Covering and Packing with Spheres by Diagonal Distortion in $\mathbb{R}^n$ *

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Abstract. We address the problem of covering $\mathbb{R}^n$ with congruent balls, while minimizing the number of balls that contain an average point. Considering the 1-parameter family of lattices defined by stretching or compressing the integer grid in diagonal direction, we give a closed formula for the covering density that depends on the distortion parameter. We observe that our family contains the thinnest lattice coverings in dimensions 2 to 5. We also consider the problem of packing congruent balls in $\mathbb{R}^n$, for which we give a closed formula for the packing density as well. Again we observe that our family contains optimal configurations, this time densest packings in dimensions 2 and 3.

Keywords. Packing, covering, spheres, balls, cubes, lattices, n-dimensional Euclidean space.

1 Introduction

The starting point for the work described in this paper is a perturbation of the integer grid designed to resolve ambiguities in the neighborhood relation of the cubes in an n-dimensional image [7]. Generalizing the perturbation to a 1-parameter family of distortions, we noted its relation with some well-known lattices in the sphere covering and packing literature; see Conway and Sloane [4], Fejes Tóth [8], and Rogers [16]. For example, in $\mathbb{R}^3$, we get the body-centered cubic, or BCC lattice by compressing with a factor $1/2$, and we get the face-centered cubic, of FCC lattice by stretching with a factor 2. We will explain the significance of these lattices for the covering and packing of congruent balls shortly.

Background. In the Euclidean plane, there is a single lattice that gives the thinnest covering of congruent disks as well as the densest packing of congruent disks. This is the hexagonal lattice, which consists of all integer combinations of the vectors

$$v_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1+\sqrt{3} \\ 1-\sqrt{3} \end{pmatrix}, \quad v_2 \;=\; \frac{1}{2\sqrt{3}} \begin{pmatrix} 1-\sqrt{3} \\ 1+\sqrt{3} \end{pmatrix}.$$

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Placing disks of radius $\sqrt{2}/3$ centered at the lattice points, we get a covering, and reducing the radius to $1/\sqrt{6}$, we get a packing. Both are optimal in the sense that no other covering achieves a smaller covering density (see Kershner [12]), and no other packing achieves a larger packing density (see Thue [21]). Elegant proofs of both results can be found in Fejes Tóth [8].

The situation gets more complicated already in $\mathbb{R}^3$, where the lattice that gives the thinnest covering is different from the one that gives the densest packing. For covering, the BCC lattice gives the smallest density of a lattice covering (see Bambah [1]), but the existence of an even thinner non-lattice covering has not yet been contradicted. For packing, the FCC lattice gives the highest density (see Gauß[10]), and the claim that no non-lattice packing can be denser has become known as the Kepler Conjecture, one of the foremost mathematical questions of our time [20]. Stated in 1611, the conjecture remained open until Hales gave a computer-assisted proof confirming Kepler's conjecture in 2005 [11].

Even less is known in dimensions beyond 3. The generalization of the BCC lattice gives thin coverings that are known to be optimal among lattice coverings in dimension 4 (see Delone and Ryskov [5]) and in dimension 5 (see Ryskov and Baranovskii [17]). The thinnest known coverings in dimensions 6 to 24 can be found in [18, 19] and the related website[1]. In contrast, the generalization of the FCC lattice fails to give the densest packing already in dimension 4. Nevertheless, the densest lattice packings are known in dimensions 4 and 5 (see Korkine and Zolotareff [13]), and in dimensions 6, 7 and 8 (see Blichfeldt [2]). No further optimality results are available until dimension 24 in which the Leech lattice, discovered independently by Witt in 1940 [22] and by Leech in 1965 [15], gives a surprisingly thin covering and dense packing. The optimality among the lattice packings has recently been established by Cohn and Kumar [3].

Results. In this paper, we give a complete analysis of the coverings and packings generated by the lattices obtained by a diagonal distortion of the integer grid. Specifically, we give closed-form expressions of the covering and packing densities as functions of $\delta > 0$, the distortion parameter. The complete analysis is possible because we get only a small number of combinatorially different Delaunay complexes for the 1-parameter family of lattices. For $0 < \delta < 1$, the distortion is a compression, and the Delaunay complex consists of copies of the Freudenthal triangulation of the unit cube. Among these lattices, we find the thinnest coverings for $\delta = 1/\sqrt{n+1}$, giving optimal covering densities among lattices for dimensions 2, 3, 4, and 5. For $\delta = 1$, the distortion is the identity, and the Delaunay complex consists of copies of the unit cube. For $1 < \delta$, the distortion stretches the integer grid, and the Delaunay complex consists of distorted diagonal slices of the unit cube. Among these lattices, we find the densest packings for $\delta = \sqrt{n+1}$, giving optimal packing densities for dimensions 2 and 3.

Outline. Section 2 introduces two decompositions of the n-cube: the Freudenthal triangulation and the slice decomposition. Section 3 explains how a lattice in $\mathbb{R}^n$ defines a covering and a packing, and how we measure their densities. Section 4 gives a complete analysis of the covering density as a function of the distortion. Section 5 does the same for the packing density. Section 6 concludes the paper.

[1] http://www.math.uni-magdeburg.de/lattice_geometry/

2 Decomposing the n-Cube

In this section, we introduce the two decompositions of the cube that are instrumental in the analysis of the covering and packing densities of the 1-parameter family of lattices.

Freudenthal triangulation. We write $[n] = \{1, 2, \ldots, n\}$ for the set of coordinate directions in $\mathbb{R}^n$ and e_i for the unit vector in the i-th coordinate direction. The n-dimensional *unit cube*, $\mathbb{U}^n = [0, 1]^n$, has 2^n vertices u_I, each corresponding to a subset $I \subseteq [n]$ such that $u_I = \sum_{i \in I} e_i$. We say u_I *precedes* u_J if $I \subseteq J$ and $I \neq J$. This defines a partial order on the vertices, with a unique smallest vertex $\mathbf{0} = u_\emptyset$, and a unique largest vertex $\mathbf{1} = u_{[n]}$. A *chain* is a sequence of distinct vertices in which each vertex precedes the next one. Its *length* is the number of vertices. Each chain of length $k + 1$ defines a k-simplex, namely the convex hull of its $k + 1$ vertices. The *Freudenthal triangulation* of the n-cube, denoted as $\mathcal{F}^n = \mathcal{F}(\mathbb{U}^n)$, is the set of all simplices defined by chains [9, 14]; see Figure 1.

