup_{n\to\infty} \left(\|w_n - t_n\|^2 - \|z_n - t_n\|^2\right) \leq (\sigma - 1)a < 0.$$

For $\epsilon = -(\sigma - 1)a/2 > 0$, there exists $N \in \mathbb{N}$ such that

$$\|w_n - t_n\|^2 - \|z_n - t_n\|^2 \leq (\sigma - 1)a + \epsilon = (\sigma - 1)a/2 < 0 \quad \forall n \in \mathbb{N}, n \geq N,$$

leading to

$$\|w_n - t_n\| < \|z_n - t_n\| \quad \forall n \in \mathbb{N}, n \geq N,$$

which is a contradiction to the definition of $z_n = P_C(w_n - Aw_n)$. Hence $a = 0$, which completes the proof. □

Next, we formulate a simple lemma that turns out to be useful for proving the strong convergence result.

Lemma 4.4. *Let Assumptions 3.1 and 3.2 hold, and let $\{x_n\}$ be the sequence generated by Algorithm 3.3. Furthermore, let $\{u_n\}$ be a sequence generated by*

$$u_n := \|x_n - z\|^2 - \theta_{n-1}\|x_{n-1} - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 + \alpha_n\|x_n - x_0\|^2$$

for all $n \in \mathbb{N}$. Then $u_n \geq 0$ for all $n \in \mathbb{N}$.

Proof. Since $\{\theta_n\}$ is non-decreasing with $0 \leq \theta_n < \frac{1}{3}$, and $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2$ for all $x, y \in H$, we have

$$\begin{aligned}
u_n &= \|x_n - z\|^2 - \theta_{n-1}\|x_{n-1} - x_n + x_n - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 + \alpha_n\|x_n - x_0\|^2 \\
&= \|x_n - z\|^2 - \theta_{n-1}\left[\|x_{n-1} - x_n\|^2 + \|x_n - z\|^2 + 2\langle x_{n-1} - x_n, x_n - z\rangle\right] \\
&\quad + 2\theta_n\|x_n - x_{n-1}\|^2 + \alpha_n\|x_n - x_0\|^2 \\
&= \|x_n - z\|^2 - \theta_{n-1}\left[2\|x_{n-1} - x_n\|^2 + 2\|x_n - z\|^2 - \|x_{n-1} - 2x_n - z\|^2\right] \\
&\quad + 2\theta_n\|x_n - x_{n-1}\|^2 + \alpha_n\|x_n - x_0\|^2 \\
&= \|x_n - z\|^2 - 2\theta_{n-1}\|x_{n-1} - x_n\|^2 - 2\theta_{n-1}\|x_n - z\|^2 + \theta_{n-1}\|x_{n-1} - 2x_n - z\|^2 \\
&\quad + 2\theta_n\|x_n - x_{n-1}\|^2 + \alpha_n\|x_n - x_0\|^2 \\
&\geq \|x_n - z\|^2 - 2\theta_n\|x_{n-1} - x_n\|^2 - \frac{2}{3}\|x_n - z\|^2 + \theta_{n-1}\|x_{n-1} - 2x_n - z\|^2 \\
&\quad + 2\theta_n\|x_n - x_{n-1}\|^2 + \alpha_n\|x_n - x_0\|^2 \\
&\geq \frac{1}{3}\|x_n - z\|^2 + \alpha_n\|x_n - x_0\|^2 \\
&\geq 0,
\end{aligned}$$

and this completes the proof. □

Before we prove our main strong convergence result for Algorithm 3.3, we state another preliminary result which provides sufficient conditions for the strong convergence of the sequence $\{x_n\}$ generated by our method to a particular solution of the variational inequality. In our strong convergence result, we will then show that these sufficient conditions automatically hold.

Lemma 4.5. *Let Assumptions 3.1 and 3.2 hold, and let $\{x_n\}$ be the sequence generated by Algorithm 3.3. Assume that*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

and

$$\lim_{n\to\infty} (\|x_{n+1} - z\|^2 - \theta_n \|x_n - z\|^2) = 0.$$

Then the entire sequence $\{x_n\}$ converges strongly to the solution z.