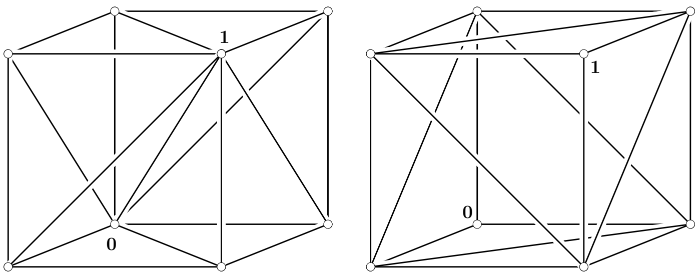

Fig. 1: Left: the Freudenthal triangulation of the 3-cube consisting of six tetrahedra sharing the edge that connects $\mathbf{0}$ with $\mathbf{1}$. Right: the slice decomposition of the 3-cube consisting of two tetrahedra sandwiching an octahedron.

Define the *silhouette* of the n-cube as its projection along the diagonal direction, which is an $(n-1)$-dimensional convex polytope. It is not difficult to see that all vertices other than $\mathbf{0}$ and $\mathbf{1}$ project to vertices of the silhouette. The faces of the silhouette have dimension between 0 and $n - 2$. We can triangulate these faces such that the join of the preimage of every $(k-2)$-simplex with the edge connecting $\mathbf{0}$ with $\mathbf{1}$ gives a k-simplex of the Freudenthal triangulation.

Slice decomposition. Let U_i be the subset of vertices u_J with $\operatorname{card} J = i$, and let H_i be the $(n - 1)$-dimensional hyperplane orthogonal to the diagonal direction that passes through the vertices of U_i, for $0 \leq i \leq n$. The $n + 1$ hyperplanes cut the n-cube into n *slices*, each of width $1/\sqrt{n}$. We call this the *slice decomposition* of the n-cube, denoted at $\mathcal{S}^n = \mathcal{S}(\mathbb{U}^n)$; see Figure 1. We note that for each edge of the n-cube, there is a unique i such that its endpoints belong to U_{i-1} and to U_i. In other words, the edge does not cross any of the hyperplanes and therefore belongs to a unique slice. It follows that the i-th slice is the convex hull of the points in $U_{i-1} \cup U_i$ and that its number of vertices is $\binom{n}{i-1} + \binom{n}{i}$. Furthermore, the i-th slice is the central reflection of the $(n-i+1)$-st slice whose vertices are the points in $U_{n-i} \cup U_{n-i+1}$.

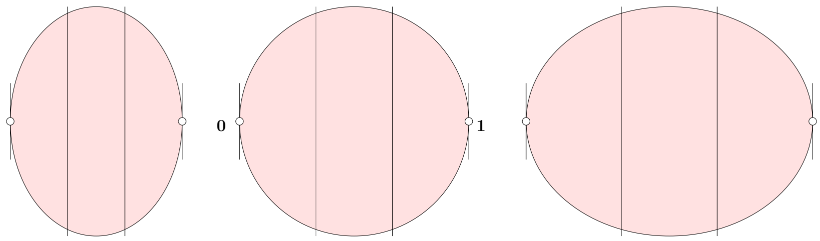

Fig. 2: The sliced circumsphere of the 3-cube in the middle, with its compressed and stretched images on the left and the right.

The hyperplanes can also be used to cut the circumscribed $(n-1)$-sphere, S, of the unit n-cube; see Figure 2. For $0 \leq i \leq n$, let $S_i = S \cap H_i$ and note that $S_0 = \mathbf{0}$, $S_n = \mathbf{1}$, and all other S_i are $(n-2)$-dimensional spheres. The radius of S is $\sqrt{n}/2$. We can therefore compute the radius of S_i as

$$r_i = \sqrt{\frac{n}{4} - \left(\frac{\sqrt{n}}{2} - \frac{i}{\sqrt{n}}\right)^2} \quad = \quad \sqrt{i - \frac{i^2}{n}}. \tag{1}$$

As n goes to infinity, the radius of S_1 converges to 1, while the radius of $S_{n/2}$ is $\sqrt{n}/2$ and thus diverges. Remarkably, the points in U_1 are nevertheless vertices of the silhouette of the n-cube. Note that the r_i are also the distances of the vertices of the silhouette from its center.

1 (Silhouette Lemma) *Let s_I and s_J be the projections of u_I and u_J. Assuming $I, J \neq \emptyset, [n]$, both are vertices of the silhouette and $\|s_I\| \leq \|s_J\|$ iff $(\operatorname{card} I - \frac{n}{2})^2 \geq (\operatorname{card} J - \frac{n}{2})^2$.*

This fact will be relevant in Section 5, where we analyze the packing density of a 1-parameter family of lattices. Now consider compressing or stretching the cube and its circumsphere along the diagonal direction. If we compress, we get an ellipsoid of *pancake* type, and the Delaunay complex of the 2^n points is the compressed Freudenthal triangulation; see [7] for a proof. If we stretch, we get an ellipsoid of *cigar* type, and the Delaunay complex of the 2^n vertices is the stretched slice decomposition; see Figure 1.

3 Lattices

In this section, we introduce the 1-parameter family of lattices and explain how they define packings and coverings. Writing V_n for the (n-dimensional) volume of the n-

dimensional unit ball, $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, we have

$$V_n = \begin{cases} \pi^{\frac{n}{2}} / \left(\frac{n}{2}\right)! & \text{if } n \text{ is even,} \\ \pi^{\frac{n-1}{2}} 2^{\frac{n+1}{2}} / n!! & \text{if } n \text{ is odd,} \end{cases}$$

where $n!! = n \cdot (n-2) \cdot \ldots \cdot 3 \cdot 1$ is the double factorial; see e.g. [4].