Proof. By assumption, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} (\|x_{n+1} - z\|^2 - \theta_n \|x_n - z\|^2) \\ &= \lim_{n\to\infty} \Big[(\|x_{n+1} - z\| + \sqrt{\theta_n} \|x_n - z\|)(\|x_{n+1} - z\| - \sqrt{\theta_n} \|x_n - z\|) \Big]. \end{aligned} \tag{30}$$

We claim that this already implies

$$\lim_{n\to\infty} (\|x_{n+1} - z\| + \sqrt{\theta_n} \|x_n - z\|) = 0,$$

from which the strong convergence of the entire sequence $\{x_n\}$ to z follows immediately. Assume this limit does not hold. Then there is a subset $K \subseteq \mathbb{N}$ and a constant $\rho > 0$ such that

$$\|x_{n+1} - z\| + \sqrt{\theta_n} \|x_n - z\| \geq \rho, \forall n \in K. \tag{31}$$

Using (30) and $\theta_n \leq \theta < 1$ by Assumption 3.2, it then follows that

$$\begin{aligned} 0 &= \lim_{n\in K} (\|x_{n+1} - z\| - \sqrt{\theta_n} \|x_n - z\|) \\ &= \limsup_{n\in K} (\|x_{n+1} - x_n + x_n - z\| - \sqrt{\theta_n} \|x_n - z\|) \\ &\geq \limsup_{n\in K} (\|x_n - z\| - \|x_{n+1} - x_n\| - \sqrt{\theta_n} \|x_n - z\|) \\ &\geq \limsup_{n\in K} ((1 - \sqrt{\theta}) \|x_n - z\| - \|x_{n+1} - x_n\|) \\ &= (1 - \sqrt{\theta}) \limsup_{n\in K} \|x_n - z\| - \lim_{n\in K} \|x_{n+1} - x_n\| \\ &= (1 - \sqrt{\theta}) \limsup_{n\in K} \|x_n - z\|. \end{aligned}$$

Consequently, we have $\limsup_{n\in K} \|x_n - z\| \leq 0$. Since $\liminf_{n\in K} \|x_n - z\| \geq 0$ obviously holds, it follows that $\lim_{n\in K} \|x_n - z\| = 0$. This implies (by (31))

$$\begin{aligned} \|x_{n+1} - x_n\| &\geq \|x_{n+1} - z\| - \|x_n - z\| \\ &= \|x_{n+1} - z\| + \sqrt{\theta_n} \|x_n - z\| - (1 + \sqrt{\theta_n}) \|x_n - z\| \\ &\geq \frac{\rho}{2} \end{aligned}$$

for all $n \in K$ sufficiently large, a contradiction to the assumption that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. This completes the proof.

□

We now verify the strong convergence of any sequence $\{x_n\}$ generated by Algorithm 3.3 to the projection of the given vector x_0 onto SOL. Hence the choice of x_0 has a direct influence on the convergence of the sequence $\{x_n\}$. Taking another vector $x_0 \in H$, we still have convergence of the entire sequence, but possibly to another solution. In particular, this means that the method is able to find different solutions. Hence, if there is an application which prefers to have a solution to belong to a certain area, this a priori information can be incorporated into the method by a suitable choice of x_0.

Theorem 4.6. *Let Assumptions 3.1 and 3.2 hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.3 strongly converges to the solution z.*

Proof. Let u_n denote the nonnegative number defined in Lemma 4.4, and let us apply Lemma 4.1. We obtain from (13) that

$$\begin{aligned} & u_{n+1} - u_n + (1 - 3\theta_{n+1} - \alpha_n)\|x_n - x_{n+1}\|^2 \\ & \quad \leq -2\alpha_n \langle x_n - z, x_n - x_0 \rangle. \end{aligned} \tag{32}$$

We now distinguish two cases.