Covering and packing. A *lattice* in $\mathbb{R}^n$ consists of all integer combinations of n linearly independent vectors v_i. Important numbers of a lattice $\mathcal{L}$ are its *determinant*, its *covering radius*, and its *packing radius*:

$$\det \mathcal{L} = \det[v_1 v_2 \ldots v_n],$$
$$R(\mathcal{L}) = \max_{x \in \mathbb{R}^n} \min_{a \in \mathcal{L}} \|x - a\|,$$
$$r(\mathcal{L}) = \min_{\mathbf{0} \neq a \in \mathcal{L}} \|a\|/2.$$

Suppose we choose a radius r and replace each point $a \in \mathcal{L}$ by the ball of radius r centered at a. The *density* of the resulting set of balls is the number of balls that contain an average point:

$$\varrho(r) = \frac{V_n r^n}{\det \mathcal{L}}. \tag{2}$$

For $r \geq R(\mathcal{L})$, we get a *covering* in which the balls cover every point at least once. The density is therefore greater than or equal to 1. For $r \leq r(\mathcal{L})$, we get a *packing* in which the balls have disjoint interiors. The density is therefore less than or equal to 1. Two lattices are *isomorphic* if they are related by a similarity. In this case, the two lattices give the same densities. We are interested in finding the lattices that give smallest possible covering density and the largest possible packing density.

The mother of all lattices is the *integer grid*, $\mathcal{L} = \mathbb{Z}^n$. We have $\det \mathcal{L} = 1$, $r(\mathcal{L}) = 1/2$, and $R(\mathcal{L}) = \sqrt{n}/2$. The corresponding packing density is $V_n/2^n$ and the corresponding covering density is $n^{\frac{n}{2}} V_n / 2^n$. For small values of n, these are given in Table 1.

n	volume of unit ball	covering density	packing density
2	$\pi = 3.141\ldots$	$\pi/2 = 1.570\ldots$	$\pi/4 = 0.785\ldots$
3	$4\pi/3 = 4.188\ldots$	$\sqrt{3}\pi/2 = 2.720\ldots$	$\pi/6 = 0.523\ldots$
4	$\pi^2/2 = 4.934\ldots$	$\pi^2/2 = 4.934\ldots$	$\pi^2/32 = 0.308\ldots$
5	$8\pi^2/15 = 5.263\ldots$	$5\sqrt{5}\pi^2/12 = 9.195\ldots$	$\pi^2/60 = 0.164\ldots$
6	$\pi^3/6 = 5.167\ldots$	$9\pi^3/16 = 17.441\ldots$	$\pi^3/384 = 0.060\ldots$
7	$16\pi^3/105 = 4.724\ldots$	$49\sqrt{7}\pi^3/120 = 33.497\ldots$	$\pi^3/840 = 0.036\ldots$
8	$\pi^4/24 = 4.058\ldots$	$2\pi^4/3 = 64.939\ldots$	$\pi^4/6144 = 0.015\ldots$

Table 1: From left to right: the volume of $\mathbb{B}^n$, the covering density of the integer grid in $\mathbb{R}^n$, and the packing density of the same grid.

Distortion. To describe a 1-parameter family of distortions of the integer grid, we introduce the *diagonal height function*, $\Delta : \mathbb{R}^n \to \mathbb{R}$, which maps every point $x = (x_1, x_2, \ldots, x_n)$ to $\Delta(x) = \langle x, \mathbf{1} \rangle = \sum_{i=1}^{n} x_i$. It is $\sqrt{n}$ times the (signed) Euclidean distance of x from the *diagonal hyperplane*, $\Delta^{-1}(0)$. For each $\delta \in \mathbb{R}$, we construct a lattice $\mathcal{L}_\delta$ by mapping the i-th unit vector to $e_i + D \cdot \mathbf{1}$, where $D = (\delta - 1)/n$. The corresponding linear transformation, $T_\delta : \mathbb{R}^n \to \mathbb{R}^n$, is given by

$$T_\delta(x) = x + D\Delta(x) \cdot \mathbf{1}. \tag{3}$$

Hence, $\mathcal{L}_\delta = T_\delta(\mathbb{Z}^n)$, and we note that $\mathcal{L}_1 = \mathbb{Z}^n$. For vanishing distortion parameter δ, we get a set of points in $\Delta^{-1}(0)$, which has only $n-1$ dimensions. This set is again a lattice and, more specifically, one in our 1-parameter family, as we now prove.

2 (Lattice Projection Lemma) *The diagonal projection of the n-dimensional integer grid, $T_0(\mathbb{Z}^n)$, is isometric to $T_\delta(\mathbb{Z}^{n-1})$, for $\delta = 1/\sqrt{n}$.*

Proof. Let L be the set of lines in $\mathbb{R}^n$ obtained by drawing a line in diagonal direction through every point in $\mathbb{Z}^n$. Intersecting L with the hyperplane G spanned by the first $n-1$ coordinate axes, we get $\mathbb{Z}^{n-1}$. Intersecting L with $H = \Delta^{-1}(0)$, wet get $T_0(\mathbb{Z}^n)$. Both are sets in $n-1$ dimensions, and we can interpolate between them by rotating the hyperplane around $G \cap H$, from G to H. This interpolation is exactly the distortion of $\mathbb{Z}^{n-1}$ defined above. It remains to show that $H \cap L$ is the distorted integer grid for $\delta = 1/\sqrt{n}$. To see this, we consider the two lines in L that pass through $\mathbf{1}$ and through $\mathbf{1}' = (1, \ldots, 1, 0)$ in $\mathbb{R}^n$. They intersect G in $\mathbf{0}$ and $\mathbf{1}'$ and they intersect H in $\mathbf{0}$ and $\mathbf{1}''$, the projection of $\mathbf{1}'$ onto H. The distance between $\mathbf{0}$ and $\mathbf{1}'$ is $\sqrt{n-1}$. To compute the distance between $\mathbf{0}$ and $\mathbf{1}''$, we consider the triangles spanned by $\mathbf{0}$, $\mathbf{1}$, $\mathbf{1}'$ and by $\mathbf{0}$, $\mathbf{1}'$, $\mathbf{1}''$; see Figure 3. The two triangles are similar, which implies that the distance between

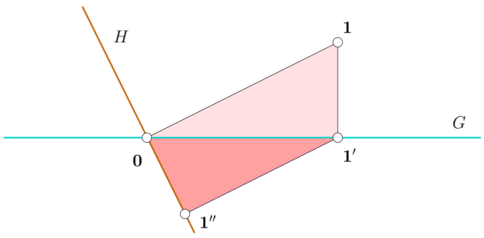

Fig. 3: Two similar right-angled triangles in $\mathbb{R}^n$.