Case 1. Suppose $\{u_n\}$ is eventually a monotonically decreasing sequence, i.e. for some $n_0 \in \mathbb{N}$ large enough, we have $u_{n+1} \leq u_n$ for all $n \geq n_0$. Then, since u_n is nonnegative for all $n \in \mathbb{N}$ by Lemma 4.4, we obviously get that $\{u_n\}$ is a convergent sequence. Consequently, it follows that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1}$. Since $\{x_n\}$ is bounded by Theorem 4.2, there exists $M > 0$ such that $2|\langle x_n - z, x_n - x_0\rangle| \leq M$. Moreover, from Assumption 3.2, it follows that there exists $N \in \mathbb{N}$ and $\gamma_1 > 0$ such that $1 - 3\theta_{n+1} - \alpha_n \geq \gamma_1$ for all $n \geq N$. Therefore, for $n \geq N$, we obtain from (32) that

$$\begin{aligned} \gamma_1 \|x_{n+1} - x_n\|^2 & \leq \alpha_n M + u_n - u_{n+1} \\ & \leq \alpha_n M + u_n - u_{n+1} \\ & \to 0 \quad \text{for } n \to \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| \to 0.$$

Together with $\alpha_n \to 0$, the boundedness of $\{x_n\}$, and the convergence of $\{u_n\}$, we therefore obtain from the definition of u_n that the limit

$$\lambda := \lim_{n\to\infty} \left(\|x_{n+1} - z\|^2 - \theta_n \|x_n - z\|^2 \right) \tag{33}$$

exists and is equal to $\lim_{n\to\infty} u_{n+1}$. In particular, Lemma 4.4 therefore implies that $\lambda \geq 0$. We will show that $\lambda = 0$ holds; then (33) together with the fact that $\theta_n \leq \theta < 1$ for all $n \in \mathbb{N}$ yields the strong convergence of the sequence $\{x_n\}$ to the solution z.

By contradiction, assume that $\lambda > 0$. Since $\{x_n\}$ is bounded by Theorem 4.2, it is easy to see that we can choose a subsequence $\{x_{n_j}\}$ which converges weakly to an element $p \in H$ and such that

$$\liminf_{n\to\infty} \langle x_n - z, z - x_0 \rangle = \lim_{j\to\infty} \langle x_{n_j} - z, z - x_0 \rangle = \langle p - z, z - x_0 \rangle.$$

We show that $p \in \mathrm{SOL}$. Observe that the updating rule for w_n implies

$$\begin{array}{rcl}\|w_n - x_n\| & = & \|\alpha_n(x_0 - x_n) + \theta_n(x_n - x_{n-1})\| \\ & \leq & \alpha_n\|x_0 - x_n\| + \theta_n\|x_n - x_{n-1}\| \to 0, \quad n \to \infty.\end{array}$$

This yields

$$\|x_{n+1} - w_n\| \leq \|x_n - w_n\| + \|x_{n+1} - x_n\| \to 0, \quad n \to \infty.$$

Then by Lemma 4.3 (b), we have that $x_n - z_n \to 0$. This implies that $z_{n_j} \rightharpoonup p$ and since $z_n \in C$, we then have that $p \in C$. Similarly, $w_{n_j} \rightharpoonup p$ since $w_n - x_n \to 0$. For all $x \in C$ and using (5), we have that (since A is monotone)

$$\begin{array}{rcl}0 & \leq & \langle z_{n_j} - w_{n_j} + Aw_{n_j}, x - z_{n_j}\rangle \\ & = & \langle z_{n_j} - w_{n_j}, x - z_{n_j}\rangle + \langle Aw_{n_j}, w_{n_j} - z_{n_j}\rangle \\ & & +\langle Aw_{n_j}, x - w_{n_j}\rangle \\ & \leq & \langle z_{n_j} - w_{n_j}, x - w_{n_j}\rangle + \langle Aw_{n_j}, w_{n_j} - z_{n_j}\rangle \\ & & +\langle Ax, x - w_{n_j}\rangle.\end{array}$$

Passing to the limit, we get

$$\langle Ax, x - p\rangle \geq 0, \quad \forall x \in C.$$

By Lemma 2.8, we have that $p \in \mathrm{SOL}$. This implies that

$$\liminf_{n\to\infty}\langle x_n - z, z - x_0\rangle = \langle p - z, z - x_0\rangle \geq 0, \tag{34}$$

where the inequality follows from the characterization (5) of a projection applied to $z = P_{\mathrm{SOL}}x_0$ and $p \in \mathrm{SOL}$. Since (33) yields

$$\liminf_{n\to\infty} \|x_{n+1} - z\|^2 \geq \lim_{n\to\infty} \left(\|x_{n+1} - z\|^2 - \theta_n\|x_n - z\|^2\right) = \lambda,$$

and since $\lambda > 0$ by assumption, we have

$$\|x_{n+1} - z\|^2 \geq \frac{1}{2}\lambda \quad \forall n \geq n_1$$

for some sufficiently large $n_1 \in \mathbb{N}$. Using the identity

$$\langle x_n - z, x_n - x_0\rangle = \|x_n - z\|^2 + \langle x_n - z, z - x_0\rangle,$$

we therefore get

$$\begin{array}{rcl}\liminf\limits_{n\to\infty}\langle x_n - z, x_n - x_0\rangle & = & \liminf\limits_{n\to\infty} \left(\|x_n - z\|^2 + \langle x_n - z, z - x_0\rangle\right) \\ & \geq & \liminf\limits_{n\to\infty} \left(\frac{1}{2}\lambda + \langle x_n - z, z - x_0\rangle\right) \\ & = & \frac{1}{2}\lambda + \liminf\limits_{n\to\infty}\langle x_n - z, z - x_0\rangle \\ & \geq & \frac{1}{2}\lambda\end{array}$$

from (34). Using once again the assumption that $\lambda > 0$, this implies

$$\langle x_n - z, x_n - x_0 \rangle \geq \frac{1}{4}\lambda \quad \forall n \geq n_2$$

for some sufficiently large $n_2 \in \mathbb{N}, n_2 \geq n_1$. From (32), we therefore obtain

$$u_{n+1} - u_n \leq -\frac{1}{2}\alpha_n \lambda \quad \forall n \geq n_2.$$

This implies

$$\frac{1}{2}\lambda \sum_{k=n_2}^{n} \alpha_k \leq u_{n_2} - u_n \leq u_{n_2} \quad \forall n \geq n_2,$$

where the second inequality follows from Lemma 4.4. Since $\lambda > 0$, this gives the summability of the sequence $\{\alpha_n\}$, a contradiction to our Assumption 3.2. Hence we must have $\lambda = 0$, and this yields the strong convergence of the sequence $\{x_n\}$ to z, cf. the above discussion.

Case 2. Assume $\{u_n\}$ is not eventually monotonically decreasing. Then let $\tau : \mathbb{N} \to \mathbb{N}$ be the map defined for all $n \geq n_0$ (for some $n_0 \in \mathbb{N}$ large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, u_k \leq u_{k+1}\}. \tag{35}$$

Clearly, $\tau(n)$ is a non-decreasing sequence such that $\tau(n) \to \infty$ for $n \to \infty$ and $u_{\tau(n)} \leq u_{\tau(n)+1}$ for all $n \geq n_0$. Hence, similar to the proof of Case 1, we therefore obtain from (32) that

$$\gamma_1 \|x_{\tau(n)+1} - x_{\tau(n)}\|^2 \leq \alpha_{\tau(n)} M \to 0 \tag{36}$$

for some constant $M > 0$. Thus,

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, \quad n \to \infty. \tag{37}$$