the two intersection points in H is

$$\|\mathbf{0} - \mathbf{1}''\| = \|\mathbf{1} - \mathbf{1}'\| \cdot \frac{\|\mathbf{0} - \mathbf{1}'\|}{\|\mathbf{0} - \mathbf{1}\|} = \sqrt{1 - \frac{1}{n}}.$$

The distortion factor is the ratio of the distance between $\mathbf{0}$ and $\mathbf{1}''$ in H and between $\mathbf{0}$ and $\mathbf{1}'$ in G, which is $\delta = 1/\sqrt{n}$. $\square$

We will see shortly that the distortion of the $(n-1)$-dimensional integer grid for $\delta = 1/\sqrt{n}$ provides the thinnest covering in the 1-parameter family we consider in this paper.

Projected Freudenthal simplex. We are interested in the diagonal projection of an n-dimensional Freudenthal simplex and the radius of its circumscribed sphere. Take the n-simplex spanned by the points $y_i = \sum_{j=1}^{i} e_j$, for $0 \leq i \leq n$, noting that $y_0 = \mathbf{0}$ and $y_n = \mathbf{1}$. The projection of y_i onto $H = \Delta^{-1}(0)$ is $x_i = T_0(y_i)$, where

$$x_i = \frac{1}{n}(n-i, \ldots, n-i, -i, \ldots, -i)$$

is a point with i equal leading coordinates and $n-i$ equal trailing coordinates. Since $x_0 = x_n$, we get only n different points which span an $(n-1)$-simplex in H, the projection of the n-simplex. Perhaps surprisingly, it is not difficult to find the center and radius of the circumsphere of the $(n-1)$-simplex. For that purpose, we consider the point

$$z = \frac{1}{n}(n-1, n-2, \ldots, 1, 0)$$

and note that $\Delta(z) = \frac{1}{n}\sum_{i=1}^{n-1} i = \frac{n-1}{2}$. The projection of z onto H is therefore $z' = T_0(z) = z - \frac{n-1}{2n} \cdot \mathbf{1}$, which gives

$$z' = \frac{1}{2n}(n-1, n-3, \ldots, -n+3, -n+1).$$

To compute the distance between the two projected points, we write the vectors of $2nx_i$, $2nz'$, and $2n(x_i - z')$:

$$\begin{aligned}
&(2n-2i, \ldots, 2n-2i \; ; -2i, \ldots, -2i),\\
&(n-1, \ldots, n-2i+1 \; ; n-2i-1, \ldots, -n+1),\\
&(n-2i+1, \ldots, n-1 \; ; -n+1, \ldots, n-2i-1),
\end{aligned}$$

showing the 1-st, i-th, $(i+1)$-st, and n-th coordinates. We can read the difference as a cyclic rotation of the vector $(-n+1, -n+3, \ldots, n-1)$. In other words, all vectors of the form $x_i - z'$ are cyclic rotations of each other, which implies that the $n+1$ points x_i all have the same distance from z'. This distance is also the radius of the circumscribed sphere of the $(n-1)$-simplex:

$$R_0 = \sqrt{\frac{(n-1)(n+1)}{12n}}. \tag{4}$$

We will use this radius in the analysis of the covering density in Section 4.

4 Covering

To compute the covering radius, we need to understand the Voronoi diagram of $\mathcal{L}_\delta$ or, equivalently, the Delaunay complex. Fortunately, there are only two types.

Radius of a slice. For $\delta > 1$, the Delaunay complex consists of distorted copies of the slice decomposition:

$$\mathrm{Del}(\mathcal{L}_\delta) = T_\delta(\mathcal{S}^n + \mathbb{Z}^n).$$

We may restrict ourselves to the slices in the decomposition of the distorted unit cube. The center of the circumsphere of every slice lies on the diagonal and between the two delimiting hyperplanes. It follows that the circumradii of the slices increase toward the middle, similar to the radii of the $(n-2)$-spheres in the Silhouette Lemma. For odd n, we have a unique middle slice, and for even n, we have two symmetric slices separated by the middle hyperplane.

Assume first that n is odd. The circumscribed $(n-1)$-sphere of the middle slice passes through two $(n-2)$-spheres of radius

$$r = \sqrt{\frac{n-1}{2} - \frac{(n-1)^2}{4n}} \quad = \quad \frac{1}{2}\sqrt{n - \frac{1}{n}}$$

and distance $2d = \delta/\sqrt{n}$ from each other; see (1). The radius of the $(n-1)$-sphere is therefore

$$R(\delta) = \sqrt{r^2 + d^2} = \frac{1}{2\sqrt{n}}\sqrt{n^2 - 1 + \delta^2}. \tag{5}$$

Now assume that n is even. The radii of the two $(n-2)$-spheres defining a slice next to the middle hyperplane are

$$r = \sqrt{\frac{n-2}{2} - \frac{(n-2)^2}{4n}} \quad = \quad \frac{1}{2}\sqrt{n - \frac{4}{n}}$$

and $\sqrt{n}/2$; see again (1). The distance between the two supporting hyperplanes is $d_1 + d_2 = \delta/\sqrt{n}$. We compute d_1 such that $r^2 + d_1^2 = \frac{n}{4} + d_2^2$. This gives $d_1 = (\delta^2+1)/2\delta\sqrt{n}$ and $d_2 = (\delta^2 - 1)/2\delta\sqrt{n}$. The radius of the circumscribed $(n-1)$-sphere is therefore

$$R(\delta) = \sqrt{\frac{n}{4} + d_2^2} = \frac{1}{2\sqrt{n}}\sqrt{\delta^2 + n^2 - 2 + \frac{1}{\delta^2}}. \tag{6}$$

Radius of a simplex. For $0 < \delta < 1$, the Delaunay complex consists of distorted copies of the Freudenthal triangulation:

$$\mathrm{Del}(\mathcal{L}_\delta) = T_\delta(\mathcal{F}^n + \mathbb{Z}^n).$$

All n-simplices are of the same type, and it suffices to compute the circumradius of the one spanned by the images of the points $y_i = \sum_{j=1}^{i} e_j$, for $0 \le i \le n$. At the beginning of the distortion, when $\delta = 1$, the circumsphere of the Freudenthal n-simplex has radius half the length of the diagonal edge, and at the end, when $\delta = 0$, the circumsphere has a radius specified in (4). We will make use of the fact that the radius of any distorted image of the n-simplex can be expressed in terms of δ and the radii at $\delta = 1$ and at $\delta = 0$. To state the result formally, we let $z(\delta)$ and $R(\delta)$ be the center and the radius of the n-simplex at distortion value $0 \le \delta \le 1$.