Using the same technique of proof as in Case 1, one can also derive the limits

$$\begin{aligned} \|x_{\tau(n)+1} - w_{\tau(n)}\| &\to 0, \quad n \to \infty, \\ \|w_{\tau(n)} - x_{\tau(n)}\| &\to 0, \quad n \to \infty, \qquad (38) \\ \|x_{\tau(n)} - z_{\tau(n)}\| &\to 0, \quad n \to \infty. \qquad (39) \end{aligned}$$

Again observe that for $j \geq 0$ by (32), we have $u_{j+1} < u_j$ when $x_j \notin \Omega := \{x \in H : \langle x - x_0, x - z \rangle \leq 0\}$ (note that this Ω is the same set as in Lemma 2.4). Hence $x_{\tau(n)} \in \Omega$ for all $n \geq n_0$ since $u_{\tau(n)} \leq u_{\tau(n)+1}$. Since $\{x_{\tau(n)}\}$ is bounded, we may choose a subsequence (which we again call $\{x_{\tau(n)}\}$) which converges weakly to some $x^* \in H$. As Ω is a closed and convex set, it is then weakly closed and so $x^* \in \Omega$. Using (39), one can see as in Case 1 that $z_{\tau(n)} \rightharpoonup x^*$ and $x^* \in \text{SOL}$. Consequently, we have $x^* \in \Omega \cap \text{SOL}$. In view of Lemma 2.4, however, the intersection $\Omega \cap \text{SOL}$ contains z as its only element. We therefore get $x^* = z$. Furthermore, we have

$$\begin{aligned} \|x_{\tau(n)} - z\|^2 &= \langle x_{\tau(n)} - x_0, x_{\tau(n)} - z \rangle - \langle z - x_0, x_{\tau(n)} - z \rangle \\ &\leq -\langle z - x_0, x_{\tau(n)} - z \rangle \end{aligned}$$

since $x_{\tau(n)} \in \Omega$. Taking lim sup in this last inequality gives

$$\limsup_{n\to\infty} \|x_{\tau(n)} - z\| \leq 0.$$

Hence

$$\|x_{\tau(n)} - z\| \to 0, \quad n \to \infty. \tag{40}$$

We claim that this implies $\lim_{n\to\infty} u_{\tau(n)+1} = 0$. By definition, $u_{\tau(n)+1}$ is equal to

$$\|x_{\tau(n)+1} - z\|^2 - \theta_{\tau(n)}\|x_{\tau(n)} - z\|^2 + 2\theta_{\tau(n)+1}\|x_{\tau(n)+1} - x_{\tau(n)}\|^2 + \alpha_{\tau(n)+1}\|x_{\tau(n)+1} - x_0\|^2.$$

Adding and subtracting $x_{\tau(n)}$ inside the norm of the first term, and using (37), (40), we see that the first term goes to zero. The second term converges to zero also in view of (40), taking into account the boundedness of $\{\theta_n\}$. The third term vanishes in the limit because of (37) and noting once again that $\{\theta_n\}$ is a bounded sequence. Finally, the last term goes to zero since $\{\alpha_n\}$ converges to zero and the sequence $\{x_n\}$ is bounded by Theorem 4.2.

We next show that we actually have $\lim_{n\to\infty} u_n = 0$. To this end, first observe that, for $n \geq n_0$, one has $u_n \leq u_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, if $\tau(n) < n$) because we necessarily have $u_j > u_{j+1}$ for $\tau(n) + 1 \leq j \leq n - 1$. It follows that for all $n \geq n_0$, we have $u_n \leq \max\{u_{\tau(n)}, u_{\tau(n)+1}\} = u_{\tau(n)+1} \to 0$, hence $\limsup_{n\to\infty} u_n \leq 0$. On the other hand, Lemma 4.4 implies that $\liminf_{n\to\infty} u_n \geq 0$. Together we obtain $\lim_{n\to\infty} u_n = 0$.

Consequently, the boundedness of $\{x_n\}$, Assumption 3.2, and (32) show that

$$\|x_n - x_{n+1}\| \to 0, \quad n \to \infty.$$

Hence the definition of u_n yields

$$\lim_{n\to\infty} \left(\|x_{n+1} - z\|^2 - \theta_n\|x_n - z\|^2\right) = 0.$$

Using Assumption 3.2, it is not difficult to see that this implies the strong convergence of the entire sequence $\{x_n\}$ to the particular solution z. The statement therefore follows from Lemma 4.5. □

5 Numerical Experiments

In this section, we discuss the numerical behaviour of Algorithm 3.3 (Alg 3.3 for short) using some example in order to illustrate the effectiveness and implementation of our method. The considered example is given in $\mathbb{R}^m$ and for this reason, there is no need to use any of algorithms that produce strong convergence to a solution of variational inequality. However, there are many problems that arise in infinite dimensional spaces and for such problems strong convergence is often much more desirable than weak convergence (see [9] and references therein). For this reason, algorithms that produce strong convergence can be better suited than Extragradient Algorithm (2) and its modifications that give weak convergence. Another reason to study algorithms that produce strong convergence is for an academic interest. In addition, our interest in this preliminary numerical investigation is to compare our proposed algorithm (which produces strong convergence) with some other already studied algorithms (see, e.g., [35, 41, 44]) in the literature where strong convergence is also obtained.

Example 5.1. This example is taken from [21] and has been considered by many authors for numerical experiments (see, for example, [24, 41, 51]). The operator A is defined by $Ax := Mx + q$, where $M = BB^T + S + D$, where $B, S, D \in \mathbb{R}^{m\times m}$ are randomly generated matrices such that S is skew-symmetric (hence the operator does not arise from an optimization problem), D is a positive definite diagonal matrix (hence the variational inequality has a unique solution) and $q = 0$. The feasible set C is described by linear inequality constraints $Kx \leq b$ for some random matrix $K \in \mathbb{R}^{k\times m}$ and a random vector $b \in \mathbb{R}^k$ with nonnegative entries. Hence the zero vector is feasible and therefore the unique solution of the corresponding variational inequality. The projections are computed by solving a quadratic optimization problem using the MATLAB solver `quadprog`. Hence, for this problem, the evaluation of A is relatively inexpensive, whereas projections are costly. We present the corresponding numerical results (number of iterations and CPU times in seconds) using different dimensions m and different numbers of inequality constraints k. $\Diamond$

We compare Alg 3.3 with the algorithms proposed in [35, 41, 44] by solving Example 5.1. For convenience of comparison, we denote the algorithm (4) in [35] by Alg 1, the algorithm (2) in [41] by Alg 2, and the algorithm defined in Theorem 3.1 in [44] by Alg 3.

Table 1: Comparison of Alg 3.3, Alg 1, Alg 2 and Alg 3 for $k = 20$.

	Iter.				CPU in second			
m	Alg 3.3	Alg 1	Alg 2	Alg 3	Alg 3.3	Alg 1	Alg 2	Alg 3
10	157	401	342	771	2.5938	14.8125	12.7813	42.2813
20	851	1338	949	3959	16.6875	47.7188	36.1563	223.5469
30	1148	4764	1584	13432	24.4688	172.5000	63.6406	767.3906

Table 2: Comparison of Alg 3.3, Alg 1, Alg 2 and Alg 3 for $k = 30$.

	Iter.				CPU in second			
m	Alg 3.3	Alg 1	Alg 2	Alg 3	Alg 3.3	Alg 1	Alg 2	Alg 3
20	913	926	974	5433	21.1094	34.5938	41.1406	331.3906
30	1092	2974	1686	8376	49.2969	119.1094	72.2813	525
40	1659	7871	1780	9604	55.1406	751.0469	127.5626	1078