3 (Distortion Lemma) *The squared radius of the circumsphere of the distorted image of the Freudenthal n-simplex satisfies $R^2(\delta) = \delta^2 R_1^2 + (1 - 2\delta^2 + \delta^4) R_0^2$.*

A proof is given in Appendix A. Using $R_1^2 = n/4$ and $R_0^2 = (n^2 - 1)/(12n)$ from (4), we get

$$\begin{aligned} R(\delta) &= \sqrt{\frac{\delta^2 n}{4} + \frac{(1 - 2\delta^2 + \delta^4)(n^2 - 1)}{12n}} \\ &= \sqrt{\frac{(n^2 - 1) + (n^2 + 2)\delta^2 + (n^2 - 1)\delta^4}{12n}}. \end{aligned} \tag{7}$$

In summary, we have three different formulas for the covering radius: the one in (5) for $1 \leq \delta$ in odd dimension, the one in (6) for $1 \leq \delta$ in even dimension, and the one in (7) for $0 \leq \delta \leq 1$.

Covering density. Given the radius $R = R(\delta)$, we get the corresponding covering density as $\gamma(\delta) = V_n R^n / \delta$ from (2). We show below that $\gamma(\delta)$ has two local minima: one in the first interval at $\delta = 1/\sqrt{n+1}$, and the other in the second interval at $\delta = \sqrt{n+1}$; see Figure 4. By comparing with the graphs for the packing density in the same figure, we note that the minima for covering coincide with the maxima for packing. We analyze γ, distinguishing between the three cases we encountered for the covering radius.

CASE 1. $0 \leq \delta \leq 1$. Then

$$\gamma(\delta) = \frac{V_n}{(12n)^{\frac{n}{2}}} \cdot \frac{A^{\frac{n}{2}}}{\delta}, \tag{8}$$

where $A = (n^2 - 1) + (n^2 + 2)\delta^2 + (n^2 - 1)\delta^4$. We compute the derivative as

$$\gamma'(\delta) = \frac{V_n}{(12n)^{\frac{n}{2}}} \cdot \frac{\frac{n}{2}\delta A^{\frac{n}{2}-1} A' - A^{\frac{n}{2}}}{\delta^2} = \frac{V_n}{(12n)^{\frac{n}{2}}} \cdot A^{\frac{n}{2}-1} \cdot a,$$

where $a = (2n^2 + n - 1)\delta^2 + (n^2 + 2) - \frac{n+1}{\delta^2}$. The only factor that can vanish is a, so we get $\gamma'(\delta) = 0$ iff $\delta^2 = \frac{1}{n+1}$. This critical point can only be a minimum.

CASE 2.1. $\delta \geq 1$ and n is odd. Then

$$\gamma(\delta) = \frac{V_n}{2^n n^{\frac{n}{2}}} \cdot \frac{B^{\frac{n}{2}}}{\delta}. \tag{9}$$

where $B = \delta^2 + n^2 - 1$. The derivative is

$$\gamma'(\delta) = \frac{V_n}{2^n n^{\frac{n}{2}}} \cdot B^{\frac{n}{2}-1} \cdot b,$$

where $b = (n-1)(1 - \frac{n+1}{\delta^2})$. The only factor that can vanish is b, so we have $\gamma'(\delta) = 0$ iff $\delta^2 = n + 1$. This can only be a minimum.

CASE 2.2. $\delta \geq 1$ and n is even. Then

$$\gamma(\delta) = \frac{V_n}{2^n n^{\frac{n}{2}}} \cdot \frac{C^{\frac{n}{2}}}{\delta}. \tag{10}$$

where $C = \delta^2 + \frac{1}{\delta^2} + n^2 - 2$. As before, we compute the derivative and get

$$\gamma'(\delta) = \frac{V_n}{2^n n^{\frac{n}{2}}} \cdot C^{\frac{n}{2}-1} \cdot c,$$

where $c = n - \frac{n}{\delta^4} - 1 - \frac{n^2-2}{\delta^2}$. The last factor that can vanish is c, so we have $\gamma'(\delta) = 0$ iff $\delta^2 = n + 1$, as in Case 2.1. Again, this can only be a minimum.

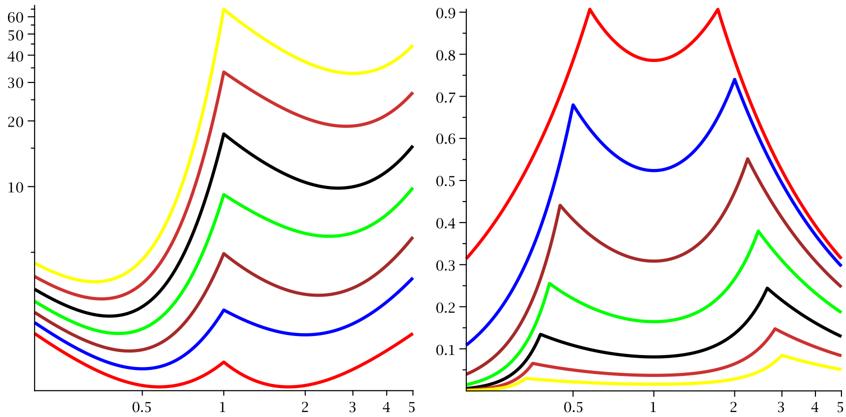

Fig. 4: Left, from bottom to top: the graphs of the covering density in dimensions 2 to 8. All functions have two local minima, the lesser at $\delta = \sqrt{n+1}$ and the global minimum at $\delta = 1/\sqrt{n+1}$. Right, from top to bottom: the graphs of the packing density in dimensions 2 to 8. All functions have two local maxima, the lesser at $\delta = 1/\sqrt{n+1}$ and the global maximum at $\delta = \sqrt{n+1}$. Some of the axes use logarithmic scale for clarity.