We take the initial point x_0 to be the unit vector in Alg 3.3, Alg 1, Alg 2 and Alg 3 and choose the stopping criterion as $\|x_n\| \leq \epsilon = 0.05$ in Tables 1 and 2. The matrices B, S, D, K and the vector b are generated randomly.
Let the Lipschitz constant L be $L = \|A\|$ in Alg 1, Alg 2 and Alg 3. In Alg 3.3, we choose $\gamma = 0.9$, $\sigma = 0.9$, $\theta_n = 0.3$ and $\alpha_n = \frac{1}{n+2}$. In Alg 1, choose $\alpha_n = \frac{1}{n+2}$, $\tau = \frac{1}{L+8}$. In Alg 2, $\lambda = \frac{1}{2L+1}$, $k = \frac{1}{1-2\lambda L} + 1$. In Alg 3, $\lambda = \frac{1}{L+2}$.

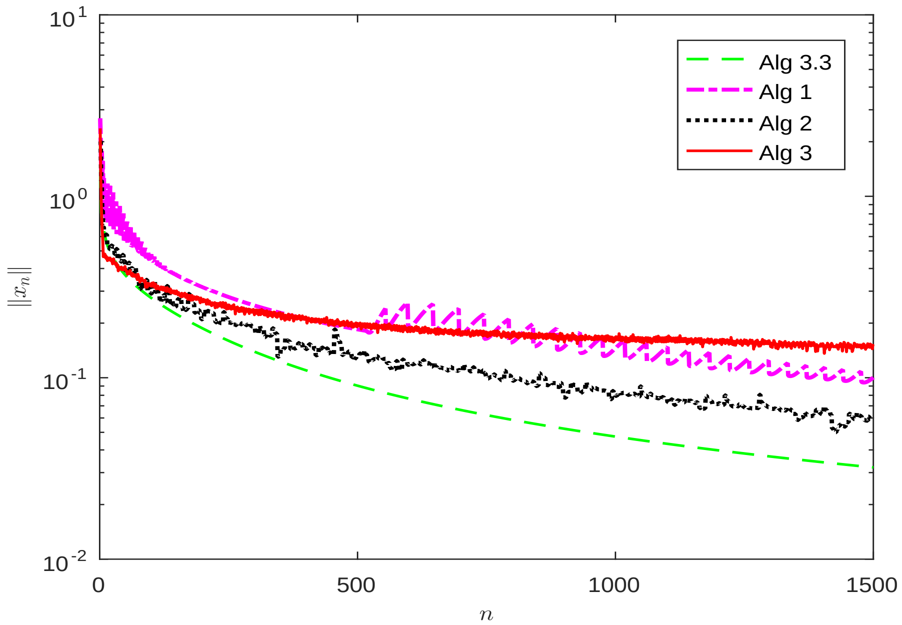

Figure 1: Comparison of Alg 3.3 with Alg 1, Alg 2, and Alg 3 for $k = 20, m = 20$.

The numerical experiment in this section validates and demonstrates the advantages of Alg 3.3 over other existing Alg 1, Alg 2 and Alg 3. The numerical results are listed in Tables 1 and 2, and Figure 1, which illustrate that Alg 3.3 converges faster than Alg 1, Alg 2 and Alg 3 in terms of the number of iterations and CPU time. In particular, CPU time of Alg 3.3 is very small compare to other algorithms and the reason may be that Alg 3.3 involves one projection onto C per each iteration and addition of inertial terms. Therefore, Alg 3.3 has numerical advantage in large-scale computations, based on our numerical example, over Alg 1, Alg 2 and Alg 3. We caution, however, that this study is a very preliminary one.

6 Final Remarks

This paper presents strong convergence result for projection-type method involving inertial extrapolation term for a monotone and uniformly continuous mapping in real Hilbert spaces. Some numerical experiments are given to show efficiency and implementation of our scheme. Our scheme gives faster convergence with an appropriate choice of θ_n when compared with other related existing strong convergence methods in the literature. Part of our future research is to consider at least one example of the real applied problem in an infinite-dimensional Hilbert space, which satisfies the basic assumptions and then give the results of the computational solution of this problem as well as the comparison with similar methods.

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