Examples. In the plane, the minimum covering density is achieved by the hexagonal lattice, with $\gamma(1/\sqrt{3}) = \gamma(\sqrt{3}) = 1.209\ldots$. More generally, we get

$$\gamma(\delta) = \begin{cases} \frac{\pi}{8}\left(\delta^3 + 2\delta + \frac{1}{\delta}\right) & \text{for } 0 \leq \delta \leq 1, \\ \frac{\pi}{8}\left(\delta + \frac{2}{\delta} + \frac{1}{\delta^3}\right) & \text{for } 1 \leq \delta, \end{cases}$$

using the formulas (8) and (10) for $n = 2$; see the lowest graph in Figure 4 on the left. Note the local maximum for the square lattice, with $\gamma(1) = 1.570\ldots$. We have $\gamma(\delta) = \gamma(1/\delta)$ for all $\delta > 0$. In $\mathbb{R}^3$, we get the thinnest covering for $\mathcal{L}_{1/2}$, with covering density $\gamma(\frac{1}{2}) = 1.463\ldots$. Compare this with $\gamma(2) = 2.094\ldots$ for the FCC lattice and with $\gamma(1) = 2.720\ldots$ for the cubic lattice. More generally, we get

$$\gamma(\delta) = \begin{cases} \frac{\pi(8+11\delta^2+8\delta^4)^{3/2}}{162\delta} & \text{for } 0 \le \delta \le 1, \\ \frac{\pi(8+\delta^2)^{3/2}}{18\sqrt{3}\delta} & \text{for } 1 \le \delta; \end{cases}$$

see the second lowest graph in Figure 4 on the left. The lattice $\mathcal{L}_{1/2}$ is isomorphic to the BCC lattice, which is commonly described as the set of integer points plus the integer points shifted by $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Recall that for $n = 2$, the two local minima correspond to the same lattice and thus give the same covering density. In contrast, for dimensions $n \ge 3$, we get a smaller density for $\delta = 1/\sqrt{n+1}$ than for $\delta = \sqrt{n+1}$. Using (4) and the Lattice Projection Lemma, we get the corresponding covering radius as the square root of $(n^2 + 2n)/(12n + 12)$. The best covering density within our 1-parameter family is therefore

$$\gamma(1/\sqrt{n+1}) = V_n \sqrt{n+1} \left(\frac{n(n+2)}{12(n+1)} \right)^{\frac{n}{2}};$$

see the left half of Table 2.

n	covering density $\gamma(\frac{1}{\sqrt{n+1}})$	covering density best	packing density $\varphi(\sqrt{n+1})$	packing density best
2	**1.209...**		**0.906...**	
3	**1.463...**		**0.740...**	
4	**1.765...**		0.551...	**0.616...**
5	**2.124...**		0.379...	**0.465...**
6	2.551...	2.464...	0.244...	**0.372...**
7	3.059...	2.900...	0.147...	**0.295...**
8	3.665...	3.142...	0.084...	**0.253...**

Table 2: Left: the covering densities of $\mathcal{L}_\delta$ for $\delta = 1/\sqrt{n+1}$ up to dimension $n = 8$, and the best known covering densities for comparison. Right: the packing densities of $\mathcal{L}_\delta$ for $\delta = \sqrt{n+1}$, and the best known packing densities for comparison. Densities that are known to be optimal for lattices are displayed in bold.

5 Packing

In this section, we give a formula for the packing density as a function of the distortion parameter.

Packing radius. To get the packing radius of $\mathcal{L}_\delta$, we consider the point $\mathbf{0}$ and find the closest other lattice point. Using the Silhouette Lemma from Section 2, we observe that there are only three possibilities:

$$\begin{aligned} T_\delta(e_1) &= (1+D, D, \ldots, D), \\ T_\delta(e_1 - e_2) &= (1, -1, 0, \ldots, 0), \\ T_\delta(\mathbf{1}) &= (\delta, \delta, \ldots, \delta). \end{aligned}$$

The distance to $T_\delta(e_1 - e_2)$ is $\sqrt{2}$, and that to $T_\delta(\mathbf{1})$ is $\delta\sqrt{n}$. The distance to the image of the first unit vector is

$$\|T_\delta(e_1)\| = \sqrt{(1+D)^2 + (n-1)D^2} = \sqrt{1 + \frac{\delta^2 - 1}{n}}.$$

Plugging $\delta^2 = n+1$ into the formula, we get $\|T_\delta(e_1)\| = \sqrt{2}$, and plugging $\delta^2 = 1/(n+1)$ into it, we get $\|T_\delta(e_1)\| = \delta\sqrt{n}$. We thus have three intervals in which the packing radius has qualitatively different behavior:

$$r(\mathcal{L}_\delta) = \begin{cases} \frac{1}{2}\delta\sqrt{n} & \text{for } 0 \leq \delta \leq \frac{1}{\sqrt{n+1}}, \\ \frac{1}{2}\sqrt{1 + \frac{\delta^2-1}{n}} & \text{for } \frac{1}{\sqrt{n+1}} \leq \delta \leq \sqrt{n+1}, \\ \frac{1}{2}\sqrt{2} & \text{for } \sqrt{n+1} \leq \delta. \end{cases}$$

Packing density. Given the radius $r = r(\mathcal{L})$, we get the corresponding packing density as $\varphi(\delta) = V_n r^n/\delta$ from (2). In the first interval, the density grows like δ^{n-1}, and in the last interval, it shrinks like $1/\delta$. We now prove that in the middle interval, φ has a single minimum, which it attains at $\delta = 1$. Indeed, we have

$$\varphi(\delta) = \frac{V_n}{2^n n^{\frac{n}{2}}} \cdot \frac{E^{\frac{n}{2}}}{\delta}, \tag{11}$$

where $E = \delta^2 + n - 1$. The derivative with respect to the distortion parameter is

$$\varphi'(\delta) = \frac{V_n}{2^n n^{\frac{n}{2}}} \cdot \frac{\frac{n}{2}\delta E^{\frac{n}{2}-1}E' - E^{\frac{n}{2}}}{\delta^2} = \frac{V_n}{2^n n^{\frac{n}{2}}} \cdot E^{\frac{n}{2}-1} \cdot e,$$

where $e = (n-1)(1 - \frac{1}{\delta^2})$. The only factor that can vanish is e. Restricting ourselves to non-negative values of the distortion parameter, we have $\varphi'(\delta) = 0$ iff $\delta = 1$. This critical point can only be a minimum. In summary, the packing density has local maxima at $\delta = 1/\sqrt{n+1}$ and $\delta = \sqrt{n+1}$, a local minimum at $\delta = 1$, and goes to zero as δ goes to 0 or to ∞; see the graphs in Figure 4.

Examples. In the plane, the maximum packing density is attained for $\delta = 1/\sqrt{3}$ and $\delta = \sqrt{3}$. For both values of the distortion parameter, $\mathcal{L}_\delta$ is isomorphic to the standard hexagonal lattice, with packing density $\varphi(1/\sqrt{3}) = \varphi(\sqrt{3}) = 0.906\ldots$. More generally, we have $\varphi(\delta) = V_2 r^2/\delta$, where $V_2 = \pi$ and $r = r(\mathcal{L}_\delta)$. Using the above formulas for the radius, we thus have

$$\varphi(\delta) = \begin{cases} \frac{\pi\delta}{2} & \text{for } 0 \leq \delta \leq \frac{1}{\sqrt{3}}, \\ \frac{\pi}{8}(\delta + \frac{1}{\delta}) & \text{for } \frac{1}{\sqrt{3}} \leq \delta \leq \sqrt{3}, \\ \frac{\pi}{2\delta} & \text{for } \sqrt{3} \leq \delta; \end{cases}$$

see the highest graph in Figure 4 on the right. Note that $\varphi(\delta) = \varphi(\frac{1}{\delta})$ for all $\delta > 0$ and that this function has a local minimum for the square lattice at $\varphi(1) = 0.785\ldots$; compare this with the graph of the covering density in the plane. In $\mathbb{R}^3$, we get local maxima at $\delta = 1/2$ and $\delta = 2$. More generally, we have

$$\varphi(\delta) = \begin{cases} \frac{\sqrt{3}\pi\delta^2}{2} & \text{for } 0 \leq \delta \leq \frac{1}{2}, \\ \frac{\pi(\delta^2+2)^{3/2}}{18\sqrt{3}\delta} & \text{for } \frac{1}{2} \leq \delta \leq 2, \\ \frac{\sqrt{2}\pi}{3\delta} & \text{for } 2 \leq \delta; \end{cases}$$

see the second highest graph in Figure 4 on the right. This function has a local minimum for the cubic lattice at $\varphi(1) = 0.523\ldots$. In contrast to the plane, the values at the two maxima are not the same and we get the higher density at $\varphi(2) = 0.740\ldots$, where $\mathcal{L}_2$ is isomorphic to the FCC lattice. Most commonly, that lattice is described as the set of integer points for which the sum of coordinates is even. This lattice differs from $\mathcal{L}_2$ by a rotation of 60° around the line that passes through **0** and **1**.

Recall that for $n = 2$, the two local maxima correspond to the same lattice and thus give the same packing density. In contrast, for dimensions $n \geq 3$, we get a higher density for $\delta = \sqrt{n+1}$ than for $\delta = 1/\sqrt{n+1}$. The best packing density within our 1-parameter family is therefore

$$\varphi(\sqrt{n+1}) = \frac{V_n}{2^{n/2}\sqrt{n+1}};$$

see the right half of Table 2.

6 Discussion

Our simple distortion of the integer grid in diagonal direction leads to a 1-parameter family of lattices that contains optimal lattice coverings in dimensions 2, 3, 4, and 5 and optimal packings in dimensions 2 ad 3. It misses the best lattices in dimensions higher than listed. We therefore pose the question whether our approach can be extended to include the other optimal lattice coverings and packings, in particular the lattices of types D and E and the Leech lattice [4], or even discover lattices with better densities than currently known. Can our 1-parameter analysis be broadened to allow for two or more independent parameters? Alternatively, can we design new 1-parameter families that are easy to analyze and explore the parameter space locally?

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Appendix A

In this appendix, we give a proof of the Distortion Lemma, which is instrumental in the analysis of the covering radius. We begin with a review of weighted points and their polar representation as hyperplanes and points; see e.g. [6].

Weighted points. We construct a convenient framework to express distance relations by generalizing spheres to allow for imaginary radii. A *weighted point* in $n-1$ dimensions is a point $x_i \in \mathbb{R}^{n-1}$ together with a weight $w_i \in \mathbb{R}$. The *power distance* of a point $z \in \mathbb{R}^{n-1}$ from the weighted point (x_i, w_i) is $\varpi_i(z) = \|z - x_i\|^2 - w_i$. Two weighted points are *orthogonal* if

$$\|x_i - x_j\|^2 = w_i + w_j. \tag{12}$$

If w_i and w_j are both positive then (12) characterizes the situation in which the spheres with centers x_i and x_j and radii $\sqrt{w_i}$ and $\sqrt{w_j}$ intersect each other in a right angle.

Let now H be a hyperplane in $\mathbb{R}^n$, z a point in H, y_i a point in $\mathbb{R}^n$, x_i the orthogonal projection of y_i onto H, and $w_i = -\|x_i - y_i\|^2$ the negative of the squared distance of y_i from H. Then it is easy to see that the square of the distance between z and y_i equals the power distance of z from the point x_i with weight w_i in H: $\|z - y_i\|^2 = \varpi_i(z)$. Letting $w = \|z - y_i\|^2$, we can rewrite this relation as $\|z - x_i\|^2 = w_i + w$. In words, the weighted points (x_i, w_i) and (z, w) in H are orthogonal. We will use this observation to reduce the n-dimensional problem of computing the circumscribed sphere of an n-simplex to the $(n-1)$-dimensional problem of computing the weighted point that is simultaneously orthogonal to n other weighted points.

Lifting and polarity. It will be convenient to recast the relation between weighted points in $\mathbb{R}^{n-1}$ in terms of hyperplanes (graphs of affine functions) and points in $\mathbb{R}^n$. Given a point $x_i \in \mathbb{R}^{n-1}$ with weight $w_i \in \mathbb{R}$, we introduce the affine function $h_i : \mathbb{R}^{n-1} \to \mathbb{R}$ via $h_i(x) = 2\langle x_i, x\rangle - \|x_i\|^2 + w_i$. Starting with two orthogonal weighted points in $\mathbb{R}^{n-1}$, we thus get

$$\begin{aligned} \|x_i - x_j\|^2 &= w_i + w_j && \text{iff} \\ \|x_i\|^2 - 2\langle x_i, x_j\rangle - w_i &= -\|x_j\|^2 + w_j && \text{iff} \\ h_i(x_j) &= \|x_j\|^2 - w_j. \end{aligned}$$

This motivates us to introduce the point $p_j = (x_j, \|x_j\|^2 - w_j) \in \mathbb{R}^n$. Traditionally, this point and the hyperplane $\mathrm{graph}(h_j)$ in $\mathbb{R}^n$ are said to be *polar* to each other. We now express what we just proved in terms of these hyperplanes and points.

4 (Ortho-dence Lemma) *The points $x_i, x_j \in \mathbb{R}^{n-1}$ with weights $w_i, w_j \in \mathbb{R}$ are orthogonal iff $p_i \in \mathrm{graph}(h_j)$ iff $p_j \in \mathrm{graph}(h_i)$.*

Proof of Distortion Lemma. We are now ready to formulate the proof of the Distortion Lemma stated in Section 4. Recall that this result concerns the Freudenthal n-simplex with vertices $\sum_{j=1}^{i} e_j$, for $0 \le i \le n$, and its distorted images under the linear transformations $T_\delta : \mathbb{R}^n \to \mathbb{R}^n$, for $0 \le \delta \le 1$. It will be convenient to translate the n-simplex so it is cut in half by the hyperplane of fixed points, $H = \Delta^{-1}(0)$. We thus define $y_i = v - \frac{1}{2}\mathbf{1} + \sum_{j=1}^{i} e_j$, with $v \cdot \mathbf{1} = 0$, for $0 \le i \le n$, and we let Y be the n-simplex spanned by the y_i. This translation does not affect our analysis because $T_\delta(Y)$ is a translate of the distorted original n-simplex, for every δ.

Let $z(\delta)$ be the center and $R(\delta)$ the radius of the circumscribed $(n-1)$-sphere of $T_\delta(Y)$. A benefit of the translation is that $z(\delta) \in H$ for all δ. Indeed, $z(\delta)$ is equally far from the distorted images of y_0 and y_n and therefore lies in the bisector of the two points, which is H. We will see that the set of points $z(\delta)$ is the line segment with endpoints $z_1 = z(1)$ and $z_0 = z(0)$. To show this, we replace each vertex $T_\delta(y_i)$ of the n-simplex by the weighted point $(x_i, w_i(\delta))$, where $x_i = T_0(y_i)$ is the orthogonal projection onto H, and $w_i(\delta) = -\delta^2 \Delta^2(y_i)/n$ is the negative of the squared distance of $T_\delta(y_i)$ from H. By what we said above, the point $z(\delta) \in H$ with weight $R^2(\delta)$ is orthogonal to $(x_i, w_i(\delta))$, for all $0 \leq i \leq n$. Note that in $\mathbb{R}^{n-1}$, we have a common orthogonal weighted point for every generic collection of n weighted points. Here there are $n+1$ weighted points, but two are the same, namely $(x_0, w_0(\delta)) = (x_n, w_n(\delta))$.

In the next step, we replace each $(x_i, w_i(\delta))$ by the affine function $h_i(\delta)$, and we replace each point $z(\delta) \in \mathbb{R}^{n-1}$ with weight $R^2(\delta)$ by the point $p(\delta) = (z(\delta), \|z(\delta)\|^2 - R^2(\delta))$ in $\mathbb{R}^n$. Since $(z(\delta), R^2(\delta))$ is orthogonal to all $(x_i, w_i(\delta))$, the point $p(\delta)$ lies on all hyperplanes of the form $\mathrm{graph}(h_i(\delta))$ in $\mathbb{R}^n$. Now observe what happens when δ changes continuously from 1 to 0. It is convenient to parametrize this motion by $\lambda = \delta^2$, which also goes from 1 to 0. Writing down the formula for the affine map:

$$h_i(\delta)(x) = 2\langle x_i, x\rangle - \|x\|^2 - \frac{\Delta^2(y_i)}{n} \cdot \lambda,$$

we note that changing λ corresponds to an affine vertical translation of each hyperplane. It follows that the common intersection, the point $p(\delta)$, traces out the line segment from $p_1 = p(1)$ to $p_0 = p(0)$ and, more specifically,

$$p(\lambda) = \lambda p_1 + (1-\lambda)p_0. \tag{13}$$

It follows that the projection to the first $n-1$ coordinates satisfies the same relationship, namely $z(\lambda) = \lambda z_1 + (1-\lambda)z_0$. Similarly, we have the same relationship for the n-th coordinate. After some rearrangements, we get the squared radius as the linear interpolation of the squared radii at the extremes plus a correction term:

$$\begin{aligned} R^2(\lambda) &= \lambda R_1^2 + (1-\lambda)R_0^2 + C, \quad \text{with} \\ C &= \|z(\lambda)\|^2 - \lambda\|z_1\|^2 - (1-\lambda)\|z_0\|^2. \end{aligned}$$

To simplify the remaining computations, we now choose the vector in the initial translation of the n-simplex as $v = -(z_1 + z_0)/2$. With this choice, the midpoint between the two centers is the origin so that $z_1 = -z_0$ and we can write $d^2 = \|z_1\|^2 = \|z_0\|^2$. Furthermore, $\|z(\lambda)\|^2 = 4(\lambda^2 - \frac{1}{2})^2 d^2$ and therefore $C = 4\lambda(\lambda-1)d^2$. On the other hand, the distance between z_1 and z_0 is $2d = R_0$, so we get $C = \lambda(1-\lambda)R_0^2$. Adding things up, we get

$$R^2(\lambda) = \lambda R_1^2 + (1 - 2\lambda + \lambda^2)R_0^2.$$

Substituting δ^2 for λ, we get the equation claimed in the Distortion Lemma